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INVOLUTIVE EQUIVALENCE BIMODULES AND INCLUSIONS OF $C^*$-ALGEBRAS WITH WATATANI INDEX2

KAZUNORI KODAKA AND TAMOTSU TERUYA

ABSTRACT. Let $A$ be a unital simple $C^*$-algebra. We shall introduce involutive $A$-$A$ equivalence bimodules and prove that the all $C^*$-algebras containing $A$ with Watatani index 2 are constructed by an involutive $A$-$A$ equivalence bimodule and $A$.

1. INTRODUCTION

V. Jones introduced index theory for $II_1$ factors. As one of his motivations of his definition of index, there is Goldman’s theorem, which says that if $[M : N] = 2$, there is a crossed product decomposition $M = \times_\alpha \mathbb{Z}/2\mathbb{Z}$.

Y. Watatani extended index theory to $C^*$-algebras. He defined indices of conditional expectations in terms of quasi-basis, which is generalization of the Pimsner-Popa basis. There is an inclusion of unital simple $C^*$-algebras with Watatani index 2, which is no written by the crossed product of a $\mathbb{Z}/2\mathbb{Z}$ action.

Equivalence bimodules for $C^*$-algebras $A$ and $B$ are introduced by M. A. Rieffel, which is a left Hilbert $A$-module as well as a right Hilbert $B$-module with full $C^*$-algebra valued inner products $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ such that $x_A(y, z) = (x, y)_{B}z$ holds.

Let $A$ be a unital simple $C^*$-algebra. We shall introduce involutive $A$-$A$ equivalence bimodules and prove that the all $C^*$-algebras containing $A$ with Watatani index 2 are constructed by an involutive $A$-$A$ equivalence bimodule and $A$.

2. PRELIMINARIES

2.1. Some results for inclusions with index 2. Let $B$ be a unital $C^*$-algebra and $A$ a $C^*$-subalgebra of $B$ with a common unit. Let $E$ be a conditional expectation of $B$ onto $A$ with $1 < \text{Index} E < \infty$. Then by Watatani [10] we have the $C^*$-basic construction $C^*(B, e_A)$ where $e_A$ is a projection induced by $E$. Let $\tilde{E}$ be the dual conditional expectation of $C^*(B, e_A)$ onto $B$ defined by

$$\tilde{E}(ae_A b) = \frac{1}{t}ab \quad \text{for any} \quad a, b \in B,$$

where $t = \text{Index} E$. Let $F$ be a linear map of $(1 - e_A)C^*(B, e_A)(1 - e_A)$ to $A(1 - e_A)$ defined by

$$F(a) = \frac{t}{t-1}(E \circ \tilde{E})(a)(1 - e_A)$$

for any $a \in (1 - e_A)C^*(B, e_A)(1 - e_A)$. By a routine computation we can see that $F$ is a conditional expectation of $(1 - e_A)C^*(B, e_A)(1 - e_A)$ onto $A(1 - e_A)$.

Lemma 2.1.1. With the above notations, let $\{(x_i, x'_i)\}_{i=1}^n$ be a quasi-basis for $E$. Then

$$\{\sqrt{t-1}(1-e_A)x_je_Ax_i(1-e_A), \sqrt{t-1}(1-e_A)x'_ie_Ax'_j(1-e_A)\}_{i,j=1}^n$$

is a quasi-basis for $F$. Furthermore $\text{Index} F = (t-1)^2(1-e_A)$.

Proof. This is immediate by a direct computation.

Date: May 31, 2001.
Corollary 2.1.1. We suppose that \( \text{Index} E = 2 \). Then
\[
(1 - e_A)C^*\langle B, e_A\rangle(1 - e_A) = A(1 - e_A) \cong A.
\]

Proof. By Lemma 2.1.1 there is a conditional expectation \( F \) of \((1 - e_A)C^*\langle B, e_A\rangle(1 - e_A)\) onto \( A(1 - e_A) \) and
\[
\text{Index} F = (\text{Index} E - 1)^2(1 - e_A).
\]
Since \( \text{Index} E = 2 \), \( \text{Index} F = 1 - e_A \). Hence by Watatani [10],
\[
(1 - e_A)C^*\langle B, e_A\rangle(1 - e_A) = A(1 - e_A).
\]
If \( a(1 - e_A) = 0 \), for \( a \in A \), then \( a = 2\tilde{E}(a(1 - e_A)) = 0 \). Therefore the map \( a \rightarrow a(1 - e_A) \) is injective. And hence \( A(1 - e_A) \cong A \). Thus we obtain the conclusion.

Lemma 2.1.2. With the same assumptions as in Lemma 2.1.1, we suppose that \( \text{Index} E = 2 \). Then for any \( b \in B \),
\[
(1 - e_A)b(1 - e_A) = E(b)(1 - e_A).
\]

Proof. By Corollary 2.1.1 there exists \( a \in A \) such that \((1 - e_A)b(1 - e_A) = a(1 - e_A)\). Therefore
\[
\begin{align*}
a &= 2\tilde{E}(a(1 - e_A)) \\
&= 2\tilde{E}((1 - e_A)b(1 - e_A)) \\
&= 2\tilde{E}(b - e_A b - b e_A + E(b)e_A) \\
&= 2(b - \frac{1}{2}b - \frac{1}{2}b + \frac{1}{2}E(b)) = E(b).
\end{align*}
\]
Thus we obtain the conclusion.

Proposition 2.1.1. With the same assumptions as in Lemma 2.1.1, we suppose that \( \text{Index} E = 2 \). Then there is a unitary element \( U \in C^*\langle B, e_A\rangle \) satisfying the followings:

(1) \( U^2 = 1 \),

(2) \( UbU^* = 2E(b) - b \) for \( b \in B \).

Hence if \( \beta = \text{Ad}(U)|_B \), \( \beta \) is an automorphism of \( B \) with \( \beta^2 = \text{id} \) and \( B^\beta = A \).

Proof. By Lemma 2.1.2, for any \( b \in B \)
\[
(1 - e_A)b(1 - e_A) = b - e_A b - b e_A + E(b)e_A
\]
\[
= E(b)(1 - e_A) = E(b) - E(b)e_A.
\]
Therefore
\[
E(b) = b - e_A b - b e_A + 2E(b)e_A.
\]
Let \( U \) be a unitary element defined by \( U = 2e_A - 1 \). Then by the above equation for any \( b \in B \)
\[
UbU^* = (2e_A - 1)b(2e_A - 1)
\]
\[
= 4E(b)e_A - 2e_A b - 2b e_A + b
\]
\[
= 2(b - e_A b - b e_A + 2E(b)e_A) - b
\]
\[
= 2E(b) - b.
\]
Thus we obtain the conclusion.

Remark 2.1.1. By the above proposition, \( E(b) = \frac{1}{2}(b + \beta(b)) \).

Lemma 2.1.3. Let \( B \) be a unital \( C^* \)-algebra and \( A \) a \( C^* \)-subalgebra of \( B \) with a common unit. Let \( E \) be a conditional expectation of \( B \) onto \( A \) with \( \text{Index} E = 2 \). Then we have
\[
C^*\langle B, e_A\rangle \cong B \times_\beta \mathbb{Z}_2.
\]
Proof. We may assume that $B \times_\beta \mathbb{Z}_2$ acts on the Hilbert space $l^2(\mathbb{Z}_2, H)$ faithfully, where $H$ is some Hilbert space on which $B$ acts faithfully. Let $W$ be a unitary element in $B \times_\beta \mathbb{Z}_2$ with $\beta = Ad(W)$, $W^2 = 1$. Let $e = \frac{1}{2}(W + 1)$. Then $e$ is a projection in $B \times_\beta \mathbb{Z}_2$ and $ebe = E(b)e$ for any $b \in B$. In fact,

$$ebe = \frac{1}{4}(W + 1)b(W + 1) = \frac{1}{4}(Wb + b)(W + 1) = (WbW + Wb + Wb + b).$$

On the other hand by Remark 2.1.1,

$$E(b)e = \frac{1}{2}(b + \beta(b))\frac{1}{2}(W + 1) = \frac{1}{4}(bW + b + \beta(b)W + \beta(b)) = \frac{1}{4}(WbW + bW + Wb + b).$$

Hence $ebe = E(b)e$ for $b \in B$. Also $A \ni a \mapsto ae \in B \times_\beta \mathbb{Z}_2$ is injective. In fact, if $ae = 0$, $aW + a = 0$. Let $\hat{\beta}$ be the dual action of $\beta$. Then $0 = \hat{\beta}(aW + a) = -a + a$. Thus $2a = 0$, i.e., $a = 0$. Thus by Watatani[10, Proposition 2.2.11], $C^*(B, e_A) \cong B \times_\beta \mathbb{Z}_2$.

**Remark 2.1.2.** (1) By the proofs of Watatani[10, Propositions 2.2.7 and 2.2.11], we see that $\kappa(b) = b$ for any $b \in B$ where $\kappa$ is the isomorphism of $C^*(B, e_A)$ onto $B \times_\beta \mathbb{Z}_2$ in Lemma 2.1.3.

(2) The above lemma is obtained in Kajiwara and Watatani [5, Theorem 5.13].

By Lemma 2.1.3 and Remark 2.1.2, we regard $\hat{\beta}$ as an automorphism of $C^*(B, e_A)$ with $\hat{\beta}(b) = b$ for any $b \in B, \hat{\beta}^2 = id$ and $\hat{\beta}(e_A) = 1 - e_A$.

**Lemma 2.1.4.** With the same assumptions as in Lemma 2.1.3,

$$C^*(B, e_A)\hat{\beta} = B.$$

**Proof.** By Lemma 2.1.3 for any $x \in C^*(B, e_A)$, we can write $x = b_1 + b_2U$, where $b_1, b_2 \in B$, We suppose that $\hat{\beta}(x) = x$. Then $b_1 - b_2U = b_1 + b_2U$. Thus $b_2 = 0$. Hence $x = b_1 \in B$. Since it is clear that $B \subset C^*(B, e_A)\hat{\beta}$, we obtain the conclusion.

### 2.2. Involutive equivalence bimodules

Let $A$ be a unital $C^*$-algebra and $X(=A^*_A)$ a complete $A$-$A$ equivalence bimodule. $X$ is **involutive** if there exists a conjugate linear map $x \rightarrow x^\sharp$ on $X$, such that

1. $(x^\sharp)^\sharp = x$, \hspace{1cm} $x \in X$,
2. $(a \cdot x \cdot b)^\sharp = b^*x^\sharp a^*$, \hspace{1cm} $x \in X, a, b \in A$,
3. $A(x, y^\sharp) = \langle x^\sharp, y \rangle_A$, \hspace{1cm} $x, y \in X$,

where $A(,)$ and $\langle, \rangle_A$ are the left and right $A$-valued inner products of $X$.

**Lemma 2.2.1.** Let $V$ be a map of $X$ onto its dual bimodule $\tilde{X}$ defined by $V(x) = \tilde{x}^\sharp$. Then $V$ is a bimodule isomorphism preserving the left and right $A$-valued inner products.

**Proof.** By $a \cdot \tilde{x} \cdot b = b^* \cdot \tilde{x} \cdot a^*$, for $a, b \in A$ and $x \in X$,

$$V(a \cdot x \cdot b) = (a \cdot x \cdot b)^\sharp$$

$$= b^* \cdot x^\sharp \cdot a^*$$

$$= a \cdot x^\sharp \cdot b = a \cdot V(x) \cdot b.$$
By $\langle x, y \rangle^\lambda = \langle x^\lambda, y \rangle_A$ and $(x^\lambda)^\lambda = x$, for $x, y \in X$,

$$A\langle V(x), V(y) \rangle^\sim = A\langle \tilde{x}^\lambda, \tilde{y}^\lambda \rangle^\sim = \langle x^\lambda, y \rangle_A = A\langle x, (y^\lambda)^\lambda \rangle = A\langle x, y \rangle.$$  

Similarly, $A\langle V(x), V(y) \rangle^\sim = \langle x, y \rangle_A$. Thus we obtain the conclusion.  

3. Correspondence between involutive equivalence bimodules and inclusions of $C^*$-algebras with Watatani index 2

Let $A$ be a unital $C^*$-algebra and we denote by $(B, E)$ a pair of a unital $C^*$-algebra $B$ including $A$ with a common unit and a conditional expectation $E$ of $B$ onto $A$ with Index $E = 2$. Let $\mathcal{L}$ be the set of all such pairs $(B, E)$. We define an equivalence relation $\sim$ in $\mathcal{L}$ as follows: For $(B, E), (B_1, E_1) \in \mathcal{L}$, $(B, E) \sim (B_1, E_1)$ if and only if there is an isomorphism $\pi$ of $B$ onto $B_1$ such that $\pi(a) = a$ for any $a \in A$ and $E_1 \circ \pi = E$. We denote by $[B, E]$ the equivalence class of $(B, E)$.

Let $\mathcal{M}$ be the set of all complete involutive $A$-$A$ equivalence bimodules. We define an equivalence relation $\sim$ in $\mathcal{M}$ as follows: For $X, Y \in \mathcal{M}$, $X \sim Y$ if and only if there is a bimodule isomorphism $\rho$ of $X$ onto $Y$ preserving the left and right $A$-valued inner products with $\rho(x^\lambda) = \rho(x)^\lambda$. We denote by $[X]$ the equivalence class of $X$. Then we have the next theorem.

Theorem 3.0.1. There is a 1-1 correspondence between $\mathcal{L}/\sim$ and $\mathcal{M}/\sim$.

4. Involutional equivalence bimodules for simple $C^*$-algebras

4.1. Construction of involutional equivalence bimodules by $2\mathbb{Z}$-inner $C^*$-dynamical systems. Let $A$ be a simple unital $C^*$-algebra and $\alpha$ an automorphism of $A$ and we suppose that $\alpha^2 = Ad(z)$ where $z$ is a unitary element in $A$ with $\alpha(z) = z$. Let $X_\alpha$ be the vector space $A$ with the obvious left action of $A$ on $X_\alpha$ and the obvious left $A$-valued inner product, but we define the right action of $A$ on $X_\alpha$ by $x \cdot a = x\alpha^{-1}(a)$ for any $x \in X_\alpha$ and $a \in A$, and the right $A$-valued inner product by $\langle x, y \rangle_A = \alpha(x^*y)$ for any $x, y \in X_\alpha$.

Proposition 4.1.1. With the above notations, Let $B_{X_\alpha}$ be a $C^*$-algebra defined by $X_\alpha$ and $L$ the linking algebra for $X_\alpha$ as defined in Section 3. Then the following conditions are equivalent:

1. $B_{X_\alpha}$ is simple,
2. $A' \cap B_{X_\alpha} = \mathbb{C} \cdot 1$,
3. $B_{X_\alpha} \cap L = \mathbb{C} \cdot 1$,
4. $\alpha$ is an outer automorphism of $A$.

Let $B$ be a unital $C^*$-algebra and $A$ a $C^*$-subalgebra of $B$ with a common unit. Let $E$ be a conditional expectation of $B$ onto $A$ with Index $E = 2$. For any $n \in \mathbb{N}$ let $M_n$ be the $n \times n$-matrix algebra over $\mathbb{C}$ and $M_n(A)$ the $n \times n$-matrix algebra over $A$. Let $\{x_i, x_i^*\}_{i=1}^n$ be a quasi-basis for $E$. We define $q = [q_{ij}] \in M_n(A)$ by $q_{ij} = E(x_i^*x_j)$. Then by Watatani [10], $q$ is a projection and $C^*(B, e_A) \simeq qM_n(A)q$. Let $\pi$ be an isomorphism of $C^*(B, e_A)$ onto $qM_n(A)q$ defined by

$$\pi(ae_Ab) = [E(x_i^*a)E(bx_j)] \in M_n(A)$$

for any $a, b \in B$. Especially for any $b \in B$,

$$\pi(b) = [E(x_i^*bx_j)]$$

since $\sum_{i=1}^n x_i e_A x_i^* = 1$. 

Proposition 4.1.2. With the above notations, the following conditions are equivalent:

(1) $e_A$ and $1 - e_A$ are equivalent in $C^*\langle B, e_A \rangle$,

(2) there exists a unitary element $u \in B$ such that $\{(1, 1), (u, u^*)\}$ is a quasi basis for $E$,

(3) there exists a $2\mathbb{Z}$-inner $C^*$-dynamical system $(A, Z, \alpha)$ such that $X_{\alpha} \sim X_B$.

Let $\theta$ be an irrational number in $(0, 1)$ and $A_\theta$ the corresponding irrational rotation $C^*$-algebra. Let $B$ be a unital $C^*$-algebra including $A_\theta$ as a $C^*$-subalgebra of $B$ with a common unit. We suppose that there is a conditional expectation $E$ of $B$ onto $A_\theta$ with $\text{Index} E = 2$ and that $A_\theta' \cap B = \mathbb{C} \cdot 1$

Proposition 4.1.3. With the above notation there is a $2\mathbb{Z}$-inner $C^*$-dynamical system $(A_\theta, Z, \alpha)$ such that $(B, E) \sim (A \times_{\alpha/2\mathbb{Z}} \mathbb{Z}, F)$, where $F$ is the canonical conditional expectation of $A \times_{\alpha/2\mathbb{Z}} \mathbb{Z}$ onto $A$.

REFERENCES