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PENTAGONAL EQUATIONS FOR OPERATORS ASSOCIATED WITH INCLUSIONS OF \( C^* \)-ALGEBRAS
(PRELIMINARY VERSION)

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1. INTRODUCTION

The pentagonal equation (PE) first appeared in the duality theory for locally compact groups. The Kac-Takesaki operator in the theory satisfies the PE (cf. [9], [29]). S. Baaj and G. Skandalis called a unitary operator on a Hilbert space a multiplicative unitary (MU) when it satisfies PE in [2]. They constructed a pair of Hopf \( C^* \)-algebras from a regular MU. M. Enock and R. Nest constructed an MU from an irreducible regular depth 2 inclusion of factors. As for measured groupoids, T. Yamanouchi constructed an analogue of the Kac-Takesaki operator in [35]. But this operator does not satisfy the PE. J. M. Vallin showed that it satisfies an equation which is a generalization of the PE in [32]. He called a unitary operator a pseudo-multiplicative unitary (PMU) when it satisfies this generalized PE. Vallin defined the generalized PE using the Connes-Sauvageot’s relative tensor products of Hilbert spaces. M. Enock and J. M. Vallin constructed a PMU from a regular depth 2 inclusion of von Neumann algebras in [10]. The basis of the PMU they studied is a (not necessarily commutative) von Neumann algebra. Recently quantum groupoids are studied by many authors. For example, see [3], [7], [18], [20], [27] and [33]. Quantum

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groupoids are related to inclusions of von Neumann algebras and PMU's. In particular, PMU's in finite-dimension were studied by G. Böhm and K. Szlachányi [3] and by J. M. Vallin [33]. They studied the PMU from the viewpoint of multiplicative isometries. Before their works, Yamanouchi studied a partial isometry which satisfies the PE in [36]. When we deal with PMU's in the theory of C*-algebras, it is useful to formulate the generalized PE in the frame work of Hilbert C*-modules. As for the usefulness of Hilbert C*-modules, for example, see the works of M. A. Rieffel [25], E. C. Lance [16], B. Blackadar [4] and Y. Watatani [34]. The author defined a PMU on a Hilbert C*-module using interior tensor products in [22]. The base algebra of the PMU defined there is a commutative C*-algebra. (When PMU is defined on a tensor product of A-modules, we will call A a base algebra. See Definition 3.1.) An analogue of the Kac-Takesaki operator for a topological groupoid becomes a PMU in the sense of [22]. Moreover, if it is a measured groupoid, that is, if it has a quasi-invariant measure, then the PMU constructed in [22] induces the PMU studied by Vallin in [32]. The author constructed in [23] a PMU in the sense of [22] from an inclusion of finite-dimensional C*-algebras when the inclusion satisfies certain conditions. There we had to assume a condition which implies a commutativity of the base algebra.

In this paper, we will study a PE in full generality. We will not distinguish a PE from a generalization of a PE and we will not distinguish an MU from a PMU. Therefore we will call a PE a generalization of a PE and we will call an operator a multiplicative operator when it satisfies a generalization of a PE. The aim of this paper is to give a definition of a PE in full generality in the framework of Hilbert C*-module and to give examples of operators which satisfies this PE. Especially, we remove the assumption of the commutativity of the base algebra, which was assumed in [22] and [23]. We meet many difficulties in defining a PE in the framework of Hilbert C*-modules. For example, we do not in general the following objects; a
flip on an interior tensor product of Hilbert $C^*$-modules, a tensor product $I \otimes x$ as operator on an interior tensor product of Hilbert $C^*$-modules for an adjointable operator $x$ on a Hilbert $C^*$-module and a modular involution on a Hilbert $C^*$-module. When the base algebra is $\mathbb{C}$, the multiplicative unitary operator (MUO) defined in this paper coincides with the MU defined by Baaj and Skandalis in [2] modulo the flip. When the base algebra is commutative, the MUO coincides with the PMU studied in [22] and [23] modulo the flip. Note that we cannot define a flip when the base algebra is not commutative.

2. PRELIMINARIES

First, we recall some definitions and notations on Hilbert $C^*$-modules. For details, we refer the reader to [16]. Let $A$ be a $C^*$-algebra. A Hilbert $A$-module is a right $A$-module $E$ with an $A$-valued inner product $\langle \cdot, \cdot \rangle$ such that $E$ is complete with respect to the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$. Note that the inner product is linear in its second variable. A Hilbert $A$-module $E$ is said to be full if the closure of the linear span of $\{\langle \xi, \eta \rangle; \xi, \eta \in E\}$ is all of $A$. Let $E$ and $F$ be Hilbert $A$-modules. We denote by $\mathcal{L}_A(E, F)$ the set of bounded adjointable operators from $E$ to $F$ and we denote by $\mathcal{K}_A(E, F)$ the closure of the linear span of $\{\theta_{\xi,\eta}; \xi \in F, \eta \in E\}$, where $\theta_{\xi,\eta}$ is the element of $\mathcal{L}_A(E, F)$ defined by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle$ for $\zeta \in E$. We abbreviate $\mathcal{L}_A(E, E)$ and $\mathcal{K}_A(E, E)$ to $\mathcal{L}_A(E)$ and $\mathcal{K}_A(E)$ respectively. We denote by $I_E$ the identity operator on $E$. We often omit the subscript $E$ for simplicity. A unitary operator $U$ of $E$ to $F$ is an adjointable operator such that $U^*U = I_E$ and $UU^* = I_F$.

Let $A$ and $B$ be $C^*$-algebras. Suppose that $E$ is a Hilbert $A$-module and that $F$ is a Hilbert $B$-module. Let $\phi$ be a $*$-homomorphism of $A$ to $\mathcal{L}_B(F)$. Then we can define the interior tensor product $E \otimes_{\phi} F$ ([16], Chapter 4). For $\xi \in E$ and $\eta \in F$, we denote by $\xi \otimes_{\phi} \eta$ the corresponding element of $E \otimes_{\phi} F$. We often omit the subscript $\phi$, writing $\xi \otimes \eta = \xi \otimes_{\phi} \eta$ for simplicity. We have $\xi a \otimes \eta = \xi \otimes \phi(a) \eta$.
for every \( a \in A \). Note that \( E \otimes_{\phi} F \) is a Hilbert \( B \)-module with a \( B \)-valued inner product such that

\[
< \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 > = < \eta_1, \phi(< \xi_1, \xi_2 >) \eta_2 >
\]

for \( \xi_1, \xi_2 \in E \) and \( \eta_1, \eta_2 \in F \). Let \( E_i \) be a Hilbert \( A_i \)-module for \( i = 1, 2, 3 \) and let \( \phi_i \) be a \( * \)-homomorphism of \( A_{i-1} \) to \( \mathcal{L}_{A_i}(E_i) \) for \( i = 2, 3 \). Define a \( * \)-homomorphism \( \phi_2 \otimes_{\phi_3} \iota \) of \( A_1 \) to \( \mathcal{L}_{A_3}(E_2 \otimes_{\phi_3} E_3) \) by \( (\phi_2 \otimes_{\phi_3} \iota)(a) = \phi_2(a) \otimes I \) for \( a \in A_1 \). We often omit the subscript \( \phi_3 \), writing \( \phi_2 \otimes \iota = \phi_2 \otimes_{\phi_3} \iota \) for simplicity. Then we have

\[
(E_1 \otimes_{\phi_2} E_2) \otimes_{\phi_3} E_3 = E_1 \otimes_{\phi_2 \otimes \iota} (E_2 \otimes_{\phi_3} E_3).
\]

We denote the above tensor product by \( E_1 \otimes_{\phi_2} E_2 \otimes_{\phi_3} E_3 \).

For \( i = 1, 2 \), let \( E_i \) be a Hilbert \( A \)-module, let \( F_i \) be a Hilbert \( B \)-module and let \( \phi_i \) be a \( * \)-homomorphism of \( A \) to \( \mathcal{L}_{B}(F_i) \). We denote by \( \mathcal{L}_B((F_1, \phi_1), (F_2, \phi_2)) \) the set of elements \( x \) of \( \mathcal{L}_B(F_1, F_2) \) such that \( x\phi_1(a) = \phi_2(a)x \) for all \( a \in A \). We abbreviate \( \mathcal{L}_B((F_1, \phi_1), (F_1, \phi_1)) \) to \( \mathcal{L}_B(F_1, \phi_1) \). We define \( \mathcal{K}_B((F_1, \phi_1), (F_2, \phi_2)) \) and \( \mathcal{K}_B(F_1, \phi_1) \) similarly. The following proposition is useful in later arguments.

**Proposition 2.1** ([22]). For \( x \in \mathcal{L}_A(E_1, E_2) \) and \( y \in \mathcal{L}_B((F_1, \phi_1), (F_2, \phi_2)) \), there exists an element \( x \otimes_{\phi_1} y \) of \( \mathcal{L}_B(E_1 \otimes_{\phi_1} F_1, E_2 \otimes_{\phi_2} F_2) \) such that \( (x \otimes_{\phi_1} y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta) \) for \( \xi \in E_1 \) and \( y \in F_1 \).

We often omit the subscript \( \phi_1 \), writing \( x \otimes y = x \otimes_{\phi_1} y \) for simplicity.

3. **PENTAGONAL EQUATIONS FOR OPERATORS ON HILBERT C*-MODULES**

Let \( A \) be a \( C^* \)-algebra, let \( E \) be a Hilbert \( A \)-module and let \( \phi \) and \( \psi \) be \( * \)-homomorphisms of \( A \) to \( \mathcal{L}_A(E) \). We assume that \( \phi \) and \( \psi \) commute, that is, \( \phi(a)\psi(b) = \psi(b)\phi(a) \) for all \( a, b \in A \). We can define \( * \)-homomorphisms \( \iota \otimes_{\phi} \phi \) and \( \iota \otimes_{\phi} \psi \) of \( A \) to \( \mathcal{L}_A(E \otimes_{\phi} E) \) by \( (\iota \otimes_{\phi} \phi)(a) = I \otimes_{\phi} \phi(a) \) and \( (\iota \otimes_{\phi} \psi)(a) = I \otimes_{\phi} \psi(a) \) respectively. We often omit the subscript \( \phi \), writing \( \iota \otimes \phi = \iota \otimes_{\phi} \phi \) and \( \iota \otimes \psi = \iota \otimes_{\phi} \psi \) for simplicity. We can also define \( * \)-homomorphisms \( \iota \otimes_{\psi} \phi \) and \( \iota \otimes_{\psi} \psi \) of \( A \) to \( \mathcal{L}_A(E \otimes_{\psi} E) \).
We often omit the subscript $\psi$. Let $W$ be an operator in $L_A(E \otimes_\psi E, E \otimes_\phi E)$. We assume that $W$ satisfies the following equations:

(3.1) \[ W(\iota \otimes_\psi \phi)(a) = (\phi \otimes_\phi \iota)(a)W, \]

(3.2) \[ W(\psi \otimes_\psi \iota)(a) = (\iota \otimes_\phi \psi)(a)W, \]

(3.3) \[ W(\phi \otimes_\psi \iota)(a) = (\psi \otimes_\phi \iota)(a)W \]

for all $a \in A$. Then, by Proposition 2.1, we can define following operators;

\[ W \otimes_\psi I \in L_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\phi E \otimes_\psi E), \]
\[ I \otimes_{\phi \otimes \iota} W \in L_A(E \otimes_\phi E \otimes_\psi E, E \otimes_\psi E \otimes_\phi E), \]
\[ W \otimes_\phi I \in L_A(E \otimes_\psi E \otimes_\phi E, E \otimes_\phi E \otimes_\phi E), \]
\[ I \otimes_{\psi \otimes \iota} W \in L_A(E \otimes_\psi E \otimes_\psi E, E \otimes_\iota \otimes_\psi (E \otimes_\phi E)), \]
\[ I \otimes_{\iota \otimes \phi} W \in L_A(E \otimes_{\iota \otimes \phi} (E \otimes_\psi E), E \otimes_\phi E \otimes_\phi E). \]

Since $\phi$ and $\psi$ commute, there exists an isomorphism $\Sigma_{12}$ of $E \otimes_{\iota \otimes \psi} (E \otimes_\phi E)$ onto $E \otimes_{\iota \otimes \phi} (E \otimes_\psi E)$ as Hilbert $A$-modules such that

\[ \Sigma_{12}(x_1 \otimes (x_2 \otimes x_3)) = x_2 \otimes (x_1 \otimes x_3) \]

for $x_i \in E \ (i = 1, 2, 3)$. Then we can define a pentagonal equation.

**Definition 3.1.** Let $W$ be an element of $L_A(E \otimes_\psi E, E \otimes_\phi E)$. Assume that $W$ satisfies the equations (3.1), (3.2) and (3.3). An operator $W$ is said to be multiplicative if it satisfies the pentagonal equation

(3.4) \[ (W \otimes_\phi I)(I \otimes_{\phi \otimes \iota} W)(W \otimes_\psi I) = (I \otimes_{\iota \otimes \phi} W) \Sigma_{12}(I \otimes_{\psi \otimes \iota} W). \]

We will call $A$ the base algebra of the multiplicative operator $W$.

**Example 3.2.** Suppose that $A = \mathbb{C}$. Then $E = H$ is a usual Hilbert space and $L_\mathbb{C}(E) = L(H)$ is the $C^*$-algebra of bounded linear operators on $H$. Let $\phi = \psi = \text{id}$,
where \( id(\lambda) = \lambda I_H \) for \( \lambda \in \mathbb{C} \). Then \( E \otimes_{id} E \) is the usual tensor product \( H \otimes H \).

Let \( \Sigma \in \mathcal{L}(H \otimes H) \) be the flip, that is, \( \Sigma(\xi \otimes \eta) = \eta \otimes \xi \). Let \( W \) be an element of \( \mathcal{L}(H \otimes H) \). Then the pentagonal equation (3.4) has the following form:

\[
(W \otimes I)(I \otimes W)(W \otimes I) = (I \otimes W)(\Sigma \otimes I)(I \otimes W).
\]

Define an operator \( \widetilde{W} \) by \( \widetilde{W} = W \Sigma \). Then \( W \) satisfies the pentagonal equation (3.5) if and only if \( \widetilde{W} \) satisfies the usual pentagonal equation:

\[
\widetilde{W}_{12} \widetilde{W}_{13} \widetilde{W}_{23} = \widetilde{W}_{23} \widetilde{W}_{13}.
\]

**Example 3.3.** In Examaple 3.2, if \( W = \Sigma \), then the equation (3.5) is the Yang-Baxter equation for the flip ([15]);

\[
(\Sigma \otimes I)(I \otimes \Sigma)(\Sigma \otimes I) = (I \otimes \Sigma)(\Sigma \otimes I)(I \otimes \Sigma).
\]

**Example 3.4.** Let \( G \) be a locally compact Hausdorff group and \( \nu \) be a right Haar measure on \( G \). Set \( H = L^2(G, \nu) \). Define an operator \( W \) on \( H \otimes H \) by \( (W \xi)(g, h) = \xi(h, gh) \) for \( \xi \in C_c(G \times G) \) and \( g, h \in G \). Then \( W \) satisfies the pentagonal equation (3.5). The operator \( \widetilde{W} \) in Example 3.2 is given by \( (\widetilde{W} \xi)(g, h) = \xi(gh, h) \), which is the Kac-Takesaki operator and satisfies the usual pentagonal equation (3.6).

Suppose that \( A = C \) is an abelian \( C^* \)-algebra. Let \( E \) be a Hilbert \( C \)-module and \( \phi \) be a \( * \)-homomorphism of \( C \) to \( \mathcal{L}_C(E) \). Define a \( * \)-homomorphism \( \psi \) of \( C \) to \( \mathcal{L}_C(E) \) by \( \psi(c) \xi = \xi c \) for \( \xi \in E \) and \( c \in C \). In this situation, we have defined a generalized pentagonal equation and we have called a unitary operator pseudo-multiplicative if it satisfies the generalized pentagonal equation in [22]. We will describe the relation between the pentagonal equation (3.4) defined in this paper and the generalized pentagonal equation defined in [22]. We wrote \( E \otimes_C E \) for \( E \otimes_{\phi} E \) in [22]. Let \( \widetilde{W} \) be a unitary operator in \( \mathcal{L}_C(E \otimes_{\phi} E, E \otimes_{\phi} E) \). Suppose that \( \widetilde{W} \) satisfies the following
There exists an isomorphism $\sigma_1$ of $E \otimes_{i \otimes \psi} (E \otimes_{\phi} E)$ onto $E \otimes_{i \otimes \phi} (E \otimes_{\psi} E)$ such that $\sigma_1(\xi \otimes (\eta \otimes \zeta)) = \eta \otimes (\xi \otimes \zeta)$. We define an operator $\tilde{W}_{13}$ in $\mathcal{L}_C(E \otimes_{i \otimes \psi} (E \otimes_{\phi} E), E \otimes_{i \otimes \phi} (E \otimes_{\phi} E))$ by $\tilde{W}_{13} = \sigma_2'(I \otimes_{i \otimes \phi} \overline{W}) \sigma_1$.

In [22], the generalized pentagonal equation was defined as follows;

\begin{equation}
(\overline{W} \otimes_{\phi} I)\tilde{W}_{13}(I \otimes_{i \otimes \psi} \overline{W}) = (I \otimes_{i \otimes \phi} \overline{W})(\overline{W} \otimes_{\psi} I).
\end{equation}

There exists the flip $\Sigma_\psi$ in $\mathcal{L}_C(E \otimes_{\psi} E)$ such that $\Sigma_\psi(\xi \otimes \eta) = \eta \otimes \xi$. Then we have the following;

**Proposition 3.5.** Let $W$ be an element of $\mathcal{L}_C(E \otimes_{\psi} E, E \otimes_{\phi} E)$. Set $\overline{W} = W \Sigma_\psi$. Then $W$ satisfies the equation (3.4) if and only if $\overline{W}$ satisfies the equation (3.9).

**Example 3.6.** Let $G$ be a second countable locally compact Hausdorff groupoid.

We denote by $s$ (resp. $r$) the source (resp. range) map of $G$. We denote by $G^{(0)}$ the unit space of $G$ and by $G^{(2)}$ the set of composable pairs. We set $G_u = s^{-1}(u)$ for $u \in G^{(0)}$. Let $\{\lambda_u; u \in G^{(0)}\}$ be a right Haar system of $G$. As for groupoids and groupoid $C^*$-algebras, see Renault [24]. (See also [19] and [22] for notations and definitions used here.) For an arbitrary topological space $X$, we denote by $C_c(X)$ the set of complex-valued continuous functions on $X$ with compact supports and by $C_0(X)$ the abelian $C^*$-algebra of continuous functions on $X$ vanishing at infinity with the supremum norm $\| \cdot \|_\infty$. Let $C$ be the abelian $C^*$-algebra $C_0(G^{(0)})$ and let $\tilde{E}$ be the linear space $C_c(G)$. Then $\tilde{E}$ is a right $C$-module with the right $C$-action defined by $(\xi c)(x) = \xi(x)c(s(x))$ for $\xi \in \tilde{E}$, $c \in C$ and $x \in G$. We define a $C$-valued
inner product of $\widetilde{E}$ by

$$<\xi, \eta> (u) = \int_G \overline{\xi(x)} \eta(x) d\lambda_u(x)$$

for $\xi, \eta \in \widetilde{E}$ and $u \in G^{(0)}$. We denote by $E$ the completion of $\widetilde{E}$ by the norm $||\xi|| = ||<\xi, \xi>||^{1/2}$. Then $E$ is a full right Hilbert $C$-module. Define non-degenerate injective $*$-homomorphisms $\phi$ and $\psi$ of $C$ to $\mathcal{L}_C(E)$ by $(\phi(c)\xi)(x) = c(r(x))\xi(x)$ and $\psi(c)\xi = \xi c$ respectively for $c \in C$, $\xi \in \overline{E}$ and $x \in G$. Set $G^2(ss) = \{(x, y) \in G^2; s(x) = s(y)\}$. We define $C$-valued inner products of $C_c(G^2(ss))$ and $C_c(G^{(2)})$ by

$$<f_1, g_1> (u) = \int\int_{G^2(ss)} \overline{f_1(x,y)} g_1(x,y) d\lambda_u(x)d\lambda_u(y),$$

$$<f_2, g_2> (u) = \int\int_{G^{(2)}} \overline{f_2(x,y)} g_2(x,y) d\lambda_r(y)(x)d\lambda_u(y)$$

respectively for $u \in G^{(0)}$, $f_1, g_1 \in C_c(G^2(ss))$ and $f_2, g_2 \in C_c(G^{(2)})$. Then $C_c(G^2(ss))$ and $C_c(G^{(2)})$ are dense pre-Hilbert $C$-submodules of $E \otimes_\phi E$ and $E \otimes_\psi E$ respectively. Define a unitary operator $W$ in $\mathcal{L}_C(E \otimes_\phi E, E \otimes_\phi E)$ by $(W\xi)(x, y) = \xi(y, xy)$ for $\xi \in C_c(G^2(ss))$, $(x, y) \in G^{(2)}$. Set $\widetilde{W} = W\Sigma_\phi$. We have $(\widetilde{W}\xi)(x, y) = \xi(xy, y)$ for $\xi \in C_c(G^2(ss))$, $(x, y) \in G^{(2)}$. It follows from [22] that $\widetilde{W}$ satisfies the equation (3.9). By Proposition 3.5, $W$ satisfies the pentagonal equation (3.4). When $G$ is a measured groupoid, that is, when there exists a quasi-invariant measure on $G^{(0)}$, we discussed in [22] the relation between the operator $\widetilde{W}$ constructed above and the fundamental operator studied by Yamanouchi in [36, §2] and by Vallin in [32, §3].

4. COPRODUCTS FOR HILBERT $C^*$-MODULES

It is known that multiplicative unitary operators give coproducts in several situations (cf. [2], [35], [31], [32], [10], [21], [22]). In this section, we study a coproduct for a Hilbert $C^*$-module associated with a multiplicative unitary operator and a fixed vector with a certain property. First we introduce a notion of coproducts for Hilbert $C^*$-modules. We denote by $E$ a Hilbert $A$-module and by $\phi$ a $*$-homomorphism of $A$ to $\mathcal{L}_A(E)$. 

Definition 4.1. Let $\delta$ be an operator in $\mathcal{L}_A(E, E \otimes_{\phi} E)$. We say that $E$ is a coproduct of $(E, \phi)$ if $\delta$ satisfies the following equations:

\begin{align*}
\delta \phi(a) &= (\phi \otimes \iota)(a) \delta & \text{for all } a \in A \\
(\delta \otimes I_E)\delta &= (I_E \otimes \delta)\delta
\end{align*}

The triplet $(E, \phi, \delta)$ is called a Hopf Hilbert $A$-module.

Suppose that $\delta$ is coproduct for $E$. For $\xi, \eta \in E$, we define a product $\xi \eta$ in $E$ by $\xi \eta = \delta^*(\xi \otimes \eta)$. It follows from (4.11) that this product is associative. Then $E$ is a right $A$-algebra with this product. Note that we have $||\xi \eta|| \leq ||\delta|| ||\xi|| ||\eta||$. Therefore, if $||\delta|| \leq 1$, then $E$ is a Banach algebra.

Let $\psi$ be a $*$-homomorphism of $A$ to $\mathcal{L}_A(E)$ such that $\phi$ and $\psi$ commute and let $W \in \mathcal{L}_A(E \otimes_{\psi} E, E \otimes_{\phi} E)$ be a multiplicative unitary operator. For an element $\xi_0$ of $E$, we say that $\xi_0$ has the property (E1) if it satisfies the following conditions:

(i) $||\xi_0|| = 1$.
(ii) $W(\xi_0 \otimes_{\psi} \xi_0) = \xi_0 \otimes_{\phi} \xi_0$.
(iii) For every $\xi \in E$, there exists an element $\pi_{\xi_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

\[ <\eta, \pi_{\xi_0}(\xi)\zeta> = <W(\xi_0 \otimes_{\psi} \eta), \xi \otimes_{\phi} \zeta> \]

for every $\eta, \zeta \in E$.

We fix an element $\xi_0$ with the property (E1). Define an operator $\delta = \delta_{\xi_0}$ in $\mathcal{L}_A(E, E \otimes_{\phi} E)$ by $\delta(\eta) = W(\xi_0 \otimes \eta)$. Then we have $||\delta|| \leq 1$ and $\delta^*(\xi \otimes \eta) = \pi_{\xi_0}(\xi)\eta$. Since $W$ satisfies the pentagonal equation, we can show that $(E, \phi, \delta)$ is a Hopf Hilbert $A$-module. We denote by $\xi \bullet \eta$ the product of $\xi$ and $\eta$ associated with $\delta$. Then we have $\pi_{\xi_0}(\xi)\eta = \xi \bullet \eta$. Moreover the map $\pi_{\xi_0}$ of $E$ to $\mathcal{L}_A(E)$ is a representation of the Banach algebra $(E, \bullet)$. We denote by $B(\xi_0)$ the closed linear subspace...
generated by elements of the form $\hat{\pi}_{\xi_0}(\xi)$ with $\xi \in E$. Then $B(\xi_0)$ is a Banach subalgebra of $\mathcal{L}_A(E)$. We denote by $C^*(B(\xi_0))$ the $C^*$-subalgebra of $\mathcal{L}_A(E)$ generated by $B(\xi_0)$.

For an element $\eta_0$ of $E$, we say that $\eta_0$ has the property (E2) if it satisfies the following conditions:

(i) $||\eta_0|| = 1$.

(ii) $W(\eta_0 \otimes \psi \eta_0) = \eta_0 \otimes \phi \eta_0$.

(iii) For every $\xi \in E$, there exists an element $\hat{\pi}_{\eta_0}(\xi)$ of $\mathcal{L}_A(E)$ such that

$$<\eta, \hat{\pi}_{\eta_0}(\xi) \zeta> = <W'(\eta_0 \otimes \phi \eta), \xi \otimes \psi \zeta>$$

for every $\eta, \zeta \in E$.

We fix an element $\eta_0$ with the property (E2). Define an operator $\hat{\delta} = \hat{\delta}_{\eta_0}$ in $\mathcal{L}_A(E, E \otimes \psi E)$ by $\hat{\delta}(\eta) = W^*(\eta_0 \otimes \eta)$. Since $W$ satisfies the pentagonal equation, we can show that $(E, \psi, \hat{\delta})$ is a Hopf Hilbert $A$-module. We denote by $\xi \circ \eta$ the product of $\xi$ and $\eta$ associated with $\hat{\delta}$. Then we have $\hat{\pi}_{\eta_0}(\xi) \eta = \xi \circ \eta$. Moreover the map $\hat{\pi}_{\eta_0}$ of $E$ to $\mathcal{L}_A(E)$ is a representation of the Banach algebra $(E, \circ)$. We denote by $\hat{B}(\eta_0)$ the closed linear subspace generated by elements of the form $\hat{\pi}_{\eta_0}(\xi)$ with $\xi \in E$. Then $\hat{B}(\eta_0)$ is a Banach subalgebra of $\mathcal{L}_A(E)$. We denote by $C^*(\hat{B}(\eta_0))$ the $C^*$-subalgebra of $\mathcal{L}_A(E)$ generated by $\hat{B}(\eta_0)$.

In the following examples, we consider a finite groupoid, an $r$-discrete groupoid and a compact groupoid. Let $G$ be a second countable locally compact Hausdorff groupoid. We keep the notations in Example 3.6 except for $C_0(G^{(0)})$. Here we denote by $A$ the $C^*$-algebra $C_0(G^{(0)})$. Let $W \in \mathcal{L}_A(E \otimes \psi E, E \otimes \phi E)$ be the multiplicative unitary operator constructed in Example 3.6. Then we have $(W \xi)(x, y) = \xi(y, xy)$ for $\xi \in C_c(G^2(ss))$ and $(x, y) \in G^{(2)}$. Note that we have $(W^* \xi)(x, y) = \xi(yx^{-1}, x)$ for $\xi \in C_c(G^{(2)})$ and $(x, y) \in G^2(ss)$. We denote by $C_r^*(G)$ the reduced groupoid
C*-algebras. (As for the definition of the reduced groupoid C*-algebra, see [19], [22].)

Example 4.2. Let G be a finite groupoid and let \{\lambda_u\} be a right Haar system such that \lambda_u is a counting measure on G_u. Then we have A = C(G^{(0)}) and E = C(G).

The A-valued inner product of E is given by \langle \xi, \eta \rangle (u) = \sum_{x \in G_u} \overline{\xi(x)} \eta(x). We have E \otimes_\phi E = C(G^2(ss)) and the A-valued inner product of E \otimes_\phi E is given by

\langle \xi, \eta \rangle (u) = \sum_{s(x) = r(y)} \overline{\xi(x,y)} \eta(x, y).

We set M = \max\{|G_u|; u \in G^{(0)}\}, where |G_u| is the number of elements of G_u.

We define an element \xi_0 of E by \xi_0(x) = M^{-1/2} for all x \in G. Then \xi_0 has the properties (E1) and (E2). We have \pi_{\xi_0}(\xi) \zeta = \xi * \zeta, where \xi * \zeta is the convolution product defined by

(\xi * \zeta)(x) = \sum_{y \in G_{s(x)}} \xi(xy^{-1}) \zeta(y).

Therefore we have \tilde{B}(\xi_0) = C^*_c(G). Since we have \hat{\pi}_{\xi_0}(\xi) = M^{1/2} \theta_{\xi,\xi_0}, we have C^*(\tilde{B}(\xi_0)) = \mathcal{K}_A(E). We define an element \eta_0 of E by \eta_0 = \chi_{G^{(0)}}, where \chi_{G^{(0)}} is the characteristic function of G^{(0)}. Then \eta_0 has the properties (E1) and (E2). Since we have \pi_{\eta_0}(\xi) = \theta_{\xi,\eta_0}, we have C^*(\tilde{B}(\eta_0)) = \mathcal{K}_A(E). We have \hat{\pi}_{\eta_0}(\xi) = m(\xi), where m(\xi) is the multiplication operator on E defined by (m(\xi)\zeta)(x) = \xi(x)\zeta(x).

Therefore we have \tilde{B}(\eta_0) = C(G).

Example 4.3. Let G be an r-discrete groupoid [24, I.2.6]. Note that G^{(0)} is open and closed in G and that G_u is discrete for every u \in G^{(0)}. Let \{\lambda_u\} be a right Haar system such that \lambda_u is the counting measure on G_u. Since we have ||\xi||_\infty \leq ||\xi||_E for \xi \in C_c(G), E is a subspace of C_0(G). Fix an element f of A such that ||f||_\infty = 1. We
define an element \( \eta_0 \) of \( E \) by \( \eta_0 = f\chi_{G^{(0)}} \). Then \( \eta_0 \) has the properties (E1) and (E2). We have \( \pi_{\eta_0}(\xi) = \theta_{\xi,\eta_0} \). If the support of \( f \) is \( G^{(0)} \), then we have \( C^*(\mathcal{B}(\eta_0)) = \mathcal{K}_A(E) \).

We have \( \hat{\pi}_{\eta}(\xi) = m(\phi(\bar{f})\xi) \), where \( m(\eta) \) is the multiplication operator on \( E \). If \( f \) is real-valued, then we have \( \hat{\pi}_{\eta}(\xi)^* = \hat{\pi}_{\eta}(\bar{\xi}) \). Therefore, if \( f \) is real-valued and the support of \( f \) is \( G^{(0)} \), then we have \( \hat{\mathcal{B}}(\eta_0) = C_0(G) \).

**Example 4.4.** Let \( G \) be a compact groupoid and let \( \{\lambda_u\} \) be a right Haar system such that \( \lambda_u(G) = 1 \) for all \( u \in G^{(0)} \). We define an element \( \xi_0 \) of \( E \) by \( \xi_0(x) = 1 \) for all \( x \in G \). Then \( \xi_0 \) has the properties (E1) and (E2). Note that \( C(G) \) is a dense subspace of \( E \). For \( \xi, \zeta \in C(G) \), we have \( \pi_{\xi_0}(\xi)\zeta = \xi \ast \zeta \), where \( \xi \ast \zeta \) is the convolution product defined by

\[
(\xi \ast \zeta)(x) = \int \xi(xy^{-1})\zeta(y) \, d\lambda_s(x)(y).
\]

Therefore we have \( B(\xi_0) = C^*_r(G) \). Since we have \( \hat{\pi}_{\xi_0}(\xi) = \theta_{\xi,\xi_0} \), we have \( C^*(\hat{\mathcal{B}}(\xi_0)) = \mathcal{K}_A(E) \).

5. **Operators associated with inclusions of \( C^* \)-algebras**

Let \( A_1 \) be a \( C^* \)-algebra and let \( A_0 \) be a \( C^* \)-subalgebra of \( A_1 \). In this section, we do not assume that \( A_1 \) and \( A_0 \) are unital. Let \( E_1 \) be a Hilbert \( A_0 \)-module and let \( \phi_1 \) be a \( * \)-homomorphism of \( A_1 \) to \( \mathcal{L}_{A_0}(E_1) \). We denote by \( \phi_0 \) the restriction of \( \phi_1 \) to \( A_0 \). Define \( E_2 = E_1 \otimes_{\phi_0} E_1 \) and define a \( * \)-homomorphism \( \phi_2 \) of \( A_1 \) to \( \mathcal{L}_{A_0}(E_2) \) by \( \phi_2 = \phi_1 \otimes \iota \). In general, we define \( E_n = E_{n-1} \otimes_{\phi_0} E_1 \). We denote by \( A \) the \( C^* \)-algebra \( \mathcal{L}_{A_0}(E_1, \phi_1) \) and by \( E \) the normed space \( \mathcal{L}_{A_0}((E_1, \phi_1), (E_2, \phi_2)) \). We define on \( E \) a structure of a right \( A \)-module by \( (xa)(\xi) = x(a\xi) \) for \( x \in E, a \in A \) and \( \xi \in E_1 \) and define on \( E \) an \( A \)-valued inner-product by \( < x, y > = x^*y \) for \( x, y \in E \). Then \( E \) becomes a Hilbert \( A \)-module. We define \( * \)-homomorphisms \( \phi \) and \( \psi \) of \( A \) to \( \mathcal{L}_A(E) \) by \( (\phi(a)x)(\xi) = (a \otimes I)x(\xi) \) and \( (\psi(a)x)(\xi) = (I \otimes a)x(\xi) \) respectively for \( a \in A, x \in E \) and \( \xi \in E_1 \). We denote by \( i \) the inclusion map of \( A \) into \( \mathcal{L}_{A_0}(E_1) \).
Proposition 5.1. There exists an $A_0$-linear bounded map $U$ of $E \otimes_i E_1$ to $E_2$ such that $U(x \otimes \xi) = x(\xi)$ for $x \in E$ and $\xi \in E_1$. Moreover the following equalities hold:

$$<U \alpha, U \beta> = <\alpha, \beta>$$ for $\alpha, \beta \in E \otimes_i E_1$,

$$U(\phi(a) \otimes I) = (a \otimes I)U$$ for $a \in A$,

$$U(\psi(a) \otimes I) = (I \otimes a)U$$ for $a \in A$.

The proof is straightforward and we omit it. Note that $U$ may not be adjointable.

We can define the following $A_0$-linear bounded operators;

$$I \otimes_{\phi \otimes \iota} U : E \otimes \phi E \otimes_i E_1 \rightarrow E \otimes_{i \otimes \iota} E_2,$$

$$U \otimes_{\phi_0} I : E \otimes_{i \otimes \iota} E_2 \rightarrow E_3,$$

$$I \otimes_{\psi \otimes \iota} U : E \otimes \psi E \otimes_i E_1 \rightarrow E \otimes_{i \otimes \iota} E_2,$$

$$I \otimes_{i \otimes \phi_0} U : E_1 \otimes_{i \otimes \phi_0} (E \otimes_i E_1) \rightarrow E_3.$$

There exists an isomorphism $S$ of $E \otimes_{i \otimes \iota} E_2$ onto $E_1 \otimes_{\iota \otimes \phi_0} (E \otimes_i E_1)$ as Hilbert $A_0$-modules such that $S(x \otimes (\xi \otimes \eta)) = \xi \otimes (x \otimes \eta)$ for $x \in E$ and $\xi, \eta \in E_1$. Define an $A_0$-linear bounded operator $V$ of $E \otimes \phi E \otimes_i E_1$ of $E_3$ by

$$V = (U \otimes_{\phi_0} I)(I \otimes_{\phi \otimes \iota} U),$$

and define an $A_0$-linear bounded operator $\bar{V}$ of $E \otimes \psi E \otimes_i E_1$ of $E_3$ by

$$\bar{V} = (I \otimes_{i \otimes \phi_0} U)S(I \otimes_{\psi \otimes \iota} U).$$

We summarize the properties of $V$ and $\bar{V}$ in the following proposition. The proof is easy and we omit it.
Proposition 5.2. The operators $V$ and $\tilde{V}$ satisfy the following equalities;

$$< V \alpha, V \beta > = < \alpha, \beta > \quad \text{for } \alpha, \beta \in E \otimes_{\phi} E \otimes_{i} E_{1},$$

$$< \tilde{V} \alpha, \tilde{V} \beta > = < \alpha, \beta > \quad \text{for } \alpha, \beta \in E \otimes_{\phi} E \otimes_{i} E_{1},$$

$$V(x \otimes y \otimes \xi) = (x \otimes_{\phi_{0}} I_{E_{1}})y(\xi) \quad \text{for } x, y \in E \text{ and } \xi \in E_{1},$$

$$\tilde{V}(x \otimes y \otimes \xi) = (I_{E_{1}} \otimes_{\phi_{0}} x)y(\xi) \quad \text{for } x, y \in E \text{ and } \xi \in E_{1}.$$  

In the rest of this section, we will prove the following theorem.

Theorem 5.3. Let $U$, $V$ and $\tilde{V}$ be as above. Suppose that $U$ is unitary and suppose that there exists an element $W$ of $\mathcal{L}_{A}(E \otimes_{\psi} E, E \otimes_{\phi} E)$ such that $V^{*}\tilde{V} = W \otimes I_{E_{1}}$. Then $W$ is a multiplicative unitary operator.

Since $U$ is unitary, $V$ and $\tilde{V}$ are also unitary operators. By straightforward calculation, we have, for every $a \in A$,

$$V(\phi(a) \otimes I_{E} \otimes I_{E_{1}}) = (a \otimes I_{E_{1}} \otimes I_{E_{1}})V,$$

$$V(I_{E} \otimes \psi(a) \otimes I_{E_{1}}) = (I_{E_{1}} \otimes I_{E_{1}} \otimes a)V,$$

$$V(\psi(a) \otimes I_{E} \otimes I_{E_{1}}) = (I_{E_{1}} \otimes a \otimes I_{E_{1}})V,$$

$$\tilde{V}(I_{E} \otimes \phi(a) \otimes I_{E_{1}}) = (a \otimes I_{E_{1}} \otimes I_{E_{1}})\tilde{V},$$

$$\tilde{V}(\psi(a) \otimes I_{E} \otimes I_{E_{1}}) = (I_{E_{1}} \otimes I_{E_{1}} \otimes a)\tilde{V},$$

$$\tilde{V}(\phi(a) \otimes I_{E} \otimes I_{E_{1}}) = (I_{E_{1}} \otimes a \otimes I_{E_{1}})\tilde{V}.$$

Therefore $W$ satisfies the equations (3.1), (3.2) and (3.3). For $n \geq 2$, we set

$$E^{\otimes_{\phi} n} = E \otimes_{\phi} \cdots \otimes_{\phi} E \quad (n \text{ times})$$
and we define $E^\otimes^n$ similarly. It follows from Proposition 5.1 that we have $U(\phi \otimes \iota)(a) = (i \otimes \iota)(a)U$ for $a \in A$. Therefore we can define the following operators;

$$I_E \otimes I_E \otimes U \in \mathcal{L}_{A_0}(E^\otimes^3 \otimes \iota, E_1, E^\otimes^2 \otimes \iota \otimes E_2),$$

$$I_E \otimes U \otimes I_{E_1} \in \mathcal{L}_{A_0}(E^\otimes^2 \otimes \iota \otimes E_2, E \otimes \iota \otimes E_3),$$

$$U \otimes I_{E_1} \otimes I_{E_1} \in \mathcal{L}_{A_0}(E \otimes \iota \otimes E_3, E_4).$$

We define an element $U_3$ in $\mathcal{L}_{A_0}(E^\otimes^3 \otimes \iota, E_1, E_4)$ by

$$U_3 = (U \otimes I_{E_1} \otimes I_{E_1})(I_E \otimes U \otimes I_{E_1})(I_E \otimes I_E \otimes U).$$

Since $U$ is unitary by the assumption, $U_3$ is also a unitary operator. To prove Theorem 5.3, it is enough to prove the following proposition.

**Proposition 5.4.** Set

$$W_1 = (W \otimes \iota)(I \otimes \phi \otimes W)(W \otimes \iota),$$

$$W_2 = (I \otimes \iota \otimes \phi W)\Sigma_{12}(I \otimes \psi \otimes \iota W).$$

Then the following equation holds;

$$U_3(W_1 \otimes I_{E_1})(x \otimes y \otimes z \otimes \xi)$$

$$= U_3(W_2 \otimes I_{E_1})(x \otimes y \otimes z \otimes \xi)$$

$$= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z \xi.$$

for $x, y, z \in E$ and $\xi \in E_1$.

In the rest of this section, we will prove Proposition 5.4. Let

$$S_\psi : E \otimes \phi E \otimes \iota \otimes E_2 \rightarrow E_1 \otimes \phi \otimes \psi (E \otimes \phi E \otimes \iota E_1)$$
be an isomorphism defined by $S_\psi(x \otimes \xi \otimes \eta) = \xi \otimes (x \otimes \eta)$ for $x \in E \otimes_\psi E$ and \( \xi, \eta \in E_1 \), and let

$$S_\phi : E \otimes_\phi E \otimes_\iota E_2 \to E_1 \otimes_\iota \otimes_\phi_0 (E \otimes_\phi E \otimes_\iota E_1)$$

be an isomorphism defined by $S_\phi(x \otimes \xi \otimes \eta) = \xi \otimes (x \otimes \eta)$ for $x \in E \otimes_\phi E$ and $\xi, \eta \in E_1$. Set $U^{(13)} = (I \otimes_{\iota \otimes \phi_0} U)S$.

**Lemma 5.5.** We have the following equalities for $x, y, z \in E$ and $\xi \in E_1$;

(5.12)
$$U_3((W \otimes_\phi I) \otimes_\iota I_{E_1}) = (\tilde{V} \otimes_{\phi_0} I_{E_1})(I_E \otimes_\phi E \otimes_\phi_0 U),$$

(5.13)
$$((I \otimes_{\phi \otimes \iota} W) \otimes_\iota I_{E_1})(W \otimes_\phi I) \otimes_\iota I_{E_1}) = (I \otimes_{\phi \otimes \iota} V^*)(I_E \otimes_{\phi_0} U^{(13)})S_\phi'(I_{E_1} \otimes_{\phi_0 \otimes \iota} V')(I_{E_1} \otimes_{\phi_0 \otimes \iota} V')(I_{E_1} \otimes x)(I_{E_1} \otimes y)z \xi,$$

(5.14)
$$(I \otimes_{\phi \otimes \iota} V^*)(I_E \otimes_{\phi \otimes \iota} U^{(13)})S_\phi'(I_{E_1} \otimes_{\phi_0 \otimes \iota} V') = (I_E \otimes_{\phi \otimes \iota} U^*)(\tilde{V}' \otimes_{\phi_0} I_{E_1}).$$

**Lemma 5.6.** We have the following equalities for $x \in E$ and $\xi_i \in E_1$ ($i = 1, 2, 3$);

(5.15)
$$W_2 \otimes I_{E_1} = (I_E \otimes_{\iota \otimes \phi \otimes \iota} V^* \tilde{V})(\Sigma_{12} \otimes_\iota I_{E_1})(I_E \otimes_{\phi \otimes \iota \otimes \phi \otimes \iota} V^* \tilde{V}),$$

(5.16)
$$U_3(I_E \otimes_{\iota \otimes \phi \otimes \iota} V^*) = U \otimes I_{E_1} \otimes I_{E_1},$$

(5.17)
$$(I_E \otimes_{\iota \otimes \phi \otimes \iota} \tilde{V})(\Sigma_{12} \otimes I_{E_1})(I_E \otimes_{\iota \otimes \phi \otimes \iota} V^*)(x \otimes (\xi_1 \otimes \xi_2 \otimes \xi_3))$$
$$= U^*(\xi_1 \otimes \xi_2) \otimes x \xi_3,$$
Proof of Proposition 5.4. Let $x$, $y$, $z$ be elements of $E$ and let $\xi$ be an element of $E_1$. It follows from Lemma 5.5 that we have

$$U_3(W_1 \otimes I_{E_1})(x \otimes y \otimes z \otimes \xi)$$

$$= U_3((W \otimes I) \otimes I_{E_1})((I \otimes \phi \otimes W) \otimes I_{E_1})((W \otimes I) \otimes I_{E_1})(x \otimes y \otimes z \otimes \xi)$$

$$= U_3((W \otimes I) \otimes I_{E_1})(I_{E \otimes E} \otimes I_0)(\tilde{V}^* \otimes \phi_0 I_{E_1})(I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi$$

$$= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi.$$  

It follows from (5.16) and (5.17) that we have

$$U_3(I_E \otimes I \otimes \phi \otimes I)(\Sigma_{12} \otimes I_{E_1})(I_E \otimes I \otimes \phi \otimes I)(x \otimes \xi_1 \otimes \xi_2 \otimes \xi_3)$$

$$= (U \otimes I_{E_1} \otimes I_{E_1})(U^* (\xi_1 \otimes \xi_2) \otimes x\xi_3)$$

$$= (I_{E_1} \otimes I_{E_1} \otimes x)(\xi_1 \otimes \xi_2 \otimes \xi_3).$$

for $\xi_i \in E_1$ $(i = 1, 2, 3)$. Then by using (5.15) we have

$$U_3(W_2 \otimes I_{E_1})(x \otimes y \otimes z \otimes \xi)$$

$$= U_3(I_E \otimes I \otimes \phi \otimes I)(\Sigma_{12} \otimes I_{E_1})(I_E \otimes I \otimes \phi \otimes I)(x \otimes y \otimes z \otimes \xi)$$

$$= U_3(I_E \otimes I \otimes \phi \otimes I)(\Sigma_{12} \otimes I_{E_1})(I_E \otimes I \otimes \phi \otimes I)(x \otimes \{(I_{E_1} \otimes y)z\xi\})$$

$$= (I_{E_1} \otimes I_{E_1} \otimes x)(I_{E_1} \otimes y)z\xi.$$  

\[\square\]

6. Inclusions of index finite-type

In this section, we study a multiplicative unitary operator associated with an inclusion of $C^*$-algebras when the inclusion is of index-finite type in the sense of Watatani [34]. Let $A_1$ be a $C^*$-algebra with the identity 1, let $A_0$ be a $C^*$-subalgebra of $A_1$ which contains 1 and let $P_1 : A_1 \rightarrow A_0$ be a faithful positive conditional
expectation. We assume that $P_1$ is of index-finite type, that is, there exists a family $u_i \in A_1$ ($i = 1, \cdots, n$) such that
\[
\sum_{i=1}^{n} u_i P_1(u_i^* a) = \sum_{i=1}^{n} P_1(au_i) u_i^* = a
\]
for every $a \in A_1$ [34, 1.2.2, 2.1.6]. Then the index of $P_1$ is given by \( \text{Index } P_1 = \sum_{i} u_i \) which is an element of the center of $A_1$. We denote by $E_1$ a right $A_0$-module $A_1$ whose right $A_0$-action is the product in $A_1$. Define an $A_0$-valued inner product $E_1$ by $<a, b> = P_1(a^* b)$ for $a, b \in E_1$. It follows from [34, 2.1.5] that there exists a positive number $\lambda$ such that
\[
\lambda ||a||_{A_1} \leq ||a||_{E_1} \leq ||a||_{A_1}
\]
for every $a \in E_1 = A_1$, where $|| \cdot ||_{A_1}$ and $|| \cdot ||_{E_1}$ denote the norms of $A_1$ and $E_1$ respectively. Therefore $E_1$ is complete and is a Hilbert $A_0$-module. Define a unital injective $\ast$-homomorphism $\psi_1 : A_1 \to \mathcal{L}_{A_0}(E_1)$ by $\psi_1(a) b = a b$ for $a \in A_1$ and $b \in E_1$, where $a b$ is the product in $A_1$. Then we can construct $A, E, \phi$ and $\psi$ as in Section 5. Moreover we can construct the operators $U, V$ and $\tilde{V}$.

We denote by $A_2$ the $C^\ast$-algebra $\mathcal{K}_{A_0}(E_1)$ (cf. [34, 2.1.2, 2.1.3]). Note that we have $\mathcal{K}_{A_0}(E_1) = \mathcal{L}_{A_0}(E_1)$. In fact, we have $I = \sum_{i=1}^{n} \theta_{u_i, u_i}$ in $\mathcal{L}_{A_0}(E_1)$. We identify $\phi(A_1)$ with $A_1$ and we have inclusions $A_0 \subset A_1 \subset A_2$, which is the basic construction ([34, 2.2.10], see also [11, Chapter 2]). Let $P_2 : A_2 \to A_1$ be the dual conditional expectation of $P_1$, that is, $P_2(\theta_{a,b}) = (\text{Index } P_1)^{-1} a b^\ast$ for $a, b \in A_1$ [34, 2.3.3]. Note that $P_2$ and $P_1 \circ P_2$ are of index-finite type [34, 1.7.1, 2.3.4]. We denote by $F_2$ a right $A_0$-module $A_2$ whose right $A_0$-action is the product in $A_2$. Define an $A_0$-valued inner product of $F_2$ by $<\xi, \eta> = P_1 \circ P_2(\xi^\ast \eta)$ for $\xi, \eta \in F_2 = A_2$. Then $F_2$ is a Hilbert $A_0$-module. Define a unital injective $\ast$-homomorphism $\tilde{\psi}_2 : A_1 \to \mathcal{L}_{A_0}(F_2)$ by $\tilde{\psi}_2(a) \xi = a \xi$ for $a \in A_1$ and $\xi \in F_2$, where $a \xi$ is the product in $A_2$. Define a
linear map $\Phi : E_2 \rightarrow F_2$ by

$$\Phi(a \otimes b) = \theta_{a,b^*} \phi_1 ((\text{Index } P_1)^{1/2})$$

for $a, b \in E_1$. Then $\Phi$ is an isomorphism between the Hilbert $A_0$-modules. Moreover, we have $\Phi(\phi_2(a_1)\xi) = \overline{\phi}_2(a_1)\Phi(\xi)$ for $a_1 \in A_1$ and $\xi \in E_2$.

We denote by $A_0' \cap A_2$ the $C^*$-algebra $\{a \in A_2; \text{ab = ba for every } b \in A_0\}$ and denote by $\overline{\text{lin}}\ A_1(A_0' \cap A_2)$ the closed linear subspace of $A_2$ generated by elements $ab$ with $a \in A_1$ and $b \in A_0' \cap A_2$. For $a \in A_1$, we denote by $C(a)$ the norm closure of the convex hull of the set consisting of elements $uau^*$ with unitary elements $u$ of $A_0$. We consider the following two conditions:

(P1) \hspace{1cm} $A_2 = \overline{\text{lin}}\ A_1(A_0' \cap A_2)$.

(P2) \hspace{1cm} $A_0' \cap C(a) \neq \emptyset$ \hspace{0.5cm} for every $a \in A_1$.

Remark. It seems that (P1) is equivalent to the condition that the inclusion $A_0 \subset A_1$ is of depth 2. The latter condition is assumed by Enock and Vallin in [10]. But I cannot prove the equivalence yet.

In the following theorem, we show that the conditions (P1) and (P2) imply the assumptions of Theorem 5.3. Thus we have a multiplicative unitary operator when these conditions are satisfied.

**Theorem 6.1.** (1) The operator $U$ is uniatry if and only if the condition (P1) is satisfied.

(2) Suppose that $U$ is unitary and that the condition (P2) is satisfied. Then there exists an element $W$ of $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\overline{V} = W \otimes I_{E_1}$.

**Corollary 6.2.** Suppose that the conditions (P1) and (P2) are satisfied. Then there exists a multiplicative unitary operator $W$ in $\mathcal{L}_A(E \otimes_\psi E, E \otimes_\phi E)$ such that $V^*\overline{V} = \ldots$
Corollary 6.3. Suppose that $A_0$ is finite-dimensional and that the codition (P1) is satisfied. Then there exists a multiplicative unitary operator $W$ in $\mathcal{L}_A(E \otimes \phi E, E \otimes \phi E)$ such that $V^* \tilde{V} = W \otimes I_{E_1}$.

The following proposition is useful to prove Theorem 6.1.

Proposition 6.4. (1) There exists a bijection $q_1$ of $A'_0 \cap A_1$ onto $A$ such that $q_1(a)b = ba$ for $a \in A'_0 \cap A_1$ and $b \in E_1$, where $ba$ is the product of $A_1$.

(2) There exists a bijection $q_2$ of $A'_0 \cap A_2$ onto $E$ such that $q_2(a)b = \Phi^{-1}(ba)$ for $a \in A'_0 \cap A_2$ and $b \in E_1$, where $ba$ is the product of $A_2$.

7. Crossed products by finite groups

Let $A_0$ be a unital $C^*$-algebra, let $G$ be a finite group and let $\alpha$ be an action of $G$ on $A$. We denote by $A_1$ the crossed product $A_0 \rtimes_\alpha G$. Then we have the inclusion $A_0 \subset A_1$ and the canonical conditional expectation $P_1$ of $A_1$ onto $A_0$. Note that $\text{Index } P_1 = |G|$. In this section, we will show that the above inclusion satisfies the condition (P1) and the assumption of Theorem 5.3. Therefore we have a multiplicative unitary operator $W$ associated with the inclusion $A_0 \subset A_0 \rtimes_\alpha G$. We can give a formula for $W$.

References


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