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On positivity and universality of templates induced from diffeomorphisms of the disk

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1. Introduction

In this note, we consider links induced from periodic orbits of orientation preserving automorphisms \( \varphi \) of \( D^2 \). We first present some basic terminologies. We denote the \( i \)-th iteration of \( \varphi \) by \( \varphi^i \). We say that \( x \in D^2 \) is a period \( k \in \mathbb{N} \) periodic point if \( \varphi^k(x) = x \) and \( \varphi^i(x) \neq x \) for \( 1 \leq i < k \). In particular, we say that \( x \) is a fixed point if \( x \) is a period 1 periodic point. For \( x \in D^2 \), \( \{\varphi^i(x) | i \in \mathbb{N}\} \) is called the orbit of \( x \) and denoted by \( O_\varphi(x) \). If \( x \) is a periodic point, then \( O_\varphi(x) \) is called the periodic orbit of \( x \).

Let \( \Phi = \{\varphi_t\}_{0 \leq t \leq 1} \) be an isotopy of \( D^2 \) such that \( \varphi_0 = id_{D^2} \), \( \varphi_1 = \varphi \). For a finite union of periodic orbits \( P \) of \( \varphi \), we define a subset of \( \tilde{V} = D^2 \times S^1 (\cong D^2 \times I/(x,0) \sim (x,1)) \), denoted by \( S_\Phi P \), as follows.

\[
S_\Phi P = \bigcup_{0 \leq t \leq 1} (\varphi_t(P) \times \{t\})/(x,0) \sim (x,1).
\]

\( S_\Phi P \) is called a suspension of \( P \) by \( \Phi \). Let \( V \) be a standardly embedded solid torus in the 3-sphere \( S^3 \). Then \( h : \tilde{V} \to V \) denotes a homeomorphism such that for a longitude \( \tilde{\ell} \) on \( \tilde{V} \), \( h(\tilde{\ell}) \) is a knot with the linking number of \( h(\tilde{\ell}) \) and the core circle of \( V \) being 1 (see Figure 1). For each \( i \in \mathbb{Z} \), \( h^i(S_\Phi P) \) is a link in \( S^3 \), where the orientation of \( S_\Phi P \) is induced from parametrization by \( t \).

Figure 1

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Definition 1.1. Let $\varphi : D^2 \to D^2$ be an orientation preserving automorphism, and $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ an isotopy of $D^2$ such that $\varphi_0 = id_{D^2}$, $\varphi_1 = \varphi$. We say that $\varphi$ induces all link types if there exists an integer $i \in \mathbb{Z}$ satisfying the following conditions.

(*) For each link $L$ in $S^3$, there exists a finite union of periodic orbits $P_L$ of $\varphi$ such that $L = h^i(S^1 P_L)$.

We note that the definition does not depend on $\Phi$. Moreover the number of integers $i$ such that $h^i$ satisfies (*) does not depend on $\Phi$ (see [8]). Hence we denote the number by $\overline{N}(\varphi)$, that is,

$$\overline{N}(\varphi) = \# \{ i \in \mathbb{Z} \mid i \text{ satisfies (*) for } \Phi \}.$$

The topological entropy $h_{top}(\varphi)$ for $\varphi$ is a measure of its dynamical complexity (see [14] for a definition of the entropy). A result of Gambaud-van Strien-Tresser ([3, Theorem A]) tells us that if $h_{top}(\varphi) = 0$, then $\varphi$ does not induce all link types, i.e., $\overline{N}(\varphi) = 0$. It is natural to ask the following problem:

Problem 1.2. Which automorphism induces all link types?

In [11], the second author researched the Smale horseshoe map [13] on Problem 1.2. The Smale horseshoe map is a fundamental example to study complicated dynamics since the invariant set is hyperbolic and is conjugate to the 2-shift, and such invariant sets are often observed in many dynamical systems [9] (see [12] for basic definitions of dynamical systems).

Theorem 1.3. [11] Let $H$ be the Smale horseshoe map. Then $\overline{N}(H) = \overline{N}(H^2) = 0$ and $\overline{N}(H^3) = 1$.

Since $h_{top}(H)$ and $h_{top}(H^2)$ are positive, Theorem 1.3 shows the existence of diffeomorphisms not inducing all link types.

We will consider Problem 1.2 for generalized horseshoe maps $G$ using twist signature $t(G)$ (see Definitions 2.1, 2.2). In Theorem 3.1, we completely determine the number $\overline{N}(G)$ by $t(G)$.

2. Generalized Horseshoe Map and Twist Signature

For definitions of generalized horseshoe map and twist signature, we first introduce some terminologies. Let $R = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset D^2$, and let $S_0, S_1$ be half disks as in Figure 2(a). For $c, c' \in [-\frac{1}{2}, \frac{1}{2}]$, we call $\ell_v = \{c\} \times [-\frac{1}{2}, \frac{1}{2}]$ (resp. $\ell_h = [-\frac{1}{2}, \frac{1}{2}] \times \{c\}$) a vertical (resp. a horizontal) line. For $[c, d], [c', d'] \subset [-\frac{1}{2}, \frac{1}{2}]$, we call $B = [c, d] \times [-\frac{1}{2}, \frac{1}{2}]$ (resp. $B' = [-\frac{1}{2}, \frac{1}{2}] \times [c', d']$) a vertical (resp. a horizontal) rectangle.

Let $B_1, B_2$ (resp. $B_1', B_2'$) be disjoint vertical (resp. disjoint horizontal) rectangles. The notation $B_1 <_1 B_2$ (resp. $B_1' <_1 B_2'$) means the first (resp. second) coordinate of a point in $B_2$ (resp. $B_2'$) is greater than that of $B_1$ (resp. $B_1'$). We denote the open rectangle which lies between $B_1$ and $B_2$ by $(B_1, B_2)$.

Definition 2.1. Let $n \geq 2$ be an integer. A generalized horseshoe map $G$ of length $n$ is an orientation preserving diffeomorphism of $D^2$ satisfying the following: There exist vertical rectangles $B_1 <_1 B_2 <_1 \cdots <_1 B_n$ and horizontal rectangles $B_1' <_2 B_2' <_2 \cdots <_2 B_n'$ such that
(1) for each $1 \leq i \leq n$, $G(B_i) = B'_j$ for some $1 \leq j \leq n$,
(2) for each $1 \leq i \leq n-1$, $G((B_i, B_{i+1})) \subset S_k$ for some $k \in \{0, 1\}$,
(3) $G$ expands the part of horizontal lines which intersects each $B_i$ uniformly, and contract the vertical lines in each $B_i$ uniformly,
(4) $G|_{S_0} : S_0 \rightarrow S_0$ is contractive,
(5) if $n$ is even (resp. odd), then $G(S_1) \subset Int S_0$ (resp. $G|_{S_1} : S_1 \rightarrow S_1$ is contractive) and
(6) $G$ has no periodic points in $D^2 \backslash R$.

Definition 2.2. Let $G$ be a generalized horseshoe map of length $n$. Twist signature $t(G)$ of $G$ is the array of $n$ integers $(a_1, \cdots, a_n)$ satisfying the following:

(1) $a_1 = 0$.
(2) For $2 \leq i \leq n$, $a_i = a_{i-1} + 1$ if $G(B_{i-1}) <_2 G(B_i)$ and $G((B_{i-1}, B_i)) \subset S_1$, or if $G(B_{i-1}) >_2 G(B_i)$ and $G((B_{i-1}, B_i)) \subset S_0$. Otherwise $a_i = a_{i-1} - 1$.

By the condition of generalized horseshoe maps, $\Lambda = \bigcap_{m \in \mathbb{Z}} G^m(B_1 \cup \cdots \cup B_n)$ is hyperbolic which is conjugate to the $n$-shift.

Notice that the Smale horseshoe map is a generalized horseshoe map of length 2 with twist signature $(0, 1)$.

Example 2.3. (1) Let $K_1$ be a generalized horseshoe map of length 3 as in Figure 2(b). Then $t(K_1) = (0, 1, 2)$.
(2) Let $K_2$ be a generalized horseshoe map of length 4 as in Figure 2(c). Then $t(K_2) =$
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(0, 1, 0, −1).

(3) Let $K_3$ be a generalized horseshoe map of length 4 as in Figure 2(d). Then $t(K_3) = (0, -1, -2, -3).

3. Statement of results

Let $G$ be a generalized horseshoe map with twist signature $(a_1, \ldots, a_n)$. We say that $G$ is positive (resp. negative) if for any $i \in \{1, \ldots, n\}$, $a_i \geq 0$ (resp. $a_i \leq 0$). We say that $G$ is mixed if $G$ is neither positive nor negative. For example, $K_1, K_2, K_3$ in Example 2.3 are positive, mixed, negative respectively.

The following is Main theorem of this note:

**Theorem 3.1.** For $x \in \mathbb{R}$, let $[x]$ be the greatest integer which does not exceed $x$. Let $G$ be a generalized horseshoe map with twist signature $(a_1, \ldots, a_n)$. Let $M_+ = \max\{a_i | 1 \leq i \leq n\}$ and $M_- = \min\{a_i | 1 \leq i \leq n\}$. If $G$ is positive, then $\overline{N}(G) = \lceil \frac{M_+ - 1}{2} \rceil$. If $G$ is negative, then $\overline{N}(G) = \lfloor \frac{-M_- - 1}{2} \rfloor + 1$. If $G$ is mixed, then $\overline{N}(G) = \lceil \frac{M_+ - 1}{2} \rceil + \lfloor \frac{-M_- - 1}{2} \rfloor + 1$.

The next corollary is a direct consequence of the above theorem:

**Corollary 3.2.** Let $G$ be a generalized horseshoe map, and $M_+$ and $M_-$ be as in Theorem 3.1. Then $G$ induces all link types, i.e., $\overline{N}(G) \geq 1$ if and only if $G$ is one of the following types.

- $G$ is positive and $M_+ \geq 3$.
- $G$ is negative and $M_- \leq -3$.
- $G$ is mixed.

Recall that $K_1, K_2, K_3$ are generalized horseshoe maps in Example 2.3. By Theorem 3.1, $\overline{N}(K_1) = 0$, $\overline{N}(K_2) = 1$ and $\overline{N}(K_3) = 1$.

The proof of Theorem 3.1 is done by using the template theory ([2], [4], [5]).

**REFERENCES**


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