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A zeta function of a smooth manifold and elliptic cohomology

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Abstract
We will propose a new definition of a zeta function of a smooth manifold, using Grothendieck’s idea of crystalline cohomology which is used to express Hasse-Weil’s congruent zeta function of a smooth projective variety defined over a finite field as an alternating product of characteristic polynomials of Frobenius. In order to compute our zeta function, we will use the theory of elliptic cohomology.

1 Motivation
The purpose of this note is to explain a main idea of [9]. Details are found in [9].

Our definition of a zeta function depends on the Grothendieck’s idea of Crystalline Cohomology, hence we first recall the definition of crystalline cohomology.

Arithmetic case
In order to avoid an unnecessary complexity, we only consider the simplest case. Let $X$ be a smooth projective variety defined over a finite prime field $\mathbb{F}_p$ of characteristic $p$. We assume an existence of a smooth model $\mathcal{X}$ of $X$ defined over $\mathbb{Z}_p$. The $i$-th crystalline cohomology of rational coefficient $H^{i}_{\text{crys}}(X/\mathbb{Z}_p) \otimes \mathbb{Q}_p$ of $X$ is defined to be

$$H^{i}_{\text{crys}}(X/\mathbb{Z}_p) \otimes \mathbb{Q}_p \overset{\text{def}}{=} H^{i}_{DR}(X/\mathbb{Z}_p) \otimes \mathbb{Q}_p. \quad (1)$$

It is well-known that $H^{i}_{\text{crys}}(X/\mathbb{Z}_p) \otimes \mathbb{Q}_p$ is independent a choice of a model $\mathcal{X}$. Also an endomorphism $\phi$ called Frobenius acts on $H^{i}_{\text{crys}}(X/\mathbb{Z}_p) \otimes \mathbb{Q}_p$. 
$\zeta_{X}(T) = \sum_{n=1}^{\infty} \frac{\left|X(F_{p}^{n})\right|}{n} T^{n}$  \hspace{1cm} (2)

be the Hasse-Weil’s congruent zeta function of $X$. $\zeta_{X}(T)$ can be expressed in terms of $\phi$. We prepare some notations.

**Notations 1.1.**

- $\text{Tr}_{\phi}^{+}(T) = \sum_{i \equiv 0(2)} \sum_{n=1}^{\infty} \text{Tr}[\phi^{n}H_{\text{cryst}}^{i}(X/Z_{p}) \otimes Q_{p}] T^{n}$

- $\text{Tr}_{\phi}^{-}(T) = \sum_{i \equiv 1(2)} \sum_{n=1}^{\infty} \text{Tr}[\phi^{n}H_{\text{cryst}}^{i}(X/Z_{p}) \otimes Q_{p}] T^{n}$

Now the following formula is due to Grothendieck and Berthelot.

**Fact 1.1.** $-T \frac{d}{dT} \log \zeta_{X}(T) = \text{Tr}_{\phi}^{+}(T) - \text{Tr}_{\phi}^{-}(T)$.

In particular, $\zeta_{X}(T) = \exp[-\int \{\text{Tr}_{\phi}^{+}(T) - \text{Tr}_{\phi}^{-}(T)\} \frac{dT}{T}]$.

**Geometric case**

Now we treat our geometric case. Let $M$ be a compact oriented manifold with $w_{2}(M) = p_{1}(M) = 0$ and let $\mathcal{LM}$ be its free loop space. Also we assume $M$ admits an almost complex structure. (In fact, this assumption is unnecessary.) Comparing to arithmetic case, $M$ corresponds to $X$ and $\mathcal{LM}$ corresponds to $\mathcal{X}$. Note that $\mathcal{LM}$ has a natural $S^{1}$-action by rotation of parameter. One can consider vector bundles $\Sigma_{+}$ and $\Sigma_{-}$ of infinite rank over $\mathcal{LM}$ which is called as a plus loop spinor bundle and minus loop spinor bundle respectively. Between them, there exists a differential operator of the first order (loop Dirac operator),

$$\Gamma(\mathcal{LM}, \Sigma_{+}) \xrightarrow{D} \Gamma(\mathcal{LM}, \Sigma_{-}).$$

\(\Sigma_{+}, \Sigma_{-}, \text{ and } D\) have the following properties ([10]);

- $\Sigma_{+}$ and $\Sigma_{-}$ admit $S^{1}$-action which is equivalent to natural one of $\mathcal{LM}$.

- $D$ is $S^{1}$-equivalent.

Therefore both $\text{Ker}D$ and $\text{Coker}D$ admit $S^{1}$-action and these correspond to the Frobenius action on $H_{\text{cryst}}^{i}(X/Z_{p}) \otimes Q_{p}$. Let

- $\text{Ker}D = \oplus_{n}H^{+}(n)$,

- $\text{Coker}D = \oplus_{n}H^{-}(n)$,

be a weight decomposition by the $S^{1}$-action and we set

$$\chi_{D}(M, q) = \sum_{n} \{\dim H^{+}(n) - \dim H^{-}(n)\} q^{n}.$$  \hspace{1cm} (4)

This corresponds to $-T \frac{d}{dT} \log \zeta_{X}(T)$.

In the following sections, we will discuss a way of calculating this invariant using elliptic cohomology.
2 Elliptic cohomology

In this section, $R$ denotes a commutative $\mathbb{Q}$-algebra. (In fact, in order to develop a theory of elliptic cohomology, it is sufficient $R$ is a commutative ring with a unit such that 6 is invertible.)

A complex oriented cohomology theory vs a formal group

A cohomology theory $H^*$ is said to be complex oriented if the cohomology ring of the classifying space $BU(1)$ of $U(1)$ is isomorphic to a formal power series ring of one variable:

$$H^*(BU(1), R) \cong R[[T]].$$ (5)

Let $\mathcal{L}$ be the universal line bundle over $BU(1)$. By universality of $\mathcal{L}$, we have a map,

$$BU(1) \times BU(1) \xrightarrow{\phi} BU(1)$$ (6)

such that $\phi^*\mathcal{L} = p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$, where $p_1$ (resp. $p_2$) is the first (resp. second) projection. The functoriality of $H^*$ induces a homomorphism

$$H^*(BU(1), R) \xrightarrow{\phi^*} H^*(BU(1), R) \otimes H^*(BU(1), R),$$ (7)

and by (5), this is a ring homomorphism

$$R[[T]] \xrightarrow{\phi^*} R[[X,Y]].$$ (8)

We set $F_H(X,Y) = \phi^*(T)$. One can easily see that $F_H(X,Y)$ satisfies the following identities.

- (commutativity) $F_H(X,Y) = F_H(Y,X)$.
- (existence of unit) $F_H(X,0) = X$.
- (associativity) $F_H(F_H(X,Y),Z) = F_H(X,F_H(Y,Z))$.
- $F_H(X,Y) = X + Y + \text{(higher order)}$.

In general, a formal power series $F(X,Y) \in R[[X,Y]]$ which satisfies the above conditions is said to be a formal group defined over $R$ ([8]). Here are some examples of formal groups.

Example 2.1. 1. (the formal group associated to additive group)

$$F(X,Y) = X + Y.$$ 2. (the formal group associated to multiplicative group)

$$F(X,Y) = X + Y + XY.$$
In this way, we associate a formal group defined over $R$ to a complex oriented cohomology theory whose coefficient ring is $R$. It is a result of Landweber ([5]) that one can also associate an $R$-coefficient complex oriented cohomology theory to a formal group defined over $R$.

A formal group vs a complex genus

We first recall results due to Lazard and Quillen.

Fact 2.1. (Lazard [6]) Let $\mathcal{L}A \overset{\text{def}}{=} \mathbb{Z}[\{z_n\}_{n=1}^{\infty}]$. ($\mathcal{L}A$ is called as Lazard's ring.) Then there exists a formal group law $F^u(X,Y)$ defined over $\mathcal{L}A$ which is universal in the following sense;

Let $F(X,Y)$ be a formal group law defined over a ring $R$. Then there exists the unique ring homomorphism, $\mathcal{L}A \overset{\theta}{\to} R$ such that $F(X,Y) = \theta(F^u(X,Y))$.

Fact 2.2. (Quillen [7]) $\mathcal{L}A$ is isomorphic to the complex cobordism ring $\Omega^U$ and the formal group determined by $F_U(X,Y)$ is isomorphic to Lazard's universal formal group.

A ring homomorphism from $\Omega^U$ to $R$ is said to be a complex genus whose values are in $R$. The above two results imply a there is a one to one correspondence between a formal group defined over $R$ and a complex genus whose values are in $R$.

Now we state our definition of elliptic genus and elliptic cohomology. Let

$$E = \{y^2 = x^3 - ax + b\}, \quad \omega = \frac{dx}{2y}. \quad (9)$$

be a pair of an elliptic curve and its invariant differential defined over $R$. We choose a formal parameter $T$ of $E$ at the origin to be

$$T = -\frac{x}{y}. \quad (10)$$

Let $\hat{O}_{E,0}$ be the formal completion of $O_E$ at the origin. The group law $E \times E \rightarrow E$ of $E$ induces a homomorphism

$$O_{E,0} \overset{\mu}{\to} O_{E,0} \hat{\otimes} O_{E,0}. \quad (11)$$

By the choice of a formal parameter $T$, $O_{E,0}$ is isomorphic to $R[[T]]$. Hence (11) becomes a homomorphism

$$R[[T]] \overset{\mu}{\to} R[[X,Y]], \quad (12)$$

and we define a formal group $F_{E,\omega}(X,Y)$ associated to $(E,\omega)$ to be $F_{E,\omega}(X,Y) \overset{\mu}{\to} (E,\omega)$ is said to be elliptic cohomology (resp. elliptic genus). Moreover if $(E,\omega)$ is defined over a ring of modular forms (of certain level), these are said to be modular. One can obtain the following proposition without difficulties.
Proposition 2.1. Let \( \phi \) be a modular elliptic genus of level \( \Gamma \). Then, for an almost complex compact manifold of dimension \( 2n \), \( \phi(M) \) is a modular form holomorphic at cusps of weight \( n \) and of level \( \Gamma \).

Particular cases of the proposition is considered in [3] and [6].

3 How to compute a zeta function (after Witten and Zagier)

In this section, we follow Witten and Zagier's argument to compute our zeta functions ([10], [11], [6]).

We first prepare some notations. For a manifold \( M \) and an indeterminate \( q \), we set

\[
S_q(TM \otimes \mathbb{C}) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \text{Sym}^k(TM \otimes \mathbb{C})q^k \in K_0(M)[[q]],
\]

where \( K_0(M) \) is the Grothendieck's group of \( M \), and \( \text{Sym}^k \) is the \( k \)-th symmetric product.

Let \( M \) be an almost complex manifold with \( w_2(M) = p_1(M) = 0 \). Witten computed \( \chi_D(M,q) \) by formally using Atiyah-Singer's fixed point formula and he obtained

\[
\chi_D(M, q) = \langle \hat{A}(M) \text{ch}(\otimes_{n=1}^{\infty} S_q^n(TM \otimes \mathbb{C})), [M] \rangle.
\]

The right hand side of (14) can be calculated more explicitly.

Let \( R = \mathbb{Q}[G_4, G_6] \), where \( G_i \) is the Eisenstein series of weight \( i \). We set

\[
Q_{WS}(T) = \exp[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(q)T^{2k}] \in \mathbb{Q}[G_4, G_6][[T]].
\]

Let \( g(T) \) be the formal inverse function of \( \frac{T}{Q(T)} \) and we write

\[
g'(T) = \sum_{i=0}^{\infty} a_n T^n, a_n \in R.
\]

We define a complex genus \( \phi_{WS} \) (which is said to be Weierstrass-Witten genus) to be

\[
\phi_{WS}(\mathbb{P}^n(\mathbb{C})) = a_n.
\]

Note that the rational complex cobordism ring \( \Omega^U \otimes \mathbb{Q} \) is generated by \( \{ \mathbb{P}^n(\mathbb{C}) \}_n \).

For an almost complex compact manifold \( M \) of dimension \( 4k \) such that \( w_2(M) = p_1(M) = 0 \), we have
\[ \chi_D(M, q) = \prod_{n=1}^{\infty} (1 - q^n)^{-4k} \phi_{WS}(M) \]

\[ \emptyset_{WS}(M) \text{ is a modular form of weight } 2k \text{ and of level 1.} \]

For a modular form \( f \) of level 1, let \( a_0(f) \) be the constant term of the Fourier expansion of \( f \). We define a zeta function of \( M \) to be the Mellin transform of \( \phi_{WS}(M) - a_0(\phi_{WS}(M)) \) (Compare Fact 1.1):

\[ \zeta_M(s) = \int_0^\infty [\phi_{WS}(M) - a_0(\phi_{WS}(M))](it)t^{s-1} \frac{dt}{t}. \]  (18)

In general, without the conditions \( w_2 = p_1 = 0 \), we define a zeta function of a smooth manifold by (16). Here are some examples.

**Example 3.1.**

1. \( \zeta_{\mathbb{P}^4(\mathbb{C})}(s) = -\frac{2^7\pi^4}{4!} \zeta(s)\zeta(s-3) \),

2. \( \zeta_{\mathbb{P}^6(\mathbb{C})}(s) = -\frac{2^83\pi^6}{6!} \zeta(s)\zeta(s-5) \).

## 4 Comments and remarks

We will briefly explain a relationship between \( \phi_{WS} \) and a series of linear representations of Monster. Details are found in [3].

It is well-known (cf.[2]) as a *Moonshine conjecture* that there is a mysterious relationship between *Monster* and \( j(q) - 744 \), where \( j(q) \) is the elliptic modular function.

Let consider the Fourier expansion of \( j(q) - 744 \),

\[ j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots. \]  (19)

Note that the first coefficient 1 is the dimension of trivial representation of Monster. It is known the dimension of the smallest non-trivial irreducible representation of Monster is 196883, and this is nothing but \( a_1(j(q) - 744) - a_{-1}(j(q) - 744) \).

(Remember that \( a_i(\cdot) \) denotes the \( i \)-th Fourier coefficient.) So it is natural to conjecture that \( j(q) - 744 \) is the generating function (in some sense) of dimension of irreducible representation of Monster. This conjecture was solved by Borchers using a vertex operator algebra. ([1], [4])

Hirzeburch proposed a problem to construct a series of irreducible representation by a geometric way. ([3], Prize Question) His plan is as follows.

1. Construct a 24 dimensional compact oriented smooth spin manifold \( M \) with \( p_1 = 0 \in H^4(M, \mathbb{Q}) \) and \( \phi_{WS}(M) = E_4^2 - 744\Delta \), where \( E_i \) is the normalized Eisenstein series of weight \( i \) and \( \Delta \) is the normalized cusp form of weight 12.

2. Find such a manifold which admits an action of Monster.
Such a manifold satisfies an identity

\[ q^{-1} \cdot \hat{A}(M, \otimes_{n=1}^{\infty}S_{q^{n}}(TM \otimes \mathbb{C})) = j(q) - 744, \]  

(20)

where \( \hat{A} \) denotes \( \hat{A} \)-genus. This identity implies

- \( \hat{A}(M) = 1 \) and \( \hat{A}(M, TM \otimes \mathbb{C}) = 0 \).
- \( \hat{A}(M, Sym^{2}(TM \otimes \mathbb{C})) = 196884 \).

Since we have a decomposition,

\[ Sym^{2}(TM \otimes \mathbb{C}) = E \oplus 1, \]  

(21)

where 1 is the trivial bundle, the smallest non-trivial irreducible representation of Monster may be realized as the cohomology group of \( E \).

Let \( M(1)_R \) be the graded ring of modular forms of full level which are holomorphic at the cusp whose Fourier coefficients are valued in a commutative ring \( R \). It is easy to see that compact smooth oriented manifolds whose dimension is divisible by 4 and which satisfy conditions \( w_2 = 0 \) and \( p_1 = 0 \in H^4(\mathbb{Q}) \) form a subring of oriented codordism ring. We denote this subring by \( \Omega^0 \). Then \( \phi_{WS} \) becomes a ring homomorphism

\[ \Omega^0 \xrightarrow{\phi_{WS}} M(1)_\mathbb{Z}. \]  

(22)

If this is surjective, we obtain a manifold which satisfies the condition 1. We have obtained the following proposition.

**Proposition 4.1.** After tensoring \( \mathbb{Z}[\frac{1}{6}] \), (22) becomes surjective.

In fact, we have constructed a compact smooth 24 dimensional manifold \( M \) satisfying the conditions and \( \phi_{WS}(M) = 144(E_4^3 - 744\Delta) \). But we do not know whether (22) is surjective or not. A problem to find a manifold which admits an action of Monster seems much more difficult.

**References**


