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Spherical 5-Designs Obtained from the Unitary Group $U_{2m}(2)$

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1 Introduction

The purpose of this talk is to give an infinite series of spherical 5-designs constructed from the unitary group over the finite field of four elements. Let $G = U_{2m}(2)$ be the unitary group of dimension $2m$ over $GF(4)$, $V = GF(4)^{2m}$ the natural module of $G$. Then $G$ acts transitively on the set $\Omega$ of (maximal) totally isotropic $m$-spaces of $V$. This permutation representation (over $\mathbb{R}$) contains an irreducible representation of dimension $d = (4^m + 2)/3$. Then one can embed the set $\Omega$ into the unit sphere $S^{d-1}$ in the Euclidean space $\mathbb{R}^d$.

Theorem 1. $\Omega \hookrightarrow S^{d-1} \subset \mathbb{R}^d$ is a spherical 5-design.

The inner product among the vectors of $\Omega$ embedded in $\mathbb{R}^d$ can be made rational-valued, so one obtains integral lattices after a suitable normalization. Shimada [5] considered a related family of lattices, and presented in a talk in January, 2000 at RIMS.

2 Preliminaries

A spherical $t$-design ($t \in \mathbb{Z}, t \geq 0$) is a finite set $\Omega \subset S^{d-1}$ such that

$$\frac{\int_{S^{d-1}} f(x) dx}{\int_{S^{d-1}} 1 dx} = \frac{1}{|\Omega|} \sum_{x \in \Omega} f(x)$$

for all polynomial $f \in \mathbb{R}[X_1, \ldots, X_d]$ of degree at most $t$. Equivalently,

$$\sum_{x,y \in \Omega} Q_i(\langle x, y \rangle) = 0 \quad (1 \leq i \leq t)$$

(1)
$Q_0(X) = 1, \quad Q_1(X) = dX,$

$$\frac{k+1}{d+2k}Q_{k+1}(X) = XQ_k(X) - \frac{d+k-3}{d+2k-4}Q_{k-1}(X)$$

are suitably normalized Gegenbauer polynomials. See [1, 4] for more details on spherical designs. In what follows we simply say a $t$-design for a spherical $t$-design.

Examples of spherical designs include the 196, 560 vectors of norm 4 in the Leech lattice (a 11-design), the 240 roots of the root system $E_8$ (a 7-design). Moreover, if $O(d, \mathbb{R}) \supset G$ is a finite irreducible subgroup, then every $G$-orbit on $S^{d-1}$ is a 2-design. Sidelnikov [6] showed that there exists a finite group $G \subset O(2^n, \mathbb{R})$ such that every $G$-orbit on $S^{2^{n-1}}$ is a 7-design. In general, the Molien series of $G$ on the space of harmonic polynomials determines $t$ for which every $G$-orbit on the unit sphere becomes a $t$-design. [1, p.102].

To see that $\Omega \mapsto S^{d-1}$ ($d = \frac{4^m + 2}{3}$) is a 5-design, we shall verify the condition (1) with $t = 5$. We note that the values of inner products $\langle x, y \rangle$ are known to be $(-2)^{-j}$, $0 \leq j \leq m$ (see Table 6.1 (C3) of [3]), and $\langle x, y \rangle = (-2)^{j}$ if and only if the dimension of the intersection of $x$ and $y$ is $m - j$ (recall that $x, y$ are $m$-dimensional subspaces of $V$). The number of pairs $(x, y) \in \Omega^2$ such that $\langle x, y \rangle = (-2)^{-j}$ is given by $|\Omega|k_j$, where

$$k_j = \prod_{h=1}^{j} \frac{2^{2h-1}(4^{m-h+1} - 1)}{4^h - 1}.$$

With these formulas at our disposal, we can verify (1) for any given values of $m$. However, we shall employ a more general framework to prove Theorem 1.

A comment on the peculiarity of this embedding can be found in [3, Remark, p.276].

3 The Q-polynomial property for the dual polar space associated to $U_{2m}(2)$

As in the previous section, we let $m$ be a fixed positive integer, and denote by $\Omega$ the set of totally isotropic $m$-spaces in the natural module $V = GF(4)^{2m}$ of
$U_{2m}(2)$. The set $\Omega$ is called the dual polar space associated to $U_{2m}(2)$, because it is a combinatorial dual of the polar space of absolute points and totally isotropic lines of the projective space $PG(V)$ with a unitary polarity. Then $U_{2m}(2)$ acts on $\Omega$, and the permutation representation (over $\mathbb{R}$) decomposes as follows:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \cdots \perp V_m,$$

(2)

where $V_0$ is the trivial module. Let $E_i \in M_{|\Omega|}(\mathbb{R})$ be the orthogonal projection of $\mathbb{R}\Omega$ onto $V_i$. If we rearrange the ordering of $V_i$'s if necessary, then there exists a polynomial $v^*_i(X)$ of degree $i$ ($0 \leq i \leq m$) such that

$$|\Omega|E_i = v^*_i(|\Omega|E_1) \quad (0 \leq i \leq m),$$

where, if

$$v^*_i(X) = \sum_{j=0}^i c_{ij} X^j,$$

then

$$v^*_i(|\Omega|E_1) = \sum_{j=0}^i c_{ij} |\Omega|E_1 \circ \cdots \circ E_1,$$

where $\circ$ denotes the entry-wise product. Roughly speaking, the existence of such polynomials is referred to as the Q-polynomial property (see [2] for details). It is known that there exist $a^*_i, b^*_i, c^*_i \in \mathbb{R}$ such that

$$Xv^*_i(X) = c^*_{i+1}v^*_{i+1}(X) + a^*_i v^*_i(X) + b^*_{i-1}v^*_{i-1}(X)$$

(3)

and $\{v^*_i(X)\}$ is a system of orthogonal polynomials.

More generally, one can define a combinatorial structure called an association scheme on which the vector space of real-valued functions on the underlying set $\Omega$ can be decomposed into a direct sum like (2), and one can define Q-polynomial property for association schemes. For precise definition, we refer to [2]. The following theorem reveals a relationship between the Q-polynomial property and spherical designs. Here we denote by $E_1(\Omega)$ the set of unit vectors obtained by normalizing the column vectors of the matrix
**Theorem 2.** Suppose that $\Omega$ is a Q-polynomial association scheme.

(i) If $a_1^* = 0$, then $E_1(\Omega)$ is a 3-design.

(ii) If moreover, $b_0^* b_1^* c_2^* + 2(b_1^* c_2^* - b_0^* b_0^* + b_0^*) = 0$, then $E_1(\Omega)$ is a 4-design.

(iii) If moreover, $a_2^* = 0$, then $E_1(\Omega)$ is a 5-design.

If $\Omega$ is the dual polar space for $U_{2m}(2)$, then all hypotheses of the theorem are satisfied, and $\Omega$ becomes a 5-design. To check this, we reproduce a more general formula for these numbers for the dual polar spaces associated with $U_{2m}(r)$, where $r$ is a prime power. They can be deduced from the formulas in [2, Section 3.5].

\[
b_i^* = \frac{(r^{2m} + r)(r^{2m+2} + (-1)^i r^{i+1})}{(r+1)(r^{2m+2} + r^{2i+1})},
\]
\[
c_i^* = \frac{r^{i-1}(r^i + (-1)^{i-1})(r^{2m} + r)}{(r+1)(r^{2m} + r^{2i-1})},
\]
\[
a_i^* = b_0^* - b_i^* - c_i.
\]

From these formulas, one checks easily that the conditions (i)–(iii) of Theorem 2 are satisfied precisely when $r = 2$.

One can find a more general formula describing these numbers for known P- and Q-polynomial association schemes [2, Section 3.5]. Thus, it is natural to consider the following problem.

**Problem.** Classify P- and Q-polynomial association scheme $\Omega$ such that $E_1(\Omega)$ is a spherical $t$-design for $t = 4, 5, 6, \ldots$.

## 4 Proof of Theorem 2

We use the orthogonality relation of the polynomials $\{v_i^*(X)\}_{i=0}^m$ given by

\[
\sum_{h=0}^{m} k_h v_i^*(\theta_h^*) v_j^*(\theta_h^*) = 0 \quad (i \neq j),
\]

where $\theta_0^* = \dim V_0 = \text{rank} E_1 = b_0^*$, and $E_1(\Omega)$ has $|\Omega| k_h$ pairs of elements with inner product $\theta_h^*/\theta_0^*$. We shall write $d$ instead of $\theta_0^*$ to simplify the notation. In view of (1), in order to prove $E_1(\Omega)$ is a $t$-design, it suffices to show

\[
\sum_{h=0}^{m} k_h Q_i(\theta_h^*/d) = 0 \quad (1 \leq i \leq t).
\]
Lemma 3. If the polynomials $Q_s(X/d)$ $(1 \leq s \leq t)$ are linear combinations of the polynomials $v_1^*(X), \ldots, v_t^*(X)$, then $E_1(\Omega)$ is a $t$-design.

Proof. Since $v_0^*(X) = 1$, the orthogonality relation (4) implies 

$$\sum_{h=0}^{m} k_h v_i^*(\theta_h^*) = 0 \quad (i > 0).$$

Then the condition (5) is seen to be satisfied. 

It follows from the definitions that $Q_1(X/dX) = X = v_1^*(X)$, so $E_1(\Omega)$ is always a 1-design. Also, one has 

$$Q_2(X/d) = \frac{d+2}{2d} (c_2^* v_2^*(X) + a_1^* v_1^*(X)),$$

and hence $E_1(\Omega)$ is always a 2-design.

To prove part (i) of Theorem 1, we assume $a_1^* = 0$, so that 

$$XQ_2(X/d) = \frac{d+2}{2d} c_2^* X v_2^*(X). \quad (6)$$

Then 

$$Q_3(X/d) = \frac{d+4}{3} \left( \frac{X}{d} Q_2(X/d) - (1 - \frac{1}{d}) Q_1(X/d) \right)$$

$$= \frac{d+4}{3d} \left( XQ_2(X/d) - (d-1)Q_1(X/d) \right)$$

$$= \frac{d+4}{3d} \left( \frac{d+2}{2d} c_2^* X v_2^*(X) - (d-1)v_1^*(X) \right)$$

$$= \frac{(d+4)(d+2)c_2^*}{6d^2} (c_3^* v_3^*(X) + a_2^* v_2^*(X))$$

$$+ \frac{(d+4)((d+2)c_2^* b_1^* - 2d(d-1))}{6d^2} v_1^*(X).$$

Thus $Q_3(X/d)$ is a linear combination of $v_1^*(X), v_2^*(X), v_3^*(X)$.

Under the assumption of (ii), we have 

$$Q_3(X/d) = \frac{(d+4)(d+2)c_2^*}{6d^2} (c_3^* v_3^*(X) + a_2^* v_2^*(X)). \quad (7)$$
\[ Q_4\left(\frac{X}{d}\right) = \frac{d+6}{4} \left( \frac{X}{d} Q_3\left(\frac{X}{d}\right) - \frac{d}{d+2} Q_2\left(\frac{X}{d}\right) \right) \]
\[ = \frac{(d+6)(d+4)(d+2)c_2^*}{24d^3} (c_3^* X v_3^*(X) + a_2^* X v_2^*(X)) - \frac{d+6}{8} c_2^* v_2^*(X). \]

It follows from (3) that \( Q_4(X/d) \) is a linear combination of \( v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X) \).

Under the assumption of (iii), we have
\[ Q_4\left(\frac{X}{d}\right) = \frac{(d+6)(d+4)(d+2)c_2^*}{24d^3} c_3^* X v_3^*(X) - \frac{d+6}{8} c_2^* v_2^*(X), \quad (8) \]
which is a linear combination of \( v_2^*(X), v_3^*(X), v_4^*(X) \) by (3). Thus \( X Q_4(X/d) \) is a linear combination of \( v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X), v_5^*(X) \) by (3). Since
\[ Q_5\left(\frac{X}{d}\right) = \frac{d+8}{5} \left( \frac{X}{d} Q_4\left(\frac{X}{d}\right) - \frac{d+1}{d+4} Q_3\left(\frac{X}{d}\right) \right) \]
and \( Q_3(X/d) \) is a scalar multiple of \( v_3^*(X) \) by (7), we see that \( Q_5(X/d) \) is a linear combination of \( v_1^*(X), v_2^*(X), v_3^*(X), v_4^*(X), v_5^*(X) \). This completes the proof of Theorem 2.

References


