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Rational unicuspidal plane curves with $\overline{\kappa} = 1$

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1 Introduction

Let $C$ be a curve on $\mathbb{P}^2 = \mathbb{P}^2(C)$. A singular point of $C$ is said to be a cusp if it is a locally irreducible singular point. We say that $C$ is cuspidal if $C$ has only cusps as its singular points. For a cusp $P$ of $C$, we denote the multiplicity sequence of $(C, P)$ by $\overline{m}_P(C)$, or simply $\overline{m}_P$. We use the abbreviation $m_k$ for a subsequence of $\overline{m}_P$ consisting of $k$ consecutive $m$'s. For example, $(2_k)$ means an $A_{2k}$ singularity. We denote by $\overline{\kappa}(\mathbb{P}^2 \setminus C)$ the logarithmic Kodaira dimension of the complement $\mathbb{P}^2 \setminus C$.

Theorem 1. If $C$ is a rational unicuspidal plane curve with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$, then there exists a unique pencil $\Lambda$ on $\mathbb{P}^2$ satisfying the following four conditions.

(i) The cusp $P$ of $C$ is a unique base point of $\Lambda$.

(ii) The pencil $\Lambda$ has a unique reducible member $C + n(C)B$. Here $B$ is a line or an irreducible conic such that $(CB)_P = (\deg B)(\deg C) - 1$.

(iii) The pencil $\Lambda$ has exactly two multiple members $\mu_A A$, $\mu_G G$, where $\mu_A, \mu_G$ are integers with $\mu_A, \mu_G \geq 2$, $A \setminus \{P\} \cong \mathbb{C}^*$, $G \setminus \{P\} \cong \mathbb{C}$.

(iv) The complement of $\{P\}$ to every member other than $\mu_A A$, $\mu_G G$ and $C + n(C)B$ is isomorphic to $\mathbb{C}^*$.

Let $C$ be a rational unicuspidal plane curve with $\overline{\kappa}(\mathbb{P}^2 \setminus C) = 1$. We say that $C$ is of type I (resp. type II) if the curve $B$ in Theorem 1 (ii) is a line (resp. an irreducible conic).
Theorem 2. Let \( C \) be a rational unicuspidal plane curve with \( \overline{\kappa}(\mathbb{P}^2 \setminus C) = 1 \). Let \( n := n(C) \). Let \( P \) be the cusp of \( C \).

(i) Type I. There exists an integer \( s \) with \( s \geq 2 \) such that \( \deg C = (n+1)^2(s-1)+1 \), \( \overline{m}_P = (n(n+1)(s-1), ((n+1)(s-1))_{2n+1}, (n+1)_{2(s-1)}) \), \( \mu_A = n+1 \) and \( \mu_G = (n+1)(s-1)+1 \). There exist \( a_2, \ldots, a_s \in \mathbb{C} \) with \( a_s \neq 0 \) such that \( C \) is projectively equivalent to the curve:

\[
((f^{s-1}y + \sum_{i=2}^{s}a_i f^{s-i}x^{(n+1)i-n})^{\mu_A} - f^{\mu_G})/x^n = 0,
\]

where \( f = x^n z + y^{n+1} \). Conversely, for arbitrary integers \( n, s \) with \( n \geq 2 \), \( s \geq 2 \) and \( a_2, \ldots, a_s \in \mathbb{C} \) with \( a_s \neq 0 \), the above equation defines a rational unicuspidal plane curve with \( \overline{\kappa}(\mathbb{P}^2 \setminus C) = 1 \). The multiplicity sequence of the cusp is equal to \( \overline{m}_P \).

(ii) Type II and \( \deg C = ((4n+1)^2+1)/2 \). We have \( \overline{m}_P = ((n(4n+1))_{4}, (4n+1)_{2n}, 3n+1, n_3), \mu_A = 4n+1 \) and \( \mu_G = 2n+1 \). The curve \( C \) is projectively equivalent to the curve:

\[
((g^n y + x^{2n+1})^{\mu_A} - (g^{2n} z + 2x^{2n} yg^n + x^{4n+1})^{\mu_G})/g^n = 0,
\]

where \( g = x z - y^2 \). Conversely, for an arbitrary integer \( n \) with \( n \geq 2 \), the above equation defines a rational unicuspidal plane curve with \( \overline{\kappa}(\mathbb{P}^2 \setminus C) = 1 \). The multiplicity sequence of the cusp is equal to \( \overline{m}_P \).

(iii) Type II and \( \deg C \neq ((4n+1)^2+1)/2 \). There exists a positive integer \( s \) such that, by setting \( m := 4n+1 \) and \( t := 4s-1 \), we have \( \deg C = (m^2t+1)/2 \),

\[
\overline{m}_P = \begin{cases} ((3mn)_4, (3m)_{2n}, (m)_3, 3n+1, n_3) & \text{if } s = 1, \\ ((tmn)_4, (tm)_{2n}, (sm)_3, (s-1)m, m_{2(s-1)}, 3n+1, n_3) & \text{if } s > 1, \end{cases}
\]

\( \mu_A = m \) and \( \mu_G = 2(ms - n) \). There exist \( a_1, \ldots, a_s \in \mathbb{C} \) with \( a_s \neq 0 \) such that \( C \) is projectively equivalent to the curve:

\[
((h^{2s-1}(g^n y + x^{2n+1}) + \sum_{i=1}^{s}a_i h^{2(s-i)}g^{mi-n})^{\mu_A} - h^{\mu_G})/g^n = 0,
\]

where \( h = g^{2n} z + 2x^{2n} yg^n + x^m \). Conversely, for an arbitrary integer \( n \) with \( n \geq 2 \), a positive integer \( s \) and \( a_1, \ldots, a_s \in \mathbb{C} \) with \( a_s \neq 0 \), the above equation defines a rational unicuspidal plane curve with \( \overline{\kappa}(\mathbb{P}^2 \setminus C) = 1 \). The multiplicity sequence of the cusp is equal to \( \overline{m}_P \).
A plane curve $C$ is said to be of type $(d, \nu)$ if the degree of $C$ is $d$ and the maximal multiplicity of $C$ is $\nu$. If $C$ is a rational cuspidal curve of type $(d, \nu)$, then the inequality $d < 3\nu$ holds true ([MS]). See also [O].

**Corollary 1.** Let $C$ be a rational unicuspidal plane curve of type $(d, \nu)$ with $\kappa(\mathbb{P}^2 \setminus C) = 1$.

(i) **Type I.** We have $1 < d/\nu \leq 5/3$. The equality holds if and only if $C$ is projectively equivalent to a curve in Theorem 2 (i) with $n = s = 2$.

(ii) **Type II.** We have $2 < d/\nu \leq 41/18$. The equality holds if and only if $C$ is projectively equivalent to the curve in Theorem 2 (ii) with $n = 2$.

**Corollary 2.** Let $C$ be a rational unicuspidal plane curve. Then $\kappa(\mathbb{P}^2 \setminus C) = 1$ if and only if the multiplicity sequence of the cusp is one of those in Theorem 2.

**Corollary 3.** Let $C$ be a rational unicuspidal plane curve. Then $\kappa(\mathbb{P}^2 \setminus C) = 2$ if and only if $n(C) \geq 2$ and the multiplicity sequence of the cusp is none of those in Theorem 2.

**Remark 4.** In [Ts1], Tsunoda claimed to have obtained the defining equations of rational unicuspidal plane curves with $\kappa = 1$. Comparing the degrees of his with ours, it seems that the equations he obtained are those of type I, $s = 2$ in Theorem 2 (i).

## 2 Proof of Theorem 1

Let $C$ be a rational unicuspidal curve on $\mathbb{P}^2$ with $\kappa(\mathbb{P}^2 \setminus C) = 1$. Let $\sigma : V \to \mathbb{P}^2$ be the composite of the shortest sequence of blowing-ups over $P$ such that the reduced total transform $D$ of $C$ is a normal crossing divisor. Let $C'$ be the strict transform of $C$. Put $D' := D - C'$. We remark that every irreducible components of $D'$ is a smooth rational curve, whose self-intersection number is less than $-1$. Let $D_0$ denote the exceptional curve of the last blowing-up of $\sigma$. The dual graph of $D$ has the following shape.

![Dual Graph](image)
As a convention, $A_1$ contains the exceptional curve of the first blowing-up. Let $A_{i1}$ denote the leftmost component of $A_i$ in the above figure. In the course of the contraction of $D'$ by $\sigma$, $A_g + D_0 + B_g$ is contracted a $(-1)$-curve $E$ and $A_{g-1} + E + B_{g-1}$ to a $(-1)$-curve, and so on. Write $\sigma = \sigma_1 \circ \cdots \circ \sigma_g$, where $\sigma_g$ contracts $A_g + D_0 + B_g$ to a $(-1)$-curve $E$, $\sigma_{g-1}$ contracts $A_{g-1} + E + B_{g-1}$ to a $(-1)$-curve, and so on. A blowing-up of $\sigma_i$ is called *sprouting* if it is done at a smooth point of the exceptional curve of the preceding blowing-ups. As a convention, the first blowing-up of $\sigma_1$ is not sprouting. Let $s_i$ denote the number of sprouting blowing-ups in $\sigma_i$.

Following [FZ], we consider a strictly minimal model $(\tilde{V}, \tilde{D})$ of $(V, D)$. We successively contract $(-1)$-curves $E$ such that $E \subset D$ and $(D - E)E \leq 2$, or $E \not\subset D$ and $DE \leq 1$. After a finite number of contractions, we have no $(-1)$-curves to contract. Let $\pi : V \to \tilde{V}$ denote the composite of the contractions. For a divisor $\Delta \subset V$, write $\tilde{\Delta} = \pi_*(\Delta)$. It is clear that $\tilde{D}$ is a divisor with only simple normal crossings and $\tilde{\kappa}(\tilde{V} \setminus \tilde{D}) = 1$. By [Ka, Theorem 2.3] and the fact that $\tilde{V} \setminus \tilde{D}$ is affine, we have the following:

**Lemma 5.** There exists a fibration $\tilde{p} : \tilde{V} \to \mathbb{P}^1$ whose general fiber $F$ is $\mathbb{P}^1$ and $DF = 2$.

It is known that a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$ is obtained from a $\mathbb{P}^1$-bundle $\tilde{p} : \Sigma \to \mathbb{P}^1$ by successive blowing-ups $\tilde{\pi} : \tilde{V} \to \Sigma$. Put $p = \tilde{p} \circ \pi$. We have the following commutative diagram.

\[
\begin{array}{ccc}
V & \xrightarrow{\pi} & \tilde{V} \\
& \searrow^{p} & \swarrow_{\tilde{p}} \rightarrow \Sigma \rightarrow \mathbb{P}^1 \\
& \tilde{p} & \\
\end{array}
\]

Following [FZ], we use the following terminology. The triple $(\tilde{V}, \tilde{D}, \tilde{p})$ is called a $C^*$-triple. A component of $\tilde{D}$ is called horizontal if the image of it under $\tilde{p}$ is 1-dimensional. Let $\tilde{H}$ be the sum of the horizontal components of $(\tilde{V}, \tilde{D}, \tilde{p})$. The $C^*$-triple $(\tilde{V}, \tilde{D}, \tilde{p})$ is called of twisted type if $\tilde{H}$ is irreducible; otherwise it is called of untwisted type. By [Kiz, Theorem 3], our $C^*$-triple is of untwisted type. (See also [M2, Theorem 4.7.1, Lemma 4.10.3].) Thus $\tilde{H}$ consists of two irreducible components $H_1, H_2$. Suppose $\tilde{p}$ has a singular fiber. The dual graph of the sum of the singular fiber and the horizontal components has the following shape (cf. [FZ, Lemma 5.5]).

\[
\begin{array}{ccc}
H_1 & \xcirc & F_1 & \xcirc & E & \xcirc & F_2 & \xcirc & H_2 \\
\end{array}
\]
Here \( E \) is a \((-1)\)-curve, which is not contained in \( \tilde{D} \). The curves \( F_1, F_2 \) are connected components of \( \tilde{D} - (H_1 + H_2) \). The fiber is contracted by \( \tilde{\pi} \) to a fiber of \( \tilde{p} \). By using [FZ, Theorem 5.8 and 5.11], we have the following lemma. (The case (B2) in [FZ, Theorem 5.8] does not occur.)

**Lemma 6.** The \( C^*\)-triple \((\tilde{V}, \tilde{D}, \tilde{p})\) has the following properties.

(i) The fibration \( \tilde{p} \) has exactly one smooth fiber \( \tilde{G} \) contained in \( \tilde{D} \) and two singular fibers \( \tilde{F}_A = \tilde{A}_1 + \tilde{E}_A + \tilde{B}_g, \tilde{F}_B = \tilde{B}_1 + \tilde{E}_B + \tilde{C}' \), where \( \tilde{E}_A \) (resp. \( \tilde{E}_B \)) is the \((-1)\)-curve in \( \tilde{F}_A \) (resp. \( \tilde{F}_B \)).

(ii) The curves \( \tilde{D}_0, \tilde{A}_2 \) are the horizontal components.

We can verify that \( \pi \) has the following properties.

**Lemma 7.** The following assertions hold true.

(i) \( \pi \) first contracts a \((-1)\)-curve \( E_G \not\subset \tilde{D} \) and every subsequent blowing-down of \( \pi \) is the contraction of a component of \( \tilde{D} \).

(ii) The curve \( E_G \) is a component of \( \pi^{-1}(\tilde{G}) \). Every blowing-up of \( \pi \) is performed at a point on the total transform of \( \tilde{G} \).

Let \( E_A, E_B \) denote the strict transforms of \( \tilde{E}_A, \tilde{E}_B \) in \( V \), respectively. Write \( A = \sigma(E_A), B = \sigma(E_B) \) and \( G = \sigma(E_G) \). Let \( \mu_A, \mu_B \) and \( \mu_G \) denote the coefficients of \( E_A, E_B \) and \( E_G \) in \( p^*(p(E_A)), p^*(p(E_B)) \) and \( p^*(p(E_G)) \), respectively. We have \( \mu_B = n(C) \) by [F, Proposition 4.8]. Since \( \pi \) does not change \( \tilde{F}_B \), it follows that \( B \) is smooth and rational with self-intersection number \( s_1 \). Thus \( B \) is a line \((s_1 = 1)\) or an irreducible conic \((s_1 = 4)\). Now it is clear that the pencil spanned by \( \mu_A A \) and \( \mu_G G \) satisfies the whole condition in Theorem 1. The uniqueness of the pencil follows from [I, Theorem 3].

### 3 Proof of Theorem 2

In order to prove Theorem 2, we determine the weighted dual graph of \( D + E_A + E_B + E_G \). By using the properties of \( \sigma, \pi \) and \( \tilde{p} \), we obtain the diagram in Figure 1, where \( n = n(C) \) and \(*\) (resp. \( \bullet \)) means a \((-1)\)-curve (resp. \((-2)\)-curve). In Theorem 2, we set \( s = s_3 \). The curves in Theorem 2 (ii) correspond to those of type II with \( g = 2 \) and the curves in (iii) to those of type II with
Figure 1: The weighted dual graph of $D + E_A + E_B + E_G$
Remark 8. Our fibration $p$ belongs to the class (D) in the sense of [Kiz]. The last two graphs in [Kiz, Figure 54] coincide with those of type I and type II with $g = 3, s_3 > 1$.

The multiplicity sequence of the cusp can be calculated from the weighted dual graph of $D'$ (cf. [BK, p.516, Theorem 12]). The degree of $C$ is calculated from $\overline{m}_P$. We calculate $\mu_A, \mu_G$ by using [F, Proposition 4.8]. The proof of the assertion for the defining equation of $C$ is based on the following fact. Let $f_A, f_B$ and $f_G$ be the defining polynomials of $A$, $B$ and $G$, respectively. Then, since $C + \mu_B B$ is a member of the pencil $\Lambda$ in Theorem 1, there exists $t \in \mathbb{C}^*$ such that $C$ is defined by the equation $(f_A^{\mu_A} + tf_G^{\mu_G})/f_B^{\mu_B} = 0$.

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References


