Spherical 5-Designs Obtained from the Unitary Group $U_{2m}(2)$

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1 Introduction

The purpose of this talk is to give an infinite series of spherical 5-designs constructed from the unitary group over the finite field of four elements. Let $G = U_{2m}(2)$ be the unitary group of dimension $2m$ over $GF(4)$, $V = GF(4)^{2m}$ the natural module of $G$. Then $G$ acts transitively on the set $\Omega$ of (maximal) totally isotropic $m$-spaces of $V$. This permutation representation (over $\mathbb{R}$) contains an irreducible representation of dimension $d = (4^m + 2)/3$. Then one can embed the set $\Omega$ into the unit sphere $S^{d-1}$ in the Euclidean space $\mathbb{R}^d$.

**Theorem 1.** $\Omega \hookrightarrow S^{d-1} \subset \mathbb{R}^d$ is a spherical 5-design.

The inner product among the vectors of $\Omega$ embedded in $\mathbb{R}^d$ can be made rational-valued, so one obtains integral lattices after a suitable normalization. Shimada [5] considered a related family of lattices, and presented in a talk in January, 2000 at RIMS.

2 Preliminaries

A spherical $t$-design ($t \in \mathbb{Z}, t \geq 0$) is a finite set $\Omega \subset S^{d-1}$ such that

$$\frac{\int_{S^{d-1}} f(x)dx}{\int_{S^{d-1}} 1dx} = \frac{1}{|\Omega|} \sum_{x \in \Omega} f(x)$$

for all polynomial $f \in \mathbb{R}[X_1, \ldots, X_d]$ of degree at most $t$. Equivalently,

$$\sum_{x,y \in \Omega} Q_i(\langle x, y \rangle) = 0 \quad (1 \leq i \leq t)$$

(1)
\[ Q_0(X) = 1, \quad Q_1(X) = dX, \]
\[ \frac{k+1}{d+2k}Q_{k+1}(X) = XQ_k(X) - \frac{d+k-3}{d+2k-4}Q_{k-1}(X) \]

are suitably normalized Gegenbauer polynomials. See [1, 4] for more details on spherical designs. In what follows we simply say a \( t \)-design for a spherical \( t \)-design.

Examples of spherical designs include the 196, 560 vectors of norm 4 in the Leech lattice (a 11-design), the 240 roots of the root system \( E_8 \) (a 7-design). Moreover, if \( O(d, \mathbb{R}) \supset G \) is a finite irreducible subgroup, then every \( G \)-orbit on \( S^{d-1} \) is a 2-design. Sidel'nikov [6] showed that there exists a finite group \( G \subset O(2^n, \mathbb{R}) \) such that every \( G \)-orbit on \( S^{2^n-1} \) is a 7-design. In general, the Molien series of \( G \) on the space of harmonic polynomials determines \( t \) for which every \( G \)-orbit on the unit sphere becomes a \( t \)-design. [1, p.102].

To see that \( \Omega \mapsto S^{d-1} (d = (4^m + 2)/3) \) is a 5-design, we shall verify the condition (1) with \( t = 5 \). We note that the values of inner products \( \langle x, y \rangle \) are known to be \((-2)^{-j}, 0 \leq j \leq m \) (see Table 6.1 (C3) of [3]), and \( \langle x, y \rangle = (-2)^j \) if and only if the dimension of the intersection of \( x \) and \( y \) is \( m - j \) (recall that \( x, y \) are \( m \)-dimensional subspaces of \( V \)). The number of pairs \( (x, y) \in \Omega^2 \) such that \( \langle x, y \rangle = (-2)^{-j} \) is given by \(|\Omega|k_j\), where

\[ k_j = \prod_{h=1}^{j} \frac{2^{2h-1}(4^m-h+1) - 1}{4^h - 1}. \]

With these formulas at our disposal, we can verify (1) for any given values of \( m \). However, we shall employ a more general framework to prove Theorem 1.

A comment on the peculiarity of this embedding can be found in [3, Remark, p.276].

3 The Q-polynomial property for the dual polar space associated to \( U_{2m}(2) \)

As in the previous section, we let \( m \) be a fixed positive integer, and denote by \( \Omega \) the set of totally isotropic \( m \)-spaces in the natural module \( V = GF(4)^{2m} \) of
The set $\Omega$ is called the dual polar space associated to $U_{2m}(2)$, because it is a combinatorial dual of the polar space of absolute points and totally isotropic lines of the projective space $PG(V)$ with a unitary polarity. Then $U_{2m}(2)$ acts on $\Omega$, and the permutation representation (over $\mathbb{R}$) decomposes as follows:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \cdots \perp V_m,$$

(2)

where $V_0$ is the trivial module. Let $E_i \in M_{|\Omega|}(\mathbb{R})$ be the orthogonal projection of $\mathbb{R}\Omega$ onto $V_i$. If we rearrange the ordering of $V_i$'s if necessary, then there exists a polynomial $v_i^*(X)$ of degree $i$ ($0 \leq i \leq m$) such that

$$|\Omega|E_i = v_i^*(|\Omega|E_1) \quad (0 \leq i \leq m),$$

where, if

$$v_i^*(X) = \sum_{j=0}^{i} c_{ij} X^j,$$

then

$$v_i^*(|\Omega|E_1) = \sum_{j=0}^{i} c_{ij} |\Omega|^j E_1 \circ \cdots \circ E_1,$$

where $\circ$ denotes the entry-wise product. Roughly speaking, the existence of such polynomials is refered to as the $Q$-polynomial property (see [2] for details). It is known that there exist $a_i^*, b_i^*, c_i^* \in \mathbb{R}$ such that

$$Xv_i^*(X) = c_{i+1}^* v_{i+1}^*(X) + a_i^* v_i^*(X) + b_{i-1}^* v_{i-1}^*(X)$$

(3)

and $\{v_i^*(X)\}$ is a system of orthogonal polynomials.

More generally, one can define a combinatorial structure called an association scheme on which the vector space of real-valued functions on the underlying set $\Omega$ can be decomposed into a direct sum like (2), and one can define $Q$-polynomial property for association schemes. For precise definition, we refer to [2]. The following theorem reveals a relationship between the $Q$-polynomial property and spherical designs. Here we denote by $E_1(\Omega)$ the set of unit vectors obtained by normalizing the column vectors of the matrix
Theorem 2. Suppose that $\Omega$ is a Q-polynomial association scheme.

(i) If $a_1^* = 0$, then $E_1(\Omega)$ is a 3-design.

(ii) If moreover, $b_0^* b_1^* c_2^* + 2(b_1^* c_2^* - b_0^* b_0^* - b_0^*) = 0$, then $E_1(\Omega)$ is a 4-design.

(iii) If moreover, $a_2^* = 0$, then $E_1(\Omega)$ is a 5-design.

If $\Omega$ is the dual polar space for $U_{2m}(2)$, then all hypotheses of the theorem are satisfied, and $\Omega$ becomes a 5-design. To check this, we reproduce a more general formula for these numbers for the dual polar spaces associated with $U_{2m}(r)$, where $r$ is a prime power. They can be deduced from the formulas in [2, Section 3.5].

\begin{align*}
    b_i^* &= \frac{(r^{2m} + r)(r^{2m+2} + (-1)^i r^{i+1})}{(r+1)(r^{2m+2} + r^{2i+1})}, \\
    c_i^* &= \frac{r^{i-1}(r^i + (-1)^i)(r^{2m} + r)}{(r+1)(r^{2m} + r^{2i-1})}, \\
    a_i^* &= b_0^* - b_i^* - c_i.
\end{align*}

From these formulas, one checks easily that the conditions (i)–(iii) of Theorem 2 are satisfied precisely when $r = 2$.

One can find a more general formula describing these numbers for known P- and Q-polynomial association schemes [2, Section 3.5]. Thus, it is natural to consider the following problem.

**Problem.** Classify P- and Q-polynomial association scheme $\Omega$ such that $E_1(\Omega)$ is a spherical $t$-design for $t = 4, 5, 6, \ldots$.

## 4 Proof of Theorem 2

We use the orthogonality relation of the polynomials $\{v_i^*(X)\}_{i=0}^m$ given by

$$\sum_{h=0}^m k_h v_i^*(\theta_h^*/d) v_j^*(\theta_h^*) = 0 \quad (i \neq j),$$

where $\theta_0^* = \dim V_0 = \text{rank} E_1 = b_0^*$, and $E_1(\Omega)$ has $|\Omega| k_h$ pairs of elements with inner product $\theta_h^*/\theta_0^*$. We shall write $d$ instead of $\theta_0^*$ to simplify the notation. In view of (1), in order to prove $E_1(\Omega)$ is a $t$-design, it suffices to show

$$\sum_{h=0}^m k_h Q_i(\theta_h^*/d) = 0 \quad (1 \leq i \leq t).$$
\textbf{Lemma 3.} If the polynomials $Q_s(X/d)$ ($1 \leq s \leq t$) are linear combinations of the polynomials $v_1^*(X), \ldots, v_i^*(X)$, then $E_1(\Omega)$ is a $t$-design.

\textit{Proof.} Since $v_0^*(X) = 1$, the orthogonality relation (4) implies
\begin{equation*}
\sum_{h=0}^{m} k_h v_i^*(\theta_h^*) = 0 \quad (i > 0).
\end{equation*}
Then the condition (5) is seen to be satisfied. \qed

It follows from the definitions that $Q_1(X/dX) = X = v_1^*(X)$, so $E_1(\Omega)$ is always a 1-design. Also, one has
\begin{equation*}
Q_2(\frac{X}{d}) = \frac{d+2}{2d} (c_2^* v_2^*(X) + a_1^* v_1^*(X)),
\end{equation*}
and hence $E_1(\Omega)$ is always a 2-design.

To prove part (i) of Theorem 1, we assume $a_1^* = 0$, so that
\begin{equation}
XQ_2(\frac{X}{d}) = \frac{d+2}{2d} c_2^* X v_2^*(X).
\end{equation}
Then
\begin{align*}
Q_3(\frac{X}{d}) &= \frac{d+4}{3} \left( \frac{X}{d} Q_2(\frac{X}{d}) - (1 - \frac{1}{d}) Q_1(\frac{X}{d}) \right) \\
&= \frac{d+4}{3d} \left( XQ_2(\frac{X}{d}) - (d - 1) Q_1(\frac{X}{d}) \right) \\
&= \frac{d+4}{3d} \left( \frac{d+2}{2d} c_2^* X v_2^*(X) - (d - 1) v_1^*(X) \right) \\
&= \frac{(d+4)(d+2)c_2^*}{6d^2} (c_3^* v_3^*(X) + a_2^* v_2^*(X)) \\
&\quad + \frac{(d+4)((d+2)c_2^* b_1^*-2d(d-1))}{6d^2} v_1^*(X).
\end{align*}
Thus $Q_3(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X)$.

Under the assumption of (ii), we have
\begin{equation}
Q_3(\frac{X}{d}) = \frac{(d+4)(d+2)c_2^*}{6d^2} (c_3^* v_3^*(X) + a_2^* v_2^*(X)).
\end{equation}
\[
Q_4\left(\frac{X}{d}\right) = \frac{d + 6}{4} \left( \frac{X}{d} Q_3\left(\frac{X}{d}\right) - \frac{d}{d + 2} Q_2\left(\frac{X}{d}\right) \right)
\]
\[
= \frac{(d + 6)(d + 4)(d + 2)c_2^*}{24d^3} (c_3^* X v_3^*(X) + a_2^* X v_2^*(X)) - \frac{d + 6}{8} c_2^* v_2^*(X).
\]

It follows from (3) that \(Q_4(X/d)\) is a linear combination of \(v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X)\).

Under the assumption of (iii), we have

\[
Q_4\left(\frac{X}{d}\right) = \frac{(d + 6)(d + 4)(d + 2)c_2^*}{24d^3} c_3^* X v_3^*(X) - \frac{d + 6}{8} c_2^* v_2^*(X),
\]

which is a linear combination of \(v_2^*(X), v_3^*(X), v_4^*(X)\) by (3). Thus \(X Q_4(X/d)\) is a linear combination of \(v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X), v_5^*(X)\) by (3). Since

\[
Q_5\left(\frac{X}{d}\right) = \frac{d + 8}{5} \left( \frac{X}{d} Q_4\left(\frac{X}{d}\right) - \frac{d + 1}{d + 4} Q_3\left(\frac{X}{d}\right) \right)
\]

and \(Q_3(X/d)\) is a scalar multiple of \(v_5^*(X)\) by (7), we see that \(Q_5(X/d)\) is a linear combination of \(v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X), v_5^*(X)\). This completes the proof of Theorem 2.

**References**


