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Kyoto University
On the smallness and the 1-bridge genus of knots

by

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Abstract. Let $K$ be a knot in $S^3$ and $g_1(K)$ the 1-bridge genus of $K$. Then P. Hoidn showed that $g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1$ for any small knots $K_1, K_2$, where a knot is small if the exterior contains no closed essential surfaces. In the present article, we show that Hoidn’s estimate is best possible, i.e., there are infinitely many pairs of small knots $K_1, K_2$ such that $g_1(K_1 \# K_2) = g_1(K_1) + g_1(K_2) - 1$.

1. Introduction

Let $S^3$ be the 3-dimensional sphere, and $K$ a knot in $S^3$. We say that $(V_1, V_2)$ is a Heegaard splitting of $S^3$ if $S^3 = V_1 \cup V_2$, $V_1 \cap V_2 = \partial V_1 = \partial V_2$ and both $V_1$ and $V_2$ are handlebodies. The genus of $V_i$ (the genus of $V_2$) is called the genus of the Heegaard splitting and the surface $\partial V_1 = \partial V_2$ is called the Heegaard surface of the Heegaard splitting. Then for any knot $K$ in $S^3$ it is well known that there is a Heegaard splitting $(V_1, V_2)$ of $S^3$ such that $K$ intersects $V_i$ in a single trivial arc in $V_i$ for both $i = 1, 2$. Hence we define the 1-bridge genus $g_1(K)$ of $K$ as the minimal genus among all such Heegaard splittings $(V_1, V_2)$ of $S^3$ (c.f. [Ho] and [MSY]).

For two knots $K_1, K_2$ in $S^3$, we denote the connected sum of $K_1$ and $K_2$ by $K_1 \# K_2$. Then by a little observation, we immediately see the following:

Fact 1.1 For any two knots $K_1$ and $K_2$ in $S^3$, $g_1(K_1 \# K_2) \leq g_1(K_1) + g_1(K_2)$.

Let $N(K)$ be a regular neighborhood of a knot $K$ in $S^3$ and $E(K) = \text{cl}(S^3 - N(K))$ the exterior of $K$. A surface $F$ (a connected 2-manifold) properly embedded in $E(K)$ is essential if $F$ is incompressible and is not parallel to $\partial E(K)$ or to a subsurface of $\partial E(K)$, and it is meridional if $\partial F \neq \emptyset$ and each component of $\partial F$ is a meridian of $K$. Then we say that $K$ is small if $E(K)$ contains no closed essential
surfaces and that $K$ is meridionally small if $E(K)$ contains no meridional essential surfaces. We note that if a knot in $S^3$ is small then it is meridionally small by [CGLS, Theorem 2.0.3].

On the problem to estimate the lower bound of $g_1(K_1 \# K_2)$, P. Hoidn showed:

**Theorem 1.2 ([Ho, Theorem])** Let $K_1, K_2$ be two knots in $S^3$. If both $K_1$ and $K_2$ are small, then $g_1(K_1 \# K_2) \geq g_1(K_1) + g(K_2) - 1$.

In the present article, we show this estimate is best possible:

**Theorem 1.3** There are infinitely many pairs of small knots $K_1, K_2$ in $S^3$ with $g_1(K_1 \# K_2) = g_1(K_1) + g(K_2) - 1$.

Moreover, as a generalization of Hoidn's theorem, we show:

**Theorem 1.4** Let $K_1, K_2$ be two knots in $S^3$. If both $K_1$ and $K_2$ are meridionally small, then $g_1(K_1 \# K_2) \geq g_1(K_1) + g(K_2) - 1$.

**Remark 1.5**

1. By [Mol, Proposition 1.6], we see that for any integer $n > 0$ there are infinitely many knots $K$ such that (i) $g_1(K) > n$, (ii) $K$ is meridionally small, (iii) $K$ is not small. This shows that Theorem 1.4 properly includes Theorem 1.2.

2. Since a small knot is meridionally small as mentioned before, the estimate in Theorem 1.4 is best possible by Theorem 1.3.

Let $t(K)$ be the tunnel number of a knot $K$ in $S^3$, i.e., $t(K)$ is the minimal number of mutually disjoint arcs $\gamma_1, \gamma_2, \ldots, \gamma_t$ properly embedded in $E(K)$ such that $cl(E(K) - N(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t))$ is a handlebody. Then by a little observation we have:

**Fact 1.6** $t(K) \leq g_1(K) \leq t(K) + 1$ for any knot $K$.

By the above inequality, we have $g_1(K) = t(K)$ or $t(K) + 1$. Let $K_1$ and $K_2$ be small knots in $S^3$, and suppose $g_1(K_i) = t(K_i)$ for both $i = 1, 2$. Then by Fact 1.1, Fact 1.6 and [MS Theorem], we have $g_1(K_1) + g_1(K_2) \geq g_1(K_1 \# K_2) \geq t(K_1) + t(K_2) = g_1(K_1) + g_1(K_2)$. Hence $g_1(K_1 \# K_2) = g_1(K_1) + g_1(K_2)$. This tells that to show Theorem 1.3 we need to find small knots $K$ with $g_1(K) = t(K) + 1$.

Let $p, q$ be coprime integers, and $r$ an arbitrarily integer. Then we consider the knot obtained by adding $r$ full twists with mutually parallel 2-strands to the $(p, q)$-torus knot as illustrated in Figure 1, and denote it by $K(p, q; r)$ (cf. [MSY]).
Then to get the candidates for Theorem 1.3, we show the following proposition and most of the present article will be devoted into the proof of this proposition.

**Proposition 1.7**  For any $p, q, r$, $K(p, q; r)$ is small.

Throughout the present article, we work in the piecewise linear category. For a manifold $X$ and subcomplex $Y$ in $X$, we denote a regular neighborhood of $Y$ in $X$ by $N(Y, X)$ or $N(Y)$ simply.

2. Proof of Theorem 1.4

To show Theorem 1.4, we need the following:

**Theorem 2.1 ([Mo1, Corollary 1.2])**  Let $K_1$ and $K_2$ be two knots in $S^3$. If both $K_1$ and $K_2$ are meridionally small, then $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$.

**Theorem 2.2 ([Mo2, Theorem 1.6])**  Let $K_1$ and $K_2$ be two meridionally small knots in $S^3$. Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if $g_i(K_i) = t(K_i) + 1$ for both $i = 1, 2$.

Suppose both $K_1$ and $K_2$ are meridionally small. Recall that $g_i(K_i) = t(K_i)$ or $t(K_i) + 1$ for $(i = 1, 2)$ by Fact 1.6.

First suppose at least one of $K_1$ and $K_2$, say $K_1$, satisfies the equality $g_1(K_1) = t(K_1)$. Then $t(K_2) \geq g_1(K_2) - 1$. Since both $K_1$ and $K_2$ are meridionally small, by the above Theorem 2.1, $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$. Hence by Fact 1.6, $g_1(K_1 \# K_2) \geq t(K_1) + t(K_2) \geq g_1(K_1) + g_1(K_2) - 1$.

Next suppose $g_1(K_i) = t(K_i) + 1$ for both $(i = 1, 2)$. Then by the above Theorem 2.2, $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$. Hence by Fact 1.6, $g_1(K_1 \# K_2) \geq t(K_1 \# K_2) = t(K_1) + t(K_2) + 1 = (g_1(K_1) - 1) + (g_1(K_2) - 1) + 1 = g_1(K_1) + g_1(K_2) - 1$. This completes the proof of Theorem 1.4. \( \square \)

3. Proof of Theorem 1.3 under Proposition 1.7

To show Theorem 1.3, we need the following:
Lemma 3.1 ([Mo3, Proposition 1.7]) Let $K$ be a knot in $S^3$. If $g_1(K) = t(K) + 1$, then $g_1(K\#K') \leq g_1(K)$ for any 2-bridge knot $K'$.

For convenience to the readers, we show the above lemma here. Let $(V_1, V_2)$ be a Heegaard splitting of a 3-sphere $S^3$ which realizes the tunnel number of $K$, i.e., $V_1$ contains $K$ as a core of a handle of $V_1$ and $g(V_1) = t(K) + 1 = g_1(K)$. Let $(B_1, \gamma_1 \cup \delta_1)$ and $(B_2, \gamma_2 \cup \delta_2)$ be a 2-bridge decomposition of $K'$ in another 3-sphere $S^3$, i.e., $(B_1, \gamma_1 \cup \delta_1)$ is a 2-string trivial tangle $(i = 1, 2)$ and $K' = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2 \subset B_1 \cup B_2 = S^3$.

Let $D$ be a meridian disk of $V_1$ which intersects $K$ in a single point and $N(D)$ a regular neighborhood of $D$ in $V_1$. Put $N(D) = D \times [0, 1]$ and $N(D) \cap K = x \times [0, 1]$, where $x$ is a point in Int$D$. Let $N(\delta_2)$ be a regular neighborhood of $\delta_2$ in $B_2$. Put $N(\delta_2) = D' \times [0, 1]$ and $\delta_2 = y \times [0, 1]$, where $D'$ is a disk and $y$ a point in Int$D'$.

Let $K\#K'$ be the connect sum of $K$ and $K'$. Then $K\#K'$ is a knot in the 3-sphere $S^3 = d(S^3 - N(D)) \cup_{\partial N(D) = \partial N(\delta_2)} d(S^3 - N(\delta_2))$. Put $W_1 = d(V_1 - N(D))$. Then, since $N(D) \cap W_1 = \partial N(D) \cap \partial W_1 = D \times \{0, 1\}$ and since $N(\delta_2) \cap B_1 = \partial N(\delta_2) \cap \partial B_1 = D' \times \{0, 1\}$, we can put $U_1 = W_1 \cup_{D \times \{0, 1\} = D' \times \{0, 1\}} B_1$. Then $U_1$ is a genus $g_1(K)$ handlebody and $(K\#K') \cap U_1$ is a trivial arc in $U_1$ because $(K\#K') \cap W_1$ is a trivial arc in $W_1$ and $(K\#K') \cap B_1 \subset B_1$ is a 2-string trivial arc in $B_1$.

On the other hand, put $W_2 = d(B_2 - N(\delta_2))$. Then, since $N(D) \cap V_2 = \partial N(D) \cap \partial V_2 = \partial D \times [0, 1]$ and since $N(\delta_2) \cap W_2 = \partial N(\delta_2) \cap \partial W_2 = \partial D' \times [0, 1]$, we can put $U_2 = V_2 \cup_{\partial D \times [0, 1] = \partial D' \times [0, 1]} W_2$. Then $U_2$ is a genus $g_1(K)$ handlebody and $(K\#K') \cap U_2$ is a trivial arc in $U_2$ because $\delta_2$ is a trivial arc in $B_2$ and $(K\#K') \cap W_2$ is a trivial arc in $W_2$.

Hence $(U_1, U_2)$ is a genus $g_1(K)$ Heegaard splitting of $S^3$ which gives a 1-bridge decomposition of $K\#K'$. This implies $g_1(K\#K') \leq g_1(K)$ and completes the proof of Lemma 3.1. \hfill $\square$

Now let's prove Theorem 1.3 under Proposition 1.7. Let $m$ be an integer and consider the knot $K_1 = K(7, 17, 5m - 2)$. Then by Proposition 1.7, $K_1$ is small, and by [MSY, Theorem 2.1], $t(K_1) = 1$ and $g_1(K_1) = 2$. Let $K_2$ be a (non-trivial) 2-bridge knot in $S^3$. Then $K_2$ is small and $g_1(K_2) = 1$. Then by the above Lemma 3.1, $g_1(K_1\#K_2) \leq g_1(K_1) = 2$. On the other hand, $g_1(K_1\#K_2) \geq 2$ because 1-bridge genus one knots are prime by [No, Sc] and Fact 1.6. Thus $g_1(K_1\#K_2) = 2$ and $g_1(K_1\#K_2) = g_1(K_1) + g_1(K_2) - 1$ for the small knots $K_1, K_2$. This completes
4. Preliminaries for the proof of Propositions 1.7

Put $K = K(p, q; r)$, $N(K)$ a regular neighborhood of $K$ in $S^3$ and $E(K) = \text{cl}(S^3 - N(K))$ the exterior. If $r = 0$, then $K$ is a $(p, q)$-torus knot and is small. Hence hereafter we assume that $r \neq 0$. Let $(W_1, W_2)$ be a genus two Heegaard splitting of $S^3$ and $(D_1, D_2) \subset (W_1, W_2)$ a cancelling disk pair, i.e., $D_i$ is a non-separating disk of $W_i$ ($i = 1, 2$) and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ is a single point. Let $N(D_1)$ be a regular neighborhood of $D_1$ in $W_1$, and regard $N(D_1)$ as a product space $D_1 \times [0, 1]$ with $D_1 = D_1 \times \{\frac{1}{2}\}$. Put $D_1^0 = D_1 \times \{0\}$, $D_1^1 = D_1 \times \{1\}$, $V_1 = d(W_1 - N(D_1))$ and $V_2 = W_2 \cup N(D_1)$. Then $V_1 \cap N(D_1) = D_1^0 \cup D_1^1$, and we can put $\partial D_2 = \gamma_1 \cup \gamma_2$, where $\partial D_2 \cap V_1 = \gamma_1$ and $\partial D_2 \cap N(D_1) = \gamma_2$.

Consider the knot $K$ as a simple closed curve in $\partial W_1 = \partial W_2$ so that those $r$ full twists are in $\partial N(D_1)$ as illustrated in Figure 2. Then we may assume that $D_1 \cap K = \partial D_1 \cap K = \text{two points}$ and $D_2 \cap K = \gamma_2 \cap K = |2r|$ points.

[ Figure 2 ]

To show Proposition 1.7, we show that $K$ is small and meridionally small simultaneously. Suppose, for a contradiction, $E(K)$ contains a meridional essential surface or a closed essential surface, say $\check{F}$. Let $F$ be a closed surface obtained from $\check{F}$ by adding meridian disks of $N(K)$ to each component of $\partial \check{F}$. Note that $F = \check{F}$ if $\check{F}$ is closed. Hereafter we consider $F$ instead of $\check{F}$. Then $F$ intersects $K$ in several points (possibly $F \cap K = \emptyset$), $F - K$ is incompressible in $S^3 - K$ and $F$ is not a 2-sphere which bounds a 3-ball intersecting $K$ in a trivial arc. By general position argument, we may assume that $D_1 \cap K \cap F = \emptyset$, and hence $N(D_1) \cap K \cap F = \emptyset$. Then by the incompressibility of $F - K$, we may assume that $D_1 \cap F$ consists of $n$ arcs for some integer $n \geq 0$ and $N(D_1) \cap F$ consists of $n$ rectangles, where each arc of $D_1 \cap F$ separates the two points $D_1 \cap K$ as illustrated in Figure 3. We assume that $n$ is minimal among all such meridional or closed essential surfaces in $E(K)$.
Put $F \cap V_1 = F_1$, $F \cap N(D_1) = F_2$, and $F \cap W_2 = F_3$, then $F \cap W_1 = F_1 \cup F_2$. By the incompressibility of $F - K$ in $S^3 - K$ and the irreducibility of $S^3 - K$, we may assume that $F_1$ is incompressible in $V_1$. Put $K \cap V_1 = cl(K - N(D_1)) = k_1 \cup k_2 = \text{two arcs in } \partial V_1$.

**Lemma 4.1**  
(1) There is no pair of a subarc $\alpha$ of $k_1 \cup k_2$ and an arc $\beta$ properly embedded in $F_1$ such that $\alpha \cap \beta = \partial \alpha = \partial \beta$ and $\alpha \cup \beta$ bounds a disk in $V_1$. (2) There is no 2-gon in $(k_1 \cup k_2) \cup \partial F_1$, which bounds a disk in $\partial V_1$.

**Proof.**  
(1) Suppose there is such a pair $\alpha, \beta$, and let $\Delta$ be the disk in $V_1$ with $\partial \Delta = \alpha \cup \beta$. Let $N(\Delta)$ be a regular neighborhood of $\Delta$ in $S^3$ such that $N(\Delta) \cap F$ is a disk which is a regular neighborhood of $\beta$ in $F$, denote it by $N(\beta, F)$. Put $c = \partial N(\beta, F)$. Then, since $c$ is a loop in $\partial N(\Delta)$, $c$ bounds a disk in $S^3 - K$. If $c$ is essential in $F - K$, then $F - K$ is compressible in $S^3 - K$, a contradiction. If $c$ is inessential in $F - K$, then $F$ is a 2-sphere which bounds a 3-ball intersecting $K$ in a trivial arc, a contradiction. Hence there is no such pair.

(2) If there is such a 2-gon in $(k_1 \cup k_2) \cup \partial F_1$, then we can find a subarc $\alpha \subset k_1 \cup k_2$ and an arc $\beta \subset F_1$ satisfying the condition (1), a contradiction. Hence there is no such 2-gon. \hfill $\square$

By noting the incompressibility of $F - K$ in $S^3 - K$, we have the next two lemmas.

**Lemma 4.2**  
$n > 0$, where $n$ is the number of the arcs $D_1 \cap F = D_1 \cap F_2$.

**Lemma 4.3**  
Each component of $F_3 \cap D_2$ is an arc connecting $\gamma_1$ and $\gamma_2$, and there are exactly two outermost arc components each of which cuts off a disk intersect $K$ in a single point and contains a point of $\partial \gamma_1 = \partial \gamma_2$ as in Figure 5. Hence the number of the points $\gamma_1 \cap \partial F_1$ is $(2|\gamma| - 1)n$.

[ Figure 4 ]

Since $F_1$ is incompressible in the solid torus $V_1$, each component of $F_1$ is a $\partial$-parallel disk, a $\partial$-parallel annulus or a meridian disk of $V_1$. Recall the notations
$D_i^0, D_i^1, k_1$ and $k_2$. Then by several arguments we have:

**Lemma 4.4** Let $G$ be a $\partial$-parallel disk component of $F_1$ and $G'$ a disk in $\partial V_1$ to which $G$ is parallel. Then one of the following folds:

1. $G'$ is a small regular neighborhood of $k_i$ in $\partial V_1$ ($i = 1, 2$),
2. $G'$ is a small regular neighborhood of $D_i^1$ in $\partial V_1$ ($i = 0, 1$),
3. $G'$ is a small regular neighborhood of $D_i^0 \cup k_1 \cup D_i^1$ in $\partial V_1$ ($i = 1, 2$).

**Lemma 4.5** Let $G$ be a $\partial$-parallel annulus component of $F_1$ and $G'$ an annulus in $\partial V_1$ to which $G$ is parallel. Then $G'$ is a small regular neighborhood of $D_i^0 \cup k_1 \cup D_i^1 \cup k_2$ in $\partial V_1$.

Moreover, concerning $\partial$-parallel disk components of $F_1$ in $V_1$, we get a stronger result than that of Lemma 4.4 as follows:

**Lemma 4.6** Let $G$ be a $\partial$-parallel disk component of $F_1$ and $G'$ a disk in $\partial V_1$ to which $G$ is parallel. Then $G'$ is a small regular neighborhood of $k_1$ or of $k_2$ in $\partial V_1$, and all such disks are mutually parallel.

5. Sketch Proof of Proposition 1.7

Recall the notations in section 4, and recall that each component of $F_1$ is a $\partial$-parallel disk, a $\partial$-parallel annulus or a meridian disk in $V_1$. Then we have the two cases. Case I: $F_1$ contains no meridian disks and Case II: $F_2$ contains a meridian disk.

Suppose we are in Case I. In this case, by Lemmas 4.5 and 4.6, $F_1$ consists of mutually parallel $\partial$-parallel disks and mutually parallel $\partial$-parallel annuli. Let $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_n$ be the disks each of which is parallel to a small regular neighborhood of $k_1$ in $\partial V_1$ and $\tilde{A} = A_1 \cup A_2 \cup \cdots \cup A_\ell$ the annuli each of which is parallel to a small regular neighborhood of $D_i^0 \cup k_1 \cup D_i^1 \cup k_2$ in $\partial V_1$. Note that $n$ is the number of the arcs $D_1 \cap F$ and $2\ell = (2|\tau| - 1)n$ by Lemma 4.3 (see Figure 5). Let $D_3$ be a meridian disk of $W_2$ such that $\partial D_3$ is a longitude of $V_1$. Since $D_i^0 \cap k_1 \cap D_i^1$ can be homotopic to a point in $\partial V_1$. We may assume that $\partial D_3 \cap (D_i^0 \cup k_1 \cup D_i^1) = \emptyset$, and hence $\partial D_3 \cap \tilde{E} = \emptyset$. A schematic picture of $(\partial \tilde{E}, \partial \tilde{A}, \partial D_3, D_i^0, D_i^1, \gamma_1, k_1, k_2)$ on $\partial V_1$ is illustrated in Figure 5.
Since we may assume that there is no 2-gon in $\partial D_3 \cup k_1 \cup k_2 \cup \partial F_1$, the arrangement of the points $\partial D_3 \cap (k_1 \cup k_2 \cup \partial F_1)$ on $\partial D_3$ is as in Figure 6, where the big points are the points of $\partial D_3 \cap (k_1 \cup k_2)$ and the small points are the points of $\partial D_3 \cap \partial F_1 = \partial D_3 \cap \partial \tilde{A}$. We note that there are some small points between any two successive big points because of $\tilde{A} \neq \emptyset$ by $2\ell = (2|r| - 1)n > 0$.

By the incompressibility of $F$ in $S^3 - K$ we may assume that each component of $D_3 \cap (F \cap W_2) = D_3 \cap F_3$ is an arc. Let $\alpha$ be an outermost arc component of $D_3 \cap F_3$ in $D_3$ and $\beta$ the corresponding arc in $\partial D_3$ with $\alpha \cap \beta = \partial \alpha = \partial \beta$. Then we have the two subcases. Case I-a : $\beta \cap (k_1 \cup k_2) \neq \emptyset$ and Case I-b : $\beta \cap (k_1 \cup k_2) = \emptyset$.

Suppose we are in Case I-a. In this case, $\alpha$ meets a single component of $\tilde{A}$, say $A_1$. Then we can take an arc, say $\alpha'$ properly embedded in $A_1$, with $\alpha \cap \alpha' = \partial \alpha = \partial \alpha'$. Since $\alpha' \cup \beta$ bounds a boundary compressing disk for $A_1$ in $V_1$, together with the outermost disk for $\alpha$ in $D_3$, $\alpha \cup \alpha'$ bounds a disk, say $\Delta$, which intersects $K$ in a single point. Perform a 2-surgery for $F$ along $\Delta$, and let $\tilde{F}$ be the surface after the surgery. Then $cl(\tilde{F} - N(K))$ is a meridional essential surface properly embedded in $E(K)$, and $A_1$ is changed to the disk in conclusion (3) of Lemma 4.4. This contradicts Lemma 4.6. Hence Case I-a does not occur.

Suppose we are in Case I-b. In this case, $\alpha$ connects two components of $\tilde{A}$, say $A_1, A_2$. Perform a boundary compression of $F_3$ along the outermost disk for $\alpha$. Let $b$ be the band in $V_1$ produced by the boundary compression. Then $A_1 \cup b \cup A_2$ is a disk with two holes, and one component of $\partial (A_1 \cup b \cup A_2)$ bounds a disk in $\partial V_1$ because $A_1$ and $A_2$ are mutually parallel. Then by the incompressibility of $F - K$, we can eliminate the components $A_1, A_2$, and we can decrease the number $n$ because of $2\ell = (2|r| - 1)n$, a contradiction. Hence Case I-b does not occur and this completes the proof of Case I.

Suppose we are in Case II. In this case, by Lemmas 4.5 and 4.6, $F_1$ consists of mutually parallel $\partial$-parallel disks and mutually parallel meridian disks. Let $\tilde{E} =$
be the \(\partial\)-parallel disks each of which is parallel to a small regular neighborhood of \(k_1\) in \(\partial V_1\) and \(\tilde{M} = M_1 \cup M_2 \cup \cdots \cup M_s\) the meridian disks. In this case \(r \geq 0\) and \(s > 0\). Let \(D_3\) be a meridian disk of \(W_2\) such that \(\partial D_3\) is a longitude of \(V_1\). Since \(M_1, M_2, \cdots, M_s\) are all mutually parallel, we can take an annulus, say \(A\) in \(\partial V_1\) such that \(\partial D_3\) is along a longitude of \(V_1\). Since \(M_1, M_2, \cdots, M_s\) are all mutually parallel, we can take an annulus, say \(A\), in \(\partial V_1\) such that \(\partial D_3\) is along a longitude of \(V_1\). Then we may assume that \(\partial D_3\) intersect \(A\) in a single essential arc properly embedded in \(A\) and \(\partial D_3\) intersects each \(\partial M_i\) in a single point. Then, since we may assume that \(\partial D_3 \cap \tilde{E} = \emptyset\), the arrangement of the intersection \(\partial D_3 \cap (k_1 \cup k_2 \cup D_1^0 \cup D_1^1 \cup \partial F_1)\) on \(\partial D_3\) is as in Figure 7, where the big points are the points of \(\partial D_3 \cap (k_1 \cup k_2)\), fat arcs are the arcs of \(\partial D_3 \cap (D_1^0 \cup D_1^1)\) and the small points are the points of \(\partial D_3 \cap \partial F_1 = \partial D_3 \cap \partial \tilde{M}\).

[Figure 7]

Let \(\alpha\) be an outermost arc component of \(D_3 \cap F_3\) in \(D_3\) and \(\beta\) the corresponding arc in \(\partial D_3\) with \(\alpha \cap \beta = \partial \alpha = \partial \beta\). Perform a boundary compression for \(F_3\) along the outermost disk for \(\alpha\), and let \(b\) be the band in \(V_1\) produced by the boundary compression. Then we may assume that \(b\) connects \(M_1\) and \(M_2\), and by observing the upper side of \(b\), we have the five cases (i) – (v) illustrated in Figure 8.

[Figure 8]

Suppose, for example, we are in case (i). In this case, we can find a pair of arcs \(\alpha, \beta\) as in Lemma 4.1, a contradiction. In the other cases, we get contradictions similarly. Hence Case II does not occur and this completes the proof of Proposition 1.7 and Theorem 1.3.
References


$K(p, q : r)$

$(p, q)$ torus braid

$r$ full twists

Figure 1

$(p, q)$ torus braid

$V_1$

$D_1$

$D_2$

$N(D_1)$

Figure 2
Figure 3

Figure 4
Figure 5

Figure 6

Figure 7
Figure 8