On positivity and universality of templates induced from diffeomorphisms of the disk

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1. Introduction

In this note, we consider links induced from periodic orbits of orientation preserving automorphisms $\varphi$ of $D^2$. We first present some basic terminologies. We denote the $i$-th iteration of $\varphi$ by $\varphi^i$. We say that $x \in D^2$ is a period $k \in \mathbb{N}$ periodic point if $\varphi^k(x) = x$ and $\varphi^i(x) \neq x$ for $1 \leq i < k$. In particular, we say that $x$ is a fixed point if $x$ is a period 1 periodic point. For $x \in D^2$, $\{\varphi^i(x) | i \in \mathbb{N}\}$ is called the orbit of $x$ and denoted by $O_\varphi(x)$. If $x$ is a periodic point, then $O_\varphi(x)$ is called the periodic orbit of $x$.

Let $\Phi = \{\varphi_t\}_{0 \leq t \leq 1}$ be an isotopy of $D^2$ such that $\varphi_0 = id_{D^2}$, $\varphi_1 = \varphi$. For a finite union of periodic orbits $P$ of $\varphi$, we define a subset of $\tilde{V} = D^2 \times S^1 (\cong D^2 \times I/(x, 0) \sim (x, 1))$, denoted by $S_\Phi P$, as follows.

$$S_\Phi P = \bigcup_{0 \leq t \leq 1} (\varphi_t(P) \times \{t\})/(x, 0) \sim (x, 1).$$

$S_\Phi P$ is called a suspension of $P$ by $\Phi$. Let $V$ be a standardly embedded solid torus in the 3-sphere $S^3$. Then $h : \tilde{V} \to V$ denotes a homeomorphism such that for a longitude $\tilde{\ell}$ on $\tilde{V}$, $h(\tilde{\ell})$ is a knot with the linking number of $h(\tilde{\ell})$ and the core circle of $V$ being 1 (see Figure 1). For each $i \in \mathbb{Z}$, $h^i(S_\Phi P)$ is a link in $S^3$, where the orientation of $S_\Phi P$ is induced from parametrization by $t$.

Figure 1

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**Definition 1.1.** Let \( \varphi : D^2 \to D^2 \) be an orientation preserving automorphism, and \( \Phi = \{ \varphi_t \}_{0 \leq t \leq 1} \) an isotopy of \( D^2 \) such that \( \varphi_0 = id_{D^2}, \varphi_1 = \varphi \). We say that \( \varphi \) induces all link types if there exists an integer \( i \in \mathbb{Z} \) satisfying the following conditions.

\((*)\) For each link \( L \) in \( S^3 \), there exists a finite union of periodic orbits \( P_L \) of \( \varphi \) such that \( L = h^i(S \circ P_L) \).

We note that the definition does not depend on \( \Phi \). Moreover the number of integers \( i \) such that \( h^i \) satisfies (*) does not depend on \( \Phi \) (see [8]). Hence we denote the number by \( \overline{N}(\varphi) \), that is,

\[ \overline{N}(\varphi) = \#\{i \in \mathbb{Z} \mid i \text{ satisfies (*) for } \Phi \}. \]

The topological entropy \( h_{top}(\varphi) \) for \( \varphi \) is a measure of its dynamical complexity (see [14] for a definition of the entropy). A result of Gambaudo-van Strien-Tresser ([3, Theorem A]) tells us that if \( h_{top}(\varphi) = 0 \), then \( \varphi \) does not induce all link types, i.e., \( \overline{N}(\varphi) = 0 \). It is natural to ask the following problem:

**Problem 1.2.** Which automorphism induces all link types?

In [11], the second author researched the Smale horseshoe map [13] on Problem 1.2. The Smale horseshoe map is a fundamental example to study complicated dynamics since the invariant set is hyperbolic and is conjugate to the 2-shift, and such invariant sets are often observed in many dynamical systems [9] (see [12] for basic definitions of dynamical systems).

**Theorem 1.3.** [11] Let \( H \) be the Smale horseshoe map. Then \( \overline{N}(H) = \overline{N}(H^2) = 0 \) and \( \overline{N}(H^3) = 1 \).

Since \( h_{top}(H) \) and \( h_{top}(H^2) \) are positive, Theorem 1.3 shows the existence of diffeomorphisms not inducing all link types.

We will consider Problem 1.2 for generalized horseshoe maps \( G \) using twist signature \( t(G) \) (see Definitions 2.1, 2.2). In Theorem 3.1, we completely determine the number \( \overline{N}(G) \) by \( t(G) \).

### 2. Generalized horseshoe map and twist signature

For definitions of generalized horseshoe map and twist signature, we first introduce some terminologies. Let \( R = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset D^2 \), and let \( S_0, S_1 \) be half disks as in Figure 2(a). For \( c, c' \in [-\frac{1}{2}, \frac{1}{2}] \), we call \( \ell_v = \{c\} \times [-\frac{1}{2}, \frac{1}{2}] \) (resp. \( \ell_h = [-\frac{1}{2}, \frac{1}{2}] \times \{c'\} \)) a vertical (resp. a horizontal) line. For \( [c, d], [c', d'] \subset [-\frac{1}{2}, \frac{1}{2}] \), we call \( B = [c, d] \times [-\frac{1}{2}, \frac{1}{2}] \) (resp. \( B' = [-\frac{1}{2}, \frac{1}{2}] \times [c', d'] \)) a vertical (resp. a horizontal) rectangle.

Let \( B_1, B_2 \) (resp. \( B'_1, B'_2 \)) be disjoint vertical (resp. disjoint horizontal) rectangles. The notation \( B_1 <_1 B_2 \) (resp. \( B'_1 <_2 B'_2 \)) means the first (resp. second) coordinate of a point in \( B_2 \) (resp. \( B'_2 \)) is greater than that of \( B_1 \) (resp. \( B'_1 \)). We denote the open rectangle which lies between \( B_1 \) and \( B_2 \) by \( (B_1, B_2) \).

**Definition 2.1.** Let \( n \geq 2 \) be an integer. A generalized horseshoe map \( G \) of length \( n \) is an orientation preserving diffeomorphism of \( D^2 \) satisfying the following: There exist vertical rectangles \( B_1 <_1 B_2 <_1 \cdots <_1 B_n \) and horizontal rectangles \( B'_1 <_2 B'_2 <_2 \cdots <_2 B'_n \) such that
(1) for each $1 \leq i \leq n$, $G(B_i) = B_j'$ for some $1 \leq j \leq n$,
(2) for each $1 \leq i \leq n - 1$, $G((B_i, B_{i+1})) \subset S_k$ for some $k \in \{0, 1\}$,
(3) $G$ expands the part of horizontal lines which intersects each $B_i$ uniformly, and contract the vertical lines in each $B_i$ uniformly,
(4) $G|_{S_0} : S_0 \to S_0$ is contractive,
(5) if $n$ is even (resp. odd), then $G(S_1) \subset \text{Int } S_0$ (resp. $G|_{S_1} : S_1 \to S_1$ is contractive) and
(6) $G$ has no periodic points in $D^2 \setminus R$.

**Definition 2.2.** Let $G$ be a generalized horseshoe map of length $n$. **Twist signature** $t(G)$ of $G$ is the array of $n$ integers $(a_1, \cdots, a_n)$ satisfying the following:

1. $a_1 = 0$.
2. For $2 \leq i \leq n$, $a_i = a_{i-1} + 1$ if $G(B_{i-1}) <_2 G(B_i)$ and $G((B_{i-1}, B_i)) \subset S_1$, or if $G(B_{i-1}) >_2 G(B_i)$ and $G((B_{i-1}, B_i)) \subset S_0$. Otherwise $a_i = a_{i-1} - 1$.

By the condition of generalized horseshoe maps $G$, $\Lambda = \bigcap_{m \in \mathbb{Z}} G^m(B_1 \cup \cdots \cup B_n)$ is hyperbolic which is conjugate to the $n$-shift.

Notice that the Smale horseshoe map is a generalized horseshoe map of length 2 with twist signature $(0, 1)$.

**Example 2.3.**
(1) Let $K_1$ be a generalized horseshoe map of length 3 as in Figure 2(b). Then $t(K_1) = (0, 1, 2)$.
(2) Let $K_2$ be a generalized horseshoe map of length 4 as in Figure 2(c). Then $t(K_2) =
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(0, 1, 0, \ldots, –1).

(3) Let $K_3$ be a generalized horseshoe map of length 4 as in Figure 2(d). Then $t(K_3) = (0, -1, -2, -3)$.

3. Statement of results

Let $G$ be a generalized horseshoe map with twist signature $(a_1, \ldots, a_n)$. We say that $G$ is positive (resp. negative) if for any $i \in \{1, \ldots, n\}$, $a_i \geq 0$ (resp. $a_i \leq 0$). We say that $G$ is mixed if $G$ is neither positive nor negative. For example, $K_1, K_2, K_3$ in Example 2.3 are positive, mixed, negative respectively.

The following is Main theorem of this note:

**Theorem 3.1.** For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the greatest integer which does not exceed $x$. Let $G$ be a generalized horseshoe map with twist signature $(a_1, \ldots, a_n)$. Let $M_+ = \max\{a_i|1 \leq i \leq n\}$ and $M_- = \min\{a_i|1 \leq i \leq n\}$. If $G$ is positive, then $\overline{N}(G) = \lfloor \frac{M_+ - 1}{2} \rfloor$. If $G$ is negative, then $\overline{N}(G) = \lfloor \frac{M_- - 1}{2} \rfloor$. If $G$ is mixed, then $\overline{N}(G) = \lfloor \frac{M_+ - 1}{2} \rfloor + \lfloor \frac{M_- - 1}{2} \rfloor + 1$.

The next corollary is a direct consequence of the above theorem:

**Corollary 3.2.** Let $G$ be a generalized horseshoe map, and $M_+$ and $M_-$ be as in Theorem 3.1. Then $G$ induces all link types, i.e., $\overline{N}(G) \geq 1$ if and only if $G$ is one of the following types.

- $G$ is positive and $M_+ \geq 3$.
- $G$ is negative and $M_- \leq -3$.
- $G$ is mixed.

Recall that $K_1, K_2, K_3$ are generalized horseshoe maps in Example 2.3. By Theorem 3.1, $\overline{N}(K_1) = 0$, $\overline{N}(K_2) = 1$ and $\overline{N}(K_3) = 1$.

The proof of Theorem 3.1 is done by using the template theory ([2], [4], [5]).

REFERENCES

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