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Multiple cover formulas for Gromov-Witten invariants and BPS states

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Abstract

In order to understand the relationship between the Gromov-Witten invariants of a Calabi-Yau 3-fold X and the enumerative geometry of X, one needs to know how multiple covers of curves contribute to the invariants.

In these lecture notes, we survey some old and new results about multiple cover formulas. We also define "BPS invariants" in terms of the Gromov-Witten invariants via the formula of Gopakumar and Vafa. These invariants are conjecturally integer valued and we show that the known multiple-cover formulas for the Gromov-Witten invariants indeed lead to integral contributions to the BPS invariants, sometimes in subtle ways. These integrality predictions lead to conjectural congruence properties of Hurwitz numbers. We prove a few of these congruences in the last section.

Ultimately, we hope the understanding of the contribution of curves in X to the BPS invariants of X will lead to an intrinsic geometric definition of the BPS invariants and that the Gopakumar-Vafa formula can be proven as a theorem (rather than a definition).

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1 Multiple cover formulas

Let X be a Calabi-Yau 3-fold (for example, the quintic hypersurface $X^3_{(5)} \subset \mathbf{CP}^4$). We wish to study the Gromov-Witten invariants of X. For $g \in \mathbf{Z}_{\geq 0}$ and $\beta \in H_2(X, \mathbf{Z})$, let $\overline{M}_g(X, \beta)$ be the moduli space of genus g, degree β, stable maps to X and let $[\overline{M}_g(X, \beta)]^{vir}$ be the virtual fundamental class (see [2], [16], or [7]). Since the virtual dimension of $\overline{M}_g(X, \beta)$ is always zero for a Calabi-Yau 3-fold, essentially the only Gromov-Witten invariants of X are the zero-point invariants which we can view simply as rational numbers $N^g_\beta(X) \in \mathbf{Q}$ which only depend on g, β, and the deformation type of X. In terms of the usual notation,

$$N^g_\beta(X) := \langle \, \rangle^X_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1 \in \mathbf{Q}$$

where this last integral is just notation for the image of $[\overline{M}_g(X, \beta)]^{vir}$ under the natural map $H_0(\overline{M}_g(X, \beta), \mathbf{Q}) \to \mathbf{Q}$.

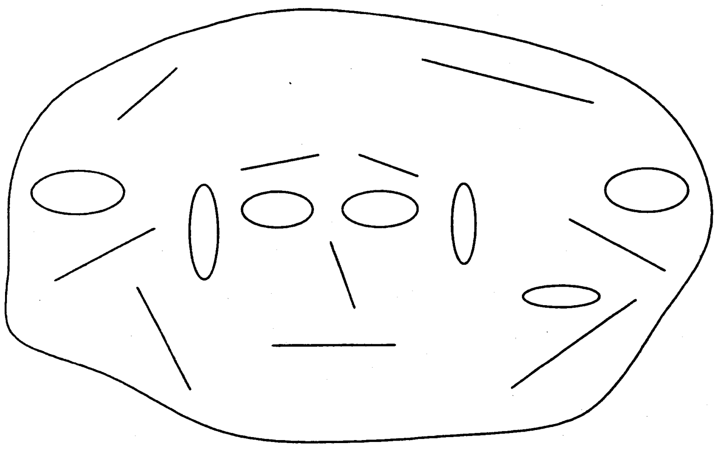
The basic question that we wish to address is:

Question 1. *How are the invariants $N^g_\beta(X)$ related to the enumerative geometry of X? In other words, how are the Gromov-Witten invariants of X related to the number of genus g curves in X in the class β and visa-versa?*

The following well known example is the proto-typical relationship between enumerative geometry and Gromov-Witten invariants. It is classically known that a generic quintic 3-fold $X^3_{(5)}$ contains 2875 lines and 609250 conics. If H denotes the class of the line, then

$$N^0_H(X^3_{(5)}) = 2875 = \#\{\text{lines}\}$$
$$N^0_{2H}(X^3_{(5)}) = \frac{4876875}{8} = \#\{\text{conics}\} + \frac{1}{8}\#\{\text{lines}\}.$$

The second term in the formula for N^0_{2H} is the contribution of maps which are degree two covers of the lines. One sees in this case that one can recover enumerative information about X from its Gromov-Witten invariants as long as one understands the contributions of multiple covers to the invariant.



There are 2875 lines
and 609250 conics on X

1 MULTIPLE COVER FORMULAS

In order to discuss multiple cover contributions in general, we make the following definition.

Definition 1.1. *Let $C \subset X$ be a curve and let $M_C \subset \overline{M}_g(X, d[C])$ be the locus of maps whose image is C; suppose that M_C is an open component of $\overline{M}_g(X, d[C])$. Then define $N_d^g(C \subset X) \in \mathbf{Q}$, the* local Gromov-Witten invariant *of C (also called the* multiple cover contribution *of C) by restricting $[\overline{M}_g(X, d[C])]^{vir}$ to $H_0(M_C, \mathbf{Q})$ and then pairing with 1.*

Note that $N_d^g(C \subset X)$ only depends on the infinitesimal neighborhood of C in X. We will sometimes just write $N_d^g(C)$ if the neighborhood of C is understood from the context.

The first multiple cover formula was conjectured by physicists ([5] or [6]), derived by Aspinwall-Morrison [1], and proved in the context of Gromov-Witten theory by Voisin [20]:

Theorem 1.2. *If $C \subset X$ is an embedded $\mathbf{CP}^1$ with $N_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (a so-called (-1,-1)-curve), then*

$$N_d^0(C \subset X) = \frac{1}{d^3}$$

If one knew that all the rational curves in X were (-1,-1)-curves, then using this formula, one could obtain the number of rational curves in each degree recursively in terms of the Gromov-Witten invariants. However, this hypothesis on the rational curves of X is somewhat strong; for example, it fails for a generic quintic 3-fold. Even if we assume the Clemen's conjecture (which states that there are a finite number of rational curves in each degree on a generic quintic) there are always rational curves with 6 nodes in degree five; Vainsencher has shown that there are 17,601,000 such curves [19].

X has 6-nodal curves in degree 5

The multiple cover formula of a nodal rational curve is *not* the same as the multiple cover formula of a smooth rational curve. Therefore, this example shows that in order to understand the relationship between the enumerative geometry and the genus 0 Gromov-Witten invariants of a quintic (and probably any other Calabi-Yau 3-fold) we must understand the multiple cover formulas for more general kinds of rational curves. The first results in this direction were proved by Bryan-Katz-Leung in [3]:

Theorem 1.3. *Let X be a Calabi-Yau 3-fold and let $C \subset X$ be a rational curve with one node and assume that C is super-rigid (see Definition 2.3). Then*

$$N_d^0(C \subset X) = \sum_{n|d} \frac{1}{n^3}.$$

Multiple cover formulas for the case when $C \subset X$ is an arbitrary contractable embedded rational curve are also proved in [3]. Such curves do not have to be (-1,-1)-curves; they can also have normal bundles $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. The multiple cover formulas for these curves do *not* just depend on the type of the normal bundle; they involve the multiplicities of certain non-reduced subschemes supported on C in their Hilbert scheme. We will state the precise results in Section 3 in the language of BPS invariants (see Theorem 3.5).

2 Multiple cover formulas in higher genus

For multiple cover formulas for higher genus Gromov-Witten invariants, there are the following basic results:

Theorem 2.1 (Faber-Pandharipande [8]). *Let $C \subset X$ be a (-1,-1)-curve, then*

$$N_d^g(C \subset X) = d^{2g-3} \frac{|B_{2g}(2g-1)|}{(2g)!}.$$

Theorem 2.2 (Pandharipande [18]). *Let $C \subset X$ be a super-rigid elliptic curve (see Definition 2.3), then*

$$N_d^g(C \subset X) = \begin{cases} 0 & \text{if } g \neq 1 \\ \frac{1}{d}\sum_{n|d} n & \text{if } g = 1 \end{cases}$$

These two formulas should be viewed as the first two in a series of multiple cover formulas for generically embedded curves of arbitrary genus.

For the rest of these notes we will mostly be interested in the multiple covers of a generically embedded genus g curve $C_g \subset X$ in a Calabi-Yau 3-fold when $g > 1$. We begin by a digression on super-rigidity and its relevance to the definition of the local invariants of C_g.

Definition 2.3. *If $M_C \subset \overline{M}_g(X, d[C])$ is an open component (c.f. Definition 1.1) and $M_C \cong \overline{M}_g(C, d)$ then we say C is (d, g)-*rigid. *If C is (d, g)-rigid for all d and g, we say C is* super-rigid.

For example, a (-1,-1)-curve is super-rigid and an elliptic curve $E \subset X$ is super-rigid if and only if $N_{E/X} \cong L \oplus L^{-1}$ where $L \to E$ is a flat line bundle such that no power of L is trivial. An example where M_C is an open component but $M_C \not\cong \overline{M}_g(C, d)$ is the case where $C \subset X$ is a contractable, smoothly embedded $\mathbf{CP}^1$ with $N_{C/X} \cong \mathcal{O} \oplus \mathcal{O}(-2)$. In this case

M_C has non-reduced structure coming from the (obstructed) infinitesimal deformations of C in the $\mathcal{O}$ direction of $N_{C/X}$.

Let $h \geq 0$ and suppose a smooth curve $C_g \subset X$ is $(d, g+h)$-rigid. Then $N_d^{g+h}(C_g \subset X)$ can be expressed as the integral of an Euler class of a bundle over $[\overline{M}_{g+h}(C_g, d)]^{vir}$. Let $\pi : U \to \overline{M}_{g+h}(C_g, d)$ be the universal curve and let $f : U \to C_g$ be the universal map. Then

$$N_d^{g+h}(C_g \subset X) \cong \int_{[\overline{M}_{g+h}(C_g,d)]^{vir}} c(R^1\pi_* f^*(N_{C/X})).$$

In fact, this integral only depends on g, h, and d since we can write

$$\begin{aligned}\int c(R^1\pi_* f^* N_{C/X}) &= \int c(R^\bullet \pi_* f^* N_{C/X}[1]) \\ &= \int c(R^\bullet \pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1])\end{aligned}$$

where all the integrals are over $[\overline{M}_{g+h}(C_g, d)]^{vir}$. The first equality holds because $(d, g+h)$-rigidity implies that $R^0\pi_* f^* N_{C/X}$ is 0; the second equality holds because $N_{C/X}$ deforms to $\mathcal{O}_C \oplus \omega_C$ (where ω_C is the canonical sheaf of C). This last integral only depends on g, h, and d and we regard it as the idealized multiple cover contribution of a genus g curve by maps of degree d and genus $g+h$.

We will denote this idealized contribution by the following notation:

$$N_d^h(g) := \int_{[\overline{M}_{g+h}(C_g,d)]^{vir}} c(R^\bullet \pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1]).$$

Whether of not there exist super-rigid curves of genus $g > 1$ in a Calabi-Yau 3-fold is a subtle question about the geometry of Calabi-Yau's. On the other hand, to construct a genus g curve that is $(d, g+h)$-rigid for any fixed g, h, and d is probably considerably less hard. Moreover, from the previous discussion, we see that $N_d^{g+h}(C_g \subset X) = N_d^h(g)$ whenever C_g is $(d, g+h)$-rigid.

One also expects that these rigidity issues are less delicate in the symplectic setting. For a generic almost complex structure on X, it is more reasonable to expect that a condition like super-rigidity will hold for any pseudo-holomorphic curve in X.

In light of this discussion, we see that the numbers $N_d^h(g)$ are natural to compute and are, in fact, critical to our understanding of the relationship of the Gromov-Witten invariants with enumerative geometry. Unfortunately, as we've already seen in Theorems 2.1 and 2.2, these are complicated rational numbers, even in the case of $g = 0$ or 1. *A priori*, there is no obvious way to organize and simplify these contributions. Remarkably, the formula of Gopakumar-Vafa and the BPS invariants seem to give a framework for understanding multiple cover contributions in terms of simpler, conjecturally integer contributions.

3 BPS invariants and the Gopakumar-Vafa formula

In [9], Gopakumar and Vafa found, via physical arguments, a relationship between the Gromov-Witten invariants and counts of certain BPS states in M-Theory. Currently, there is no mathematically rigorous geometric definition of the BPS state counts (although there have been some positive results in this direction, c.f. Remark 3.3). However, one can use the Gopakumar-Vafa formula to *define* the BPS state counts *in terms* of the Gromov-Witten invariants.

Definition 3.1. *We define the* BPS invariants $n^r_\beta(X)$ *by the formula:*

$$\sum_{\beta\neq 0}\sum_{r\geq 0} N^r_\beta(X)t^{2r-2}q^\beta = \sum_{\beta\neq 0}\sum_{r\geq 0} n^r_\beta(X)\sum_{k>0}\tfrac{1}{k}\left(2\sin(\tfrac{kt}{2})\right)^{2r-2}q^{k\beta}.$$

Matching the coefficients of the two series yields equations determining $n^r_\beta(X)$ recursively in terms of $N^r_\beta(X)$.

From the above definition, there is no (mathematical) reason to expect $n^r_\beta(X)$ to be an integer. Thus, the physics makes the following prediction.

Conjecture 3.2. *The BPS invariants are integers:*

$$n^r_\beta(X)\in\mathbf{Z}.$$

Remark 3.3. According to physics, the BPS invariants should have a geometric definition along the following lines: there should be a moduli space of D-branes $\hat{M}\to M$ where M parameterizes curves in X in the class β and the fiber of $\hat{M}\to M$ over some curve $C\in M$ parameterized flat line bundles on C. Furthermore, there should be an $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$ representation on $H^*(\hat{M},\mathbf{C})$ so that the diagonal action is the usual $\mathfrak{sl}_2$ Lefschetz representation (assuming $\hat{M}$ is compact, smooth, and Kähler). The BPS state counts $n^g_\beta(X)$ should then be the coefficients in a certain kind of decomposition of $H^*(\hat{M},\mathbf{C})$ as a $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$ representation. The correct general definition of the D-brane moduli space is unknown, although there has been recent progress in the case when the curves move in a surface $S\subset X$ (see [12], [13], [15]). The nature of the correct D-brane moduli space in the case where there are non-reduced curves in the family M (e.g. any multiple of a curve class) is currently poorly understood. In this case, the fiber of $\hat{M}\to M$ over a point corresponding to a non-reduced curve may involve higher rank bundles on the reduction of the curve.

The physical discussion suggests that the BPS invariants will be a sum of integral contributions coming from each component of the D-brane moduli space (whatever it is). Consequently, we expect that various contributions to the Gromov-Witten invariants arising from components in the moduli space of stable maps (e.g. the local invariants $N^g_d(C\subset X)$) should lead to integral contributions to the BPS invariants. Thus we define (again via the Gopakumar-Vafa formula) the local BPS invariants corresponding to the local Gromov-Witten invariants.

Definition 3.4. *Define the* local BPS invariants $n_d^g(C \subset X)$ *in terms of the local Gromov-Witten invariants by the formula*

$$\sum_{\beta\neq 0}\sum_{g\geq 0} N_d^g(C\subset X)t^{2g-2}q^d = \sum_{d\neq 0}\sum_{g\geq 0} n_d^g(C\subset X)\sum_{k>0}\tfrac{1}{k}\left(2\sin(\tfrac{kt}{2})\right)^{2g-2}q^{kd}.$$

Note that $n_d^g(C \subset X)$ is well defined whenever the local invariants $N_{d'}^{g'}(C \subset X)$ are defined for all $g' \leq g$ and $d'|d$. We also use the notation $n_d^h(g)$ for the local BPS invariants obtained from $N_d^h(g)$.

The local BPS invariants are much simpler than the corresponding local Gromov-Witten invariants in the known cases. The multiple cover formulas stated in the previous section can be restated as follows[1]:

$$n_d^g(\text{(-1,-1)-curve}) = \begin{cases} 1 & \text{for } g=0 \text{ and } d=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$n_d^g(\text{super-rigid elliptic curve}) = \begin{cases} 1 & \text{for } g=1 \text{ and any } d \\ 0 & \text{for } g\neq 1, \end{cases}$$

$$n_d^g(\text{super-rigid 1-nodal rational curve}) = \begin{cases} 1 & \text{for } g=0 \text{ and any } d \\ 1\# & \text{for } g=1 \text{ and any } d \\ 0\# & \text{for } g>1, \end{cases}$$

Amazingly, the Gopakumar-Vafa formula has magically encoded all the complicated rational numbers, sums over divisors, etc. that occur in the N_d^g's into the few simple integers occurring in the n_d^g's !

The local BPS invariants *can* have values other than 0 or 1 as shown in the following result of Bryan-Katz-Leung [3] for embedded contractable $\mathbf{CP}^1$'s:

Theorem 3.5. *Suppose $C \subset X$ is a smoothly embedded, contractable $\mathbf{CP}^1$ in a Calabi-Yau 3-fold X. That is, there exists a map $\pi : X \to Y$ with Y normal such that $\pi(C) = \{p\}$ and π induces an isomorphism $X\backslash C \cong Y\backslash p$. Define subschemes $C = C_1 \subset C_2 \subset \cdots \subset C_l$ by their ideal sheaves as follows. Let $\mathcal{I}_{C_l} = \pi^{-1}(\mathcal{I}_p)$ and define $\mathcal{I}_{C_i}$, $1 \leq i \leq l$ to be the sheaf of functions on X that vanish to order i along C when restricted to the pullback of a generic hyperplane section in Y passing through p. Note that i is the length of C_i over C (c.f. Definition 1.6 in [3]).*

Let k_i be the multiplicity of C_i in its corresponding Hilbert scheme. Then

$$n_d^g(C\subset X) = \begin{cases} k_d & \textit{if } g=0 \textit{ and } d\in\{1,\ldots,l\} \\ 0 & \textit{otherwise.} \end{cases}$$

[1]The numbers marked by # are not proved but are based on reasonable conjectures in Gromov-Witten theory.

The number l is Kóllar's invariant "length". If $N_{C/X} \cong \mathcal{O} \oplus \mathcal{O}(-2)$, then $l = 1$ and $k_1 > 1$ is Reid's invariant "width". For $N_{C/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-3)$, the length can be 2, 3, 4, 5, or 6. No other normal bundles are possible for a contractable $\mathbf{CP}^1$ (see [14]).

4 BPS invariants of higher genus curves

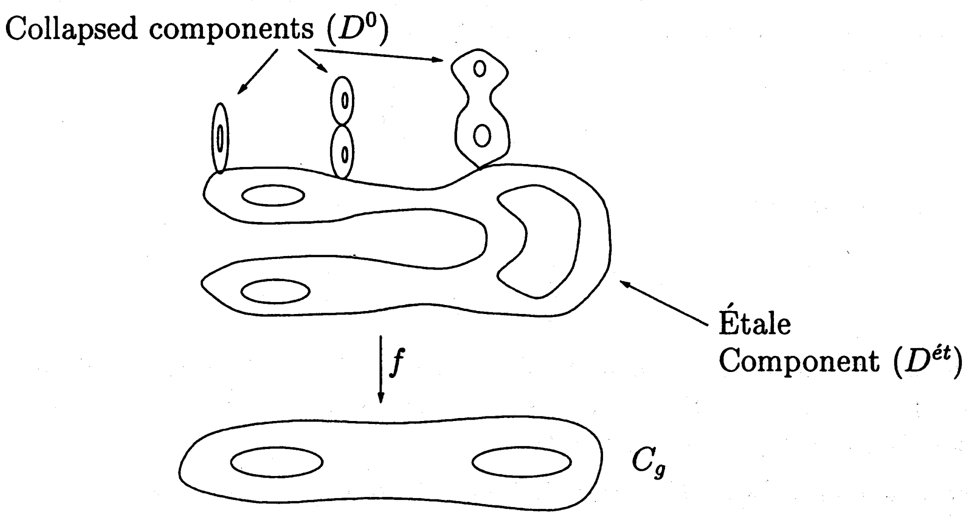
In this section we will discuss results for the degree d, genus $g + h$ multiple covers of embedded genus g curves. That is, we wish to compute the (idealized) local Gromov-Witten invariants $N_d^h(d)$ and the corresponding local BPS invariants $n_d^h(g)$.

In general, there are few, if any, techniques to compute the integral required for $N_g^h(g)$. When $g = 0$ this can be done by Graber-Pandharipande localization [10] since there is a $\mathbf{C}^*$ action on the moduli space induced by the action on $\mathbf{P}^1$. When $g = 1$, there is the action of the elliptic curve itself on the moduli space induced by translation, and this action plays a crucial role in Pandharipande's computation[18]. But for $g > 1$ the usual techniques for computing Gromov-Witten invariants do not apply.

4.1 Contributions from maps with a single étale component.

The moduli space $\overline{M}_{g+h}(C_g, d)$ is extremely complicated in general; it can be singular with many different components of different dimensions intersecting along complicated subschemes. However, it does have some open components where the integral defining $N_d^h(g)$ is computable. One such component that is important is the locus of maps consisting of a single étale component with simply attached collapsing components.

Definition 4.1. *Let $M^{\acute{e}t} \subset \overline{M}_{g+h}(C_g, d[C_g])$ be the locus of maps $f : D \to C_g$ with $D = D^{\acute{e}t} \cup D^0$ where $D^{\acute{e}t}$ is connected, $f : D^{\acute{e}t} \to C_g$ is étale, and f is constant on the components of D^0. Furthermore, we require that the components of D^0 are simply attached to $D^{\acute{e}t}$, that is in the dual graph of D no cycle contains the vertex corresponding to $D^{\acute{e}t}$.*



The above figure illustrates a map in $M^{ét}$.

A deformation argument shows that $M^{ét}$ is an open component of $\overline{M}_{g+h}(C_g, d)$. Since $M^{ét}$ is an open component, it has a virtual fundamental class obtained by restricting the virtual class of $M_{g+h}(C_g, d)$ to it. Define $N_d^h(g)^{ét}$ to be the contribution to $N_d^h(g)$ obtained by integrating over this component, that is

$$N_d^h(g)^{ét} := \int_{[M^{ét}]^{vir}} c(R^\bullet \pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1]).$$

Define $n_d^h(g)^{ét}$ to be the corresponding contribution to the BPS invariants.

Remark 4.2. *A priori*, there is no obvious reason, even physically, for the numbers $n_d^h(g)^{ét}$ to be integers (except for the range of d, h, and g where $\overline{M}_{g+h}(C_g, d) = M^{ét}$ so that $n_d^h(g)^{ét} = n_d^h(g)$). However, in [4] Bryan and Pandharipande compute a closed formula for $n_d^h(g)^{ét}$ and they show that $n_d^h(g)^{ét} \in \mathbf{Z}$ *for all* d, g, and h. This is highly suggestive that the correct D-brane moduli space for multiples of a rigid curve has a distinguished component corresponding to the étale contributions $n_d^h(g)^{ét}$.

The formula for $n_d^h(g)^{ét}$ is as follows:

Theorem 4.3 (see [4]). *For any fixed d and g, the étale BPS invariants $n_d^h(g)^{ét}$ are given by the follow generating function:*

$$\sum_{h \geq 0} n_d^h(g)^{ét} y^{h+g} = \sum_{k|d} \frac{k}{d} \mu(\frac{d}{k}) C_{k,g} \left(y P_{\frac{d}{k}}(y)\right)^{k(g-1)}$$

where $\mu(a)$, $C_{k,g}$, and $P_l(y)$ are defined below.

$\mu(a)$ is the Möbius function, *i.e.*

$$\mu(a) = \begin{cases} 0 & \text{if } a \text{ is not square-free} \\ (-1)^l & \text{if } a \text{ is a product of } l \text{ distinct primes.} \end{cases}$$

$C_{k,g}$ is the number of connected, étale, degree k, covers of a genus g curve, each counted by the reciprocal of the number of automorphisms. Finally, $P_l(y)$ is the polynomial defined by the equation

$$P_l(4\sin^2(t)) = \frac{\sin^2(lt)}{\sin^2(t)};$$

it is given explicitly by

$$P_l(y) = \sum_{a=0}^{l-1} \frac{l}{a+1} \binom{a+l}{2a+1} (-y)^a.$$

It is not obvious from the formula in the theorem that $n_d^h(g)^{ét} \in \mathbf{Z}$. However, it is true and also proved in [4]. The proof relies on somewhat delicate properties of the rational numbers $C_{k,g}$ and the polynomials $P_l(y)$.

Theorem 4.4 (see [4]). *The étale BPS invariants $n_d^h(g)^{ét}$ are integers.*

4.2 Contribution from maps with 2 ramifications

There is another situation where $\overline{M}_{g+h}(C_g, d)$ has a distinguished open component. If

$$h = (d-1)(g-1) + 1$$

then there are exactly two open components, namely the étale component $M^{\acute{e}t}$ and one other $\widetilde{M} \subset \overline{M}_{g+h}(C_g, d)$. In this subsection we fix d, g, and h so that the above equation holds. The generic points of $\widetilde{M}$ correspond to maps of smooth curves with exactly two simple ramification points. Let $\widetilde{N}^h_g(d)$ be the corresponding contribution to the Gromov-Witten invariants so that

$$N^h_d(g) = N^h_d(g)^{\acute{e}t} + \widetilde{N}^h_d(g).$$

The component $\widetilde{M}$ has a finite map to $\mathrm{Sym}^2(C_g)$ given by pointwise by sending a map to its branched locus (see [8] for the existence of such a morphism).

The invariant $\widetilde{N}^h_d(g)$ is computed by Bryan and Pandharipande in [4] by a Grothendieck-Riemann-Roch (GRR) computation. The relative Todd class required by GRR is computed using the formula of Mumford [17] adapted to the context of stable maps (c.f. [8] Section 1.1). The intersections in the GRR formula are computed by pushing forward to $\mathrm{Sym}^2(C_g)$. The result of this computation is the following:

Theorem 4.5.

$$\widetilde{N}^h_d(g) = \int_{\widetilde{M}} c(R^\bullet \pi_* f^*(\mathcal{O}_{C_g} \oplus \omega_{C_g})[1]) = \frac{g-1}{8}\Big((g-1)D_{d,g} - D^*_{d,g} - \frac{1}{27}D^{**}_{d,g}\Big).$$

The numbers $D_{d,g}$, $D^*_{d,g}$, and $D^{**}_{d,g}$ are the following Hurwitz numbers of covers.

$D_{d,g}$ = the number of connected, degree d covers of C_g simply branched over 2 distinct fixed points of C_g.

$D^*_{d,g}$ = the number of connected, degree d, covers of C_g with 1 node lying over a fixed point of C_g.

$D^{**}_{d,g}$ = the number of connected, degree d covers of C_g with 1 double ramification point over a fixed point of C_g.

The covers are understood to be étale away from the imposed ramification. Also, $D_{d,g}$, $D^*_{d,g}$, and $D^{**}_{d,g}$ are all counts weighted by the reciprocal of the number of automorphisms of the covers.

There is an additional Hurwitz number which is natural to consider here:

$D^{***}_{d,g}$ = the number of connected, degree d covers of C_g with 2 distinct ramification points in the domain lying over a fixed point of C_g.

However, $D^{***}_{d,g}$ is determined from the previous Hurwitz numbers by the degeneration relation:

$$D_{d,g} = D^{*}_{d,g} + \frac{1}{3}D^{**}_{d,g} + D^{***}_{d,g} \tag{1}$$

(see [11]). Theorem 4.5 therefore involves all of the independent covering numbers which appear in this 2 branch point geometry.

As $\overline{M}_{g+h}(C_g, d)$ has the two component decomposition $\widetilde{M} \cup M^{ét}$, the corresponding Gromov-Witten invariant $N^h_d(g) = N^h_d(g)^{ét} + \widetilde{N}^h_d(g)$ is determined by Theorem 4.3 and Theorem 4.5. Since the BPS invariant $n^h_g(g)$ only depends on $N^{h'}_{d'}(g)$ for $h' \leq h$ and $d'|d$, when $h = (d-1)(g-1)+1$ we can completely determine $n^h_d(g)$ if d is prime; it only gets contributions from the étale contributions and $\widetilde{N}^h_d(g)$. Thus we get the following:

Corollary 4.6. *Let $h = (d-1)(g-1)+1$ then*

$$N^h_d(g) = \frac{g-1}{24}\Big(- 2dC_{d,g} + (3g-3)D_{d,g} - 3D^{*}_{d,g} - \frac{1}{9}D^{**}_{d,g}\Big).$$

Suppose also that d is prime, then

$$n^h_d(g) = n^h_d(g)^{ét} + \widetilde{N}^h_d(g).$$

Since $n^h_d(g)^{ét}$ is integral by Theorem 4.4, the integrality conjecture predicts that $\widetilde{N}^h_d(g) \in \mathbf{Z}$ when d is prime. Using our computation of $\widetilde{N}^h_d(g)$, this can be rephrased as the following conjectural congruence properties about Hurwitz numbers:

Conjecture 4.7. *Let*

$$\Upsilon_{d,g} = (g-1)\left(27(g-1)D_{d,g} - 27D^{*}_{d,g} - D^{**}_{d,g}\right).$$

Then for d prime,

$$\Upsilon_{d,g} \equiv 0 \pmod{216}.$$

Although $D_{d,g}$, $D^{*}_{d,g}$, and $D^{**}_{d,g}$ are not *a priori* integers, we will prove in Section 5 that $\Upsilon_{d,g} \in \mathbf{Z}$. We will then verify the conjecture for $d = 2$ and 3.

Remark 4.8. Various congruence properties of $C_{d,g}$ (the number of degree d connected étale covers) were also used in the proof of the integrality of the étale BPS invariants $n^h_d(g)^{ét}$ (see [4]). We speculate that these and the above conjecture are the beginning of a series of congruence properties of general Hurwitz numbers that are encoded in the integrality of the local BPS invariants.

5 Congruence properties of Hurwitz numbers

In this section we prove Conjecture 4.7 for $d = 2$ and 3.

Let $D_{d,g}$, $D^*_{d,g}$, $D^{**}_{d,g}$, and $D^{***}_{d,g}$ be the Hurwitz numbers defined in Subsection 4.2. We begin by showing that $\Upsilon_{d,g}$ is an integer. This immediately follows from the degeneration relation (Equation 1) and the following lemma.

Lemma 5.1. *The numbers $D^*_{d,g}$, $D^{**}_{d,g}$, and $D^{***}_{d,g}$ are integers.*

PROOF: Let $\widetilde{D}^{**}_{d,g}$ and $\widetilde{D}^{**}_{d,g}$ be the Hurwitz numbers analogous to $D^{**}_{d,g}$ and $D^{***}_{d,g}$ but where we allow covers that are *not necessarily connected* in the count. Similarly, let $\widetilde{C}_{d,g}$ be the analog of $C_{d,g}$, *i.e.* the number of (not necessarily connected) étale covers (this is called $a_{d,g}$ in [4]).

These numbers are more natural from the point of view of group theory. A cover of a genus g curve ramified over at most one point is determined by the monodromy of the $2g$ generators of the once punctured surface. The ramification type is determined by the monodromy around the puncture. Thus we get:

$$\widetilde{D}^{***}_{d,g} = \frac{1}{d!}\#\{(a_1,\ldots,a_g,b_1,\ldots,b_g) \in (S_d)^{2g} | \prod_{i=1}^{g}[a_i,b_i] \text{ is 2 disjoint 2-cycles}\},$$

$$\widetilde{D}^{**}_{d,g} = \frac{1}{d!}\#\{(a_1,\ldots,a_g,b_1,\ldots,b_g) \in (S_d)^{2g} | \prod_{i=1}^{g}[a_i,b_i] \text{ is a single 3-cycle}\},$$

$$\widetilde{C}_{d,g} = \frac{1}{d!}\#\{(a_1,\ldots,a_g,b_1,\ldots,b_g) \in (S_d)^{2g} | \prod_{i=1}^{g}[a_i,b_i] = 1\ \}.$$

By the proof of Lemma C1 in [4], the numbers $\widetilde{D}^{**}_{d,g}$, $\widetilde{D}^{*}_{d,g}$, and $\widetilde{C}_{d,g}$ are all integers. The relationship between the above Hurwitz numbers and their analogs for connected covers is easily derived geometrically. Clearly,

$$\widetilde{D}^{**}_{d,g} = \sum_{j=0}^{d} D^{**}_{j,g}\widetilde{C}_{d-j,g} \tag{2}$$

$$\widetilde{D}^{***}_{d,g} = \sum_{j=0}^{d} D^{***}_{j,g}\widetilde{C}_{d-j,g}.$$

These formulas imply inductively that $D^{**}_{d,g}, D^{***}_{d,g} \in \mathbf{Z}$. By taking the normalization of the covers counted by $D^*_{d,g}$, we get the following relationship:

$$\binom{d}{2}\widetilde{C}_{d,g} = \sum_{l=2}^{d} D^*_{l,g}\widetilde{C}_{d-l,g}. \tag{3}$$

(See [4] for the relationship between $\widetilde{C}_{d,g}$ and $C_{d,g}$). Multiplying the above equation by q^d, summing over d, and re-indexing, we arrive at the following identity of formal power series:

$$\frac{1}{2}q^2\frac{d^2}{dq^2}\left(\sum_{d\geq 1}\widetilde{C}_{d,g}q^d\right) = \left(\sum_{l\geq 2}D^*_{l,g}q^l\right)\left(\sum_{m\geq 0}\widetilde{C}_{m,g}q^m\right).$$

The series on the left is an integer series and since $\sum_{m=0}^{\infty}\widetilde{C}_{m,g}q^m$ is an integer series beginning with 1, it has an inverse series that is integral. Thus $\sum D^*_{l,g}q^l$ is an integer series and the lemma is proved. □

Theorem 5.2. *Conjecture 4.7 holds for $d = 2$ and $d = 3$. That is $\Upsilon_{2,g} \equiv \Upsilon_{3,g} \equiv 0 \pmod{216}$.*

Proof when $d = 2$: In this case, the Hurwitz numbers can be determined explicitly. Since $d = 2$, $D^{**}_{2,g} = D^{***}_{2,g} = 0$ and so $D_{2,g} = D^*_{2,g}$ which can be counted as follows. The normalization $\widetilde{C}' \to C'$ of a double cover $C' \to C_g$ with one node over $p \in C_g$ is étale. Conversely, any étale double cover $\widetilde{C}' \to C_g$ gives rise to a connected cover with one node over p by gluing. The number of such covers is 2^{2g} and since they all have a $\mathbf{Z}/2$ automorphism, we have

$$D_{2,g} = 2^{2g-1}.$$

Thus

$$\Upsilon_{2,g} = 27(g-1)(g-2)2^{2g-1}$$

which is 0 modulo 216.

Proof for $d = 3$: In this case, $D^{***}_{3,g} = 0$ and $D_{3,g} = D^*_{3,g} + \frac{1}{3}D^{**}_{3,g}$ so

$$\Upsilon_{3,g} = (g-1)(27(g-2)D^*_{3,g} + (9g-10)D^{**}_{3,g}).$$

From Equation 2 we see that $D^{**}_{3,g} = \widetilde{D}^{**}_{3,g}$ and from Equation 3 we deduce that

$$\begin{aligned} D^*_{3,g} &= 3\widetilde{C}_{3,g} - \widetilde{C}_{2,g} \\ &= 3\widetilde{C}_{3,g} - 2^{2g-1}. \end{aligned}$$

We need to show that $\Upsilon_{3,g} \equiv 0 \pmod 8$ and $\Upsilon_{3,g} \equiv 0 \pmod{27}$. We first compute modulo 8:

$$\begin{aligned} \Upsilon_{3,g} &\equiv (g-1)(3(g-2)3\widetilde{C}_{3,g} + (g-2)\widetilde{D}^{**}_{3,g}) \pmod 8 \\ &\equiv (g-1)(g-2)(\widetilde{C}_{3,g} + \widetilde{D}^{**}_{3,g}) \pmod 8. \end{aligned}$$

Since a product of commutators is an even cycle and the only even cycles in S_3 are the identity and the 3-cycles, we see that

$$\begin{aligned}\widetilde{N}^{**}_{3,g} + \widetilde{C}_{3,g} &= \frac{1}{3!}\#\{(a_1,\dots,a_g,b_1,\dots,b_g) \in (S_3)^{2g}\}\\ &= 6^{2g-1}.\end{aligned}$$

Thus

$$\begin{aligned}\Upsilon_{3,g} &\equiv (g-1)(g-2)6^{2g-1} \pmod 8\\ &\equiv 0 \pmod 8.\end{aligned}$$

It remains to prove that $\Upsilon_{3,g} \equiv 0 \pmod{27}$. Since

$$\Upsilon_{3,g} \equiv (g-1)(9g-10)\widetilde{D}^{**}_{3,g} \pmod{27}$$

it suffices to prove that $\widetilde{D}^{**}_{3,g} \equiv 0 \pmod{27}$ for $g > 1$.

Let $\sigma \in S_3$ be a non-trivial 3-cycle. For any $\alpha \in S_3$ define

$$D^{\alpha}_g = \#\{(a_1,\dots,a_g,b_1,\dots,b_g) \in (S_3)^{2g} | \prod_{i=1}^{g}[a_i,b_i] = \alpha\}.$$

Note that $\widetilde{C}_{3,g} = \frac{1}{6}D^1_g$ and $\widetilde{D}^{**}_{3,g} = \frac{1}{6}(D^{\sigma}_g + D^{\sigma^2}_g) = \frac{1}{3}D^{\sigma}_g$. A direct count shows that $D^1_1 = 18$ and $D^{\sigma}_1 = D^{\sigma^2}_1 = 9$. So we have

$$\begin{aligned}D^1_g &= D^1_{g-1}D^1_1 + D^{\sigma}_{g-1}D^{\sigma^2}_1 + D^{\sigma^2}_{g-1}D^{\sigma}_1\\ &= 18D^1_{g-1} + 18D^{\sigma}_{g-1},\end{aligned}$$

and therefore

$$\widetilde{C}_{3,g} = 18\widetilde{C}_{3,g-1} + 9\widetilde{D}^{**}_{3,g-1}.$$

Similarly,

$$\begin{aligned}\widetilde{D}^{**}_{3,g} &= \frac{1}{3}D^{\sigma}_g\\ &= \frac{1}{3}(D^{\sigma}_{g-1}D^1_1 + D^1_{g-1}D^{\sigma}_1 + D^{\sigma^2}_{g-1}D^{\sigma^2}_1)\\ &= 9D^{\sigma}_{g-1} + 3D^1_{g-1}\\ &= 27\widetilde{D}^{**}_{3,g-1} + 18\widetilde{C}_{3,g-1}.\end{aligned}$$

Using these formulas, induction on g then easily implies that $\widetilde{D}^{**}_{3,g} \equiv 0 \pmod{27}$ for $g > 1$ and the theorem is proved. □

References

[1] Paul S. Aspinwall and David R. Morrison. Topological field theory and rational curves. *Comm. Math. Phys.*, 151(2):245–262, 1993.

[2] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[3] Jim Bryan, Sheldon Katz, and Naichung Conan Leung. Multiple covers and the integrality conjecture for rational curves in Calabi-Yau threefolds. Preprint, math.AG/9911056, To appear in *Jour. of Alg. Geom.*

[4] Jim Bryan and Rahul Pandharipande. BPS states of curves in Calabi-Yau 3-folds. In preparation.

[5] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.

[6] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. In *Essays on mirror manifolds*, pages 31–95. Internat. Press, Hong Kong, 1992.

[7] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry.* American Mathematical Society, Providence, RI, 1999.

[8] C. Faber and R. Pandharipande. Hodge integrals and Gromov-Witten theory. *Invent. Math.*, 139(1):173–199, 2000.

[9] Rajesh Gopakumar and Cumrun Vafa. M-theory and topological strings–II, 1998. Preprint, hep-th/9812127.

[10] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[11] J. Harris and I. Morrison. Slopes of effective divisors on the moduli space of stable curves. *Invent. Math.*, 99(2):321–355, 1990.

[12] S. Hosono, M.-H. Saito, and A. Takahashi. Holomorphic anomaly equation and BPS state counting of rational elliptic surface. *Adv. Theor. Math. Phys.*, 3(1):177–208, 1999.

[13] Sheldon Katz, Albrecht Klemm, and Cumrun Vafa. M-theory, topological strings and spinning black holes. Preprint: hep-th/9910181.

[14] Sheldon Katz and David R. Morrison. Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups. *J. Algebraic Geom.*, 1(3):449–530, 1992.

REFERENCES

[15] Albrecht Klemm and Eric Zaslow. Local Mirror Symmetry at Higher Genus. IASSNS-HEP-99-55.

[16] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.

[17] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, pages 271–328. Birkhäuser Boston, Boston, Mass., 1983.

[18] R. Pandharipande. Hodge integrals and degenerate contributions. *Comm. Math. Phys.*, 208(2):489–506, 1999.

[19] Israel Vainsencher. Enumeration of n-fold tangent hyperplanes to a surface. *J. Algebraic Geom.*, 4(3):503–526, 1995.

[20] Claire Voisin. A mathematical proof of a formula of Aspinwall and Morrison. *Compositio Math.*, 104(2):135–151, 1996.