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Spherical 5-Designs Obtained from the Unitary Group $U_{2m}(2)$

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1 Introduction

The purpose of this talk is to give an infinite series of spherical 5-designs constructed from the unitary group over the finite field of four elements. Let $G = U_{2m}(2)$ be the unitary group of dimension $2m$ over $GF(4)$, $V = GF(4)^{2m}$ the natural module of G. Then G acts transitively on the set Ω of (maximal) totally isotropic m-spaces of V. This permutation representation (over $\mathbb{R}$) contains an irreducible representation of dimension $d = (4^m + 2)/3$. Then one can embed the set Ω into the unit sphere S^{d-1} in the Euclidean space $\mathbb{R}^d$.

Theorem 1. $\Omega \hookrightarrow S^{d-1} \subset \mathbb{R}^d$ is a spherical 5-design.

The inner product among the vectors of Ω embedded in $\mathbb{R}^d$ can be made rational-valued, so one obtains integral lattices after a suitable normalization. Shimada [5] considered a related family of lattices, and presented in a talk in January, 2000 at RIMS.

2 Preliminaries

A spherical t-design ($t \in \mathbf{Z}$, $t \geq 0$) is a finite set $\Omega \subset S^{d-1}$ such that

$$\frac{\int_{S^{d-1}} f(x)dx}{\int_{S^{d-1}} 1dx} = \frac{1}{|\Omega|} \sum_{x \in \Omega} f(x)$$

for all polynomial $f \in \mathbb{R}[X_1, \ldots, X_d]$ of degree at most t. Equivalently,

$$\sum_{x,y \in \Omega} Q_i(\langle x, y \rangle) = 0 \qquad (1 \leq i \leq t) \tag{1}$$

$$Q_0(X) = 1, \qquad Q_1(X) = dX,$$
$$\frac{k+1}{d+2k} Q_{k+1}(X) = XQ_k(X) - \frac{d+k-3}{d+2k-4} Q_{k-1}(X)$$

are suitably normalized Gegenbauer polynomials. See [1, 4] for more details on spherical designs. In what follows we simply say a t-design for a spherical t-design.

Examples of spherical designs include the 196,560 vectors of norm 4 in the Leech lattice (a 11-design), the 240 roots of the root system E_8 (a 7-design). Moreover, if $O(d, \mathbb{R}) \supset G$ is a finite irreducible subgroup, then every G-orbit on S^{d-1} is a 2-design. Sidel'nikov [6] showed that there exists a finite group $G \subset O(2^n, \mathbb{R})$ such that every G-orbit on S^{2^n-1} is a 7-design. In general, the Molien series of G on the space of harmonic polynomials determines t for which every G-orbit on the unit sphere becomes a t-design. [1, p.102].

To see that $\Omega \hookrightarrow S^{d-1}$ ($d = (4^m + 2)/3$) is a 5-design, we shall verify the condition (1) with $t = 5$. We note that the values of inner products $\langle x, y \rangle$ are known to be $(-2)^{-j}$, $0 \le j \le m$ (see Table 6.1 (C3) of [3]), and $\langle x, y \rangle = (-2)^j$ if and only if the dimension of the intersection of x and y is $m - j$ (recall that x, y are m-dimensional subspaces of V). The number of pairs $(x, y) \in \Omega^2$ such that $\langle x, y \rangle = (-2)^{-j}$ is given by $|\Omega| k_j$, where

$$k_j = \prod_{h=1}^{j} \frac{2^{2h-1}(4^{m-h+1} - 1)}{4^h - 1}.$$

With these formulas at our disposal, we can verify (1) for any given values of m. However, we shall employ a more general framework to prove Theorem 1.

A comment on the peculiarity of this embedding can be found in [3, Remark, p.276].

3 The Q-polynomial property for the dual polar space associated to $U_{2m}(2)$

As in the previous section, we let m be a fixed positive integer, and denote by Ω the set of totally isotropic m-spaces in the natural module $V = GF(4)^{2m}$ of

$U_{2m}(2)$. The set Ω is called the dual polar space associated to $U_{2m}(2)$, because it is a combinatorial dual of the polar space of absolute points and totally isotropic lines of the projective space $PG(V)$ with a unitary polarity. Then $U_{2m}(2)$ acts on Ω, and the permutation representation (over $\mathbb{R}$) decomposes as follows:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \cdots \perp V_m, \tag{2}$$

where V_0 is the trivial module. Let $E_i \in M_{|\Omega|}(\mathbb{R})$ be the orthogonal projection of $\mathbb{R}\Omega$ onto V_i. If we rearrange the ordering of V_i's if necessary, then there exists a polynomial $v_i^*(X)$ of degree i $(0 \le i \le m)$ such that

$$|\Omega|E_i = v_i^*(|\Omega|E_1) \qquad (0 \le i \le m),$$

where, if

$$v_i^*(X) = \sum_{j=0}^{i} c_{ij} X^j,$$

then

$$v_i^*(|\Omega|E_1) = \sum_{j=0}^{i} c_{ij} |\Omega|^j \underbrace{E_1 \circ \cdots \circ E_1}_{j},$$

where $\circ$ denotes the entry-wise product. Roughly speaking, the existence of such polynomials is refered to as the Q-polynomial property (see [2] for details). It is known that there exist $a_i^*, b_i^*, c_i^* \in \mathbb{R}$ such that

$$X v_i^*(X) = c_{i+1}^* v_{i+1}^*(X) + a_i^* v_i^*(X) + b_{i-1}^* v_{i-1}^*(X) \tag{3}$$

and $\{v_i^*(X)\}$ is a system of orthogonal polynomials.

More generally, one can define a combinatorial structure called an association scheme on which the vector space of real-valued functions on the underlying set Ω can be decomposed into a direct sum like (2), and one can define Q-polynomial property for association schemes. For precise definition, we refer to [2]. The following theorem reveals a relationship between the Q-polynomial property and spherical designs. Here we denote by $E_1(\Omega)$ the set of unit vectors obtained by normalizing the column vectors of the matrix

Theorem 2. Suppose that Ω is a Q-polynomial association scheme.
(i) If $a_1^* = 0$, then $E_1(\Omega)$ is a 3-design.
(ii) If moreover, $b_0^* b_1^* c_2^* + 2(b_1^* c_2^* - b_0^{*2} + b_0^*) = 0$, then $E_1(\Omega)$ is a 4-design.
(iii) If moreover, $a_2^* = 0$, then $E_1(\Omega)$ is a 5-design.

If Ω is the dual polar space for $U_{2m}(2)$, then all hypotheses of the theorem are satisfied, and Ω becomes a 5-design. To check this, we reproduce a more general formula for these numbers for the dual polar spaces associated with $U_{2m}(r)$, where r is a prime power. They can be deduced from the formulas in [2, Section 3.5].

$$b_i^* = \frac{(r^{2m} + r)(r^{2m+2} + (-1)^i r^{i+1})}{(r+1)(r^{2m+2} + r^{2i+1})},$$
$$c_i^* = \frac{r^{i-1}(r^i + (-1)^{i-1})(r^{2m} + r)}{(r+1)(r^{2m} + r^{2i-1})},$$
$$a_i^* = b_0^* - b_i^* - c_i.$$

From these formulas, one checks easily that the conditions (i)–(iii) of Theorem 2 are satisfied precisely when $r = 2$.

One can find a more general formula describing these numbers for known P- and Q-polynomial association schemes [2, Section 3.5]. Thus, it is natural to consider the following problem.

Problem. Classify P- and Q-polynomial association scheme Ω such that $E_1(\Omega)$ is a spherical t-design for $t = 4, 5, 6, \ldots$.

4 Proof of Theorem 2

We use the orthogonality relation of the polyomials $\{v_i^*(X)\}_{i=0}^m$ given by

$$\sum_{h=0}^{m} k_h v_i^*(\theta_h^*) v_j^*(\theta_h^*) = 0 \quad (i \neq j), \tag{4}$$

where $\theta_0^* = \dim V_0 = \mathrm{rank} E_1 = b_0^*$, and $E_1(\Omega)$ has $|\Omega| k_h$ pairs of elements with inner product θ_h^*/θ_0^*. We shall write d instead of θ_0^* to simplify the notation. In view of (1), in order to prove $E_1(\Omega)$ is a t-design, it suffices to show

$$\sum_{h=0}^{m} k_h Q_i(\theta_h^*/d) = 0 \qquad (1 \leq i \leq t). \tag{5}$$

Lemma 3. If the polynomials $Q_s(X/d)$ $(1 \le s \le t)$ are linear combinations of the polynomials $v_1^*(X), \dots, v_t^*(X)$, then $E_1(\Omega)$ is a t-design.

Proof. Since $v_0^*(X) = 1$, the orthogonality relation (4) implies

$$\sum_{h=0}^{m} k_h v_i^*(\theta_h^*) = 0 \quad (i > 0).$$

Then the condition (5) is seen to be satisfied. □

It follows from the definitions that $Q_1(X/dX) = X = v_1^*(X)$, so $E_1(\Omega)$ is always a 1-design. Also, one has

$$Q_2(\frac{X}{d}) = \frac{d+2}{2d}(c_2^* v_2^*(X) + a_1^* v_1^*(X)),$$

and hence $E_1(\Omega)$ is always a 2-design.

To prove part (i) of Theorem 1, we assume $a_1^* = 0$, so that

$$XQ_2(\frac{X}{d}) = \frac{d+2}{2d} c_2^* X v_2^*(X). \tag{6}$$

Then

$$\begin{aligned}
Q_3(\frac{X}{d}) &= \frac{d+4}{3}\left(\frac{X}{d}Q_2(\frac{X}{d}) - (1 - \frac{1}{d})Q_1(\frac{X}{d})\right) \\
&= \frac{d+4}{3d}\left(XQ_2(\frac{X}{d}) - (d-1)Q_1(\frac{X}{d})\right) \\
&= \frac{d+4}{3d}\left(\frac{d+2}{2d}c_2^* X v_2^*(X) - (d-1)v_1^*(X)\right) \\
&= \frac{(d+4)(d+2)c_2^*}{6d^2}(c_3^* v_3^*(X) + a_2^* v_2^*(X)) \\
&\quad + \frac{(d+4)((d+2)c_2^* b_1^* - 2d(d-1))}{6d^2} v_1^*(X).
\end{aligned}$$

Thus $Q_3(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X)$.

Under the assumption of (ii), we have

$$Q_3(\frac{X}{d}) = \frac{(d+4)(d+2)c_2^*}{6d^2}(c_3^* v_3^*(X) + a_2^* v_2^*(X)). \tag{7}$$

$$Q_4(\frac{X}{d}) = \frac{d+6}{4}\left(\frac{X}{d}Q_3(\frac{X}{d}) - \frac{d}{d+2}Q_2(\frac{X}{d})\right)$$
$$= \frac{(d+6)(d+4)(d+2)c_2^*}{24d^3}(c_3^*Xv_3^*(X) + a_2^*Xv_2^*(X)) - \frac{d+6}{8}c_2^*v_2^*(X).$$

It follows from (3) that $Q_4(X/d)$ is a linear combination of $v_1^*(X)$, $v_2^*(X)$, $v_3^*(X)$, $v_4^*(X)$.

Under the assumption of (iii), we have

$$Q_4(\frac{X}{d}) = \frac{(d+6)(d+4)(d+2)c_2^*}{24d^3}c_3^*Xv_3^*(X) - \frac{d+6}{8}c_2^*v_2^*(X), \qquad (8)$$

which is a linear combination of $v_2^*(X)$, $v_3^*(X)$, $v_4^*(X)$ by (3). Thus $XQ_4(X/d)$ is a linear combination of $v_1^*(X)$, $v_2^*(X)$, $v_3^*(X)$, $v_4^*(X)$, $v_5^*(X)$ by (3). Since

$$Q_5(\frac{X}{d}) = \frac{d+8}{5}\left(\frac{X}{d}Q_4(\frac{X}{d}) - \frac{d+1}{d+4}Q_3(\frac{X}{d})\right)$$

and $Q_3(X/d)$ is a scalar multiple of $v_3^*(X)$ by (7), we see that $Q_5(X/d)$ is a linear combination of $v_1^*(X)$, $v_2^*(X)$, $v_3^*(X)$, $v_4^*(X)$, $v_5^*(X)$. This completes the proof of Theorem 2.

References

[1] 坂内英一, 坂内悦子, 「球面上の代数的組合せ論」シュプリンガー・フェアラーク東京, 1999.

[2] E. Bannai and T. Ito, "Algebraic Combinatorics I : Association schemes," Benjamin/Cummings, Menlo Park, Calif., 1984.

[3] A. E. Brouwer, A. M. Cohen and A. Neumaier, "Distance-Regular Graphs," Springer, Berlin-Heidelberg, 1989.

[4] P. Delsarte, J.-M. Goethals and J. J. Seidel, Spherical codes and designs, Geom. Dedicata 6 (1977), 363-388.

[5] I. Shimada, Lattices of algebraic cycles on Fermat varieties in positive characteristics, Proc. London Math. Soc. (3) 82 (2001), 131-172.

[6] V. M. Sidelnikov, Spherical 7-designs in 2^n-dimensional Euclidean space, J. Algebraic Combin. 10 (1999), 279-288.