Surgery along a projective plane in a 4-manifold and $D_4$-singularity

山田 裕一 (Yuichi YAMADA)

(University of Electro-Communications, Math. Dept.)

(2001年5/7から6/29まで京都大学理学部数学科に滞在 長期研究員)

1 Introduction

The Price surgery has been defined in [P, KSTY, Y3] as a cut and paste of a 4-manifold $N_2$ in the 4-sphere $S^4$ and also in general 4-manifolds, where $N_2$ is defined as a total space of a non-orientable $D^2$-bundle over a projective plane with normal Euler number 2 (see [M1, M2, L1, Y1]). It may be expected to make a fake pair of 4-manifolds, which means a pair that are homotopy equivalent but non-diffeomorphic to each other, but such a trial seems not to be succeeded yet except the non-orientable example [A1, A2] (see also [KSTY]: Gluck surgery ([Gl]) is realized by Price surgery).

The 4-manifold $N_2$ is represented by the framed link (see [Ki, GoS]) in Figure 1(1). The boundary $\partial N_2$ is homeomorphic to the quaternion space $Q$, which is the quotient space of the unit sphere $S^3$ of the quaternion field $\mathbb{H} \cong \mathbb{R}^4$ by the quaternion group of order 8. This space $Q$ is also homeomorphic to the linking 3-manifold of $D_4$-singularity: $S^5 \cap \{f^{-1}(0)\}$, where

$$f: \mathbb{C}^3 \rightarrow \mathbb{C}$$

$$(x, y, z) \mapsto x^2 + y^3 + z^3,$$

and we regard $S^5$ as the unit sphere (the boundary of the unit disk $D^6$) in $\mathbb{C}^3 \cong \mathbb{R}^8$. Throughout the paper, by the notation $D_4$ we denote the compact 4-manifold obtained from $D^6 \cap \{f^{-1}(0)\}$ by resolve the singularity minimally, which is represented by the framed link in Figure 1(2).

\(^{0}\)2000 Mathematics Subject Classification: Primary 57Q45, Secondary 57R65, 57N13.

Title of the author's talk on 31 May was slightly different from that of this report.
The boundaries of the two 4-manifolds $N_2$ and $D_4$ are homeomorphic to each other and also to $Q$, thus we can define

**Operation**: “Cut $D_4$ off and paste $N_2$ on” a 4-manifold,

but the resulting 4-manifold is not well-defined because of the ambiguity of the gluing map (self-homeomorphisms on $Q$). Thus, for a given $D_4$ in an original 4-manifold $M$, we study the set $\Omega_M(D_4)$ (consisting of at most three elements, see Section 2) of diffeomorphic class of resulting 4-manifolds.

This operation changes some topological invariants of the ambient 4-manifold: it decreases the Euler characteristic number $\chi$ by 4, the negative second Betti number $\beta_2^-$ by 4 and do not change the positive second Betti number $\beta_2^+$, thus increases the signature $\sigma$ of the 4-manifold by 4.

In this paper, we will report two lemmas related to the operation. One is Lemma 3.1 in § 3, which says that a certain operation consisting of four blowing up's and the operation above is reduced to Price surgery. The other is Lemma 4.1 in § 4 on the resulting manifolds of the operation on the simple elliptic surfaces. Before stating the results, in the next section, we will recall some facts on Price surgery in general 4-manifolds. In §5, we will show some key lemmas by “relative Kirby calculus” (see Section 5.5 in [GoS]), but we do not give the complete proof.

The author would like to express his thanks to the people at Research Institute for Mathematical Sciences, Kyoto University for their hospitality during his stay for two months.

この研究のきっかけは1999年夏和歌山での研究集会「いろいろなカテゴリーでの多様体のトポロジーと特異点」で奥間智弘氏に初歩的な質問をしてみたことでした。また、今回の講演後には何人かの初対面の先生方に複素曲面や特異点の構成について教えていただきました。この場をお借りして感謝致します。ありがとうございました。
Price surgery

We recall notations and facts on Price surgery from [KSTY, Y3].

1. We denote by $N_2$ the total space of a non-orientable $D^2$-bundle over a projective plane with normal Euler number 2, which is a compact oriented 4-manifold with a boundary, and which is described by the Kirby diagram in Figure 1(1). Note that $N_2$ has a handlebody decomposition with one 0-handle, one 1-handle and one 2-handle.

2. The boundary $\partial N_2$ is diffeomorphic to the quaternion space $Q$, which admits a Seifert fibered structure whose Seifert invariants in the sense of [O, §5.2] are given by \{-1; (0_1, 0); (2, 1), (2, 1), (2, 1)\}. We call the three singular fibers $c_{-1}, c_0, c_1$.

3. In [P], Price has investigated the self-diffeomorphisms of the quaternion space $Q$ and has shown that the mapping class group $\mathcal{M}(Q)$ (the group of isotopy classes of orientation preserving self-diffeomorphisms) is isomorphic to $S_3$, the symmetric group on three letters \{-1, 0, 1\}. For each element $\sigma$ in $S_3$, there is a self-diffeomorphism $f_\sigma$ of $Q$ which preserves the Seifert fibered structure and satisfies $f_\sigma(c_i) = c_{\sigma(i)}$. Each map $f_\sigma$ represents the class of $\mathcal{M}(Q)$.

4. Price has also shown that there is a self-diffeomorphism $g$ ($g_1$ in [P, p.116]) of $Q = \partial N_2$ whose order is two in $\mathcal{M}(Q)$ and that can extend over $N_2$ as a self-diffeomorphism. (In fact, $g$ is a bundle isomorphism “−” : $N_2 \rightarrow N_2$ which maps each vector $\vec{v}$ to $-\vec{v}$.) Thus, for a given oriented 4-manifold $E$ whose boundary is $-Q$, we have at most only three 4-manifolds up to diffeomorphism $E \cup_{i\circ \varphi} N_2$ obtained by gluing $N_2$ to $E$ along the boundary. where we use the compositions of a fixed orientation reversing map $i$ from $\partial N_2$ to $\partial E$ and an orientation preserving self-diffeomorphism $\varphi$ on $Q$ as the gluing map. The three 4-manifolds correspond to the classes of $\varphi$ in the right coset $\mathcal{M}(Q)/\{1, g\}$, which consists of three elements.
3 Equivalence of two operations

Let $M$ be a closed oriented 4-manifold and $K$ a smoothly embedded 2-sphere in $M$ whose normal bundle is trivial. We define two operations $A$ and $B$ on $M$ along $K$.

Operation $A$: Taking a pairwise connected sum of $(M, K)$ with the (positive) standard projective plane $(S^4, P_0)$ (see [PR], [L1], [Y1]), we have an embedded projective plane $(M, K \# P_0)$ in $M$ whose normal Euler number 2. The tubular neighborhood $N(K \# P_0)$ is diffeomorphic to $N_2$. Let $\Pi_M(K \# P_0)$ be the set of diffeomorphic class of 4-manifolds obtained by pasting $N_2$ to the exterior $M \setminus \text{int}N(K \# P_0)$ along the boundary. The original manifold $M$ itself and the Gluck surgery $\Sigma_M(K)$ of $M$ along $K$, by Theorem 4.1 in [KSTY], are contained in the set $\Pi_M(K \# P_0)$. By (4) in Section 2, $\Pi_M(K \# P_0)$ consists of at most three elements.

Operation $B$: This operation consists of five steps, see Figure 2: (1) Blow up at a point in $K$. (2) Blow up at the intersection point of the proper lift of $K$ and the exceptional curve. (3) Blow up at a point on the newest exceptional curves. (4) Blow up at a point on the newest exceptional curves again. After this step, we have a $D_4$ in the ambient 4-manifold $M \# 4\mathbb{CP}^2$. (5) Do the operation "Cut $D_4$ off and paste $N_2$ on" the 4-manifold. By $\Omega_M(D_4(K))$, we denote the set of the of diffeomorphic class of the resulting 4-manifolds.

![Figure 2](image)

Lemma 3.1 Two operations $A$ and $B$ along $K$ on $M$ are equivalent, i.e., it holds that $\Pi_M(K \# P_0) = \Omega_M(D_4(K))$ as sets.
4 Operation on elliptic surfaces

Let $E(n)$ be the simply connected elliptic surface (with section) whose Euler characteristic is $12n$, $(E(1) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. $E(n)$ is the fiber sum of $n$ copies of $E(1)$. $E(2) \cong \text{the K3 surface}$, $\ldots$). In [BGo], using a method "relative Kirby diagram" (see Section 5.5 in [GoS]), a decomposition of $E(n)$ as a union of $n + 1$ pieces $N_n \cup W_n \cup W_{n-1} \cup \cdots W_1$ has been shown, where $N_n$ is the nuclei of $E(n)$ ([Go]) and $W_1$ is the $E_8$-plumbing. Each $W_j (j \geq 2)$ is a cobordism represented by the relative Kirby diagram in Figure 3 (modified from Figure 27 in [BGo]), which clearly contains one $E_8$-plumbing. An $E_8$-plumbing contains an obvious $D_4$. Thus we can do the operation "Cut $D_4$ off and paste $N_2$ on" $E(n)$ at most $n$ times. To study the resulting 4-manifolds, we do the operation on $W_j$. For $W_1$, see Lemma 5.2.

**Lemma 4.1** The resulting 4-manifold of the operation "Cut $D_4$ off and paste $N_2$ on" $W_j (j \geq 2)$ does not depend on the gluing map of $\partial N_2$ and is diffeomorphic to $W_j \# 4\overline{\mathbb{CP}^2}$, where $W_j$ is the 4-manifold represented by the relative Kirby diagram in Figure 4.

![Figure 3: Wj](image)

![Figure 4: Wj](image)

Note that $W_j$ is, thus $W_j$ is also a cobordism from the Seifert homology 3-sphere $-\Sigma(2, 3, 6(j-1)-1)$ to $\Sigma(2, 3, 6j-1)$ for $j \geq 2$. We conjecture that all the resulting 4-manifolds, the (non-trivial) union of possible $W_j$’s and $W_j$’s capped by $N_n$ and their "logarithmic transformation" (as a 4-manifold, not as a complex surface) in $N_n$ are all diffeomorphic to $\beta_2^+ (\mathbb{CP}^2) \# \beta_2^- (\overline{\mathbb{CP}^2})$.

5 Key of the proof

We show some key lemmas for Lemma 3.1 and give a proof of Lemma 4.1. They are shown by (ordinary) Kirby calculus and relative Kirby calculus (see Section 5.5 in [GoS]).
Lemma 5.1  See the Kirby calculus from the diagram (A) to (B) of 3-manifolds in Figure 5. It corresponds to a homeomorphism $\varphi$ from the boundary $\partial D_4$ of $D_4$ to $\partial N_2$. Calculating the curves $c_i$'s with 0-framing in (A) during the process of the Kirby calculus, we get the curves $c_i$'s with framings ($\cdot$) in (B). They are $\varphi(c_i)$'s in $\partial N_2$. Thus (under some conditions) the local change from (A) to (B) in a Kirby diagram of a 4-manifold $M$ corresponds to (one of) the operation “Cut $D_4$ off and paste $N_2$ on” $M$.

Of course, another Kirby calculus from the diagram (A) to (B) corresponds to another homeomorphism from $\partial D_4$ to $\partial N_2$. To prove Lemma 4.1 completely, we need every (six or three) calculus from the diagram (A) to (B) for each element of the mapping class group $\mathcal{M}(Q)$ of order six, but in this paper, we omit the other calculus.

Now we use Lemma 5.1 to study the resulting 4-manifold of the operation “Cut $D_4$ off and paste $N_2$ on” the obvious $D_4$ in the $E_8$-plumbing $W_1$.

Lemma 5.2  The resulting 4-manifold is diffeomorphic to $W_1 \# 3\overline{CP^2}$, where $W_1$ is the 4-manifold represented by the final Kirby diagram ($-1$-framed left-hand trefoil) in Figure 6.
Proof. See the Kirby calculus in Figure 6. □

Figure 6

Lemma 4.1 is shown by application of such method.

Note that the action of $\mathcal{M}(Q) (\cong S_3)$ on $\partial D_4$ is obvious. Thus we can calculate every resulting 4-manifold of the operation on $D_4$ in the $E_8$-plumbing for each choice of the gluing map in $\mathcal{M}(Q)$. For a smoothly embedded 2-sphere $K$ in $S^4$, we can also study the resulting 4-manifold of the operation cut the $D_4$ and paste an exterior $-X(P_0 \| K)$ of a projective plane $P_0 \| K$ in $S^4$ instead of $N_2$ ($N_2 \cong -X(P_0)$, see [PR, P, L1, L2, Y1, Y2]) by the method "circle with a dot and with a symbol $K$" in Kirby diagrams introduced in Appendix of [KSTY]. They are all diffeomorphic to $\mathcal{W}_1 \| 3\overline{CP^2}$. Note that the Gluck surgery $\Sigma(K)$ along any $K$ in $S^4$ satisfies that $\Sigma(K) \| \overline{CP^2} \cong \overline{CP^2}$.

Outline of the proof of Lemma 3.1: See the Kirby calculus in Figure 2 again. It describes the process of operation $B$ near the 2-sphere $K$, but we have not done the final step yet. Doing the change in Lemma 5.1 to the final diagram, we finish the operation $B$ and get the first diagram in Figure 7 (The dotted circle corresponds to a meridian to $K$ in $M$. The thin circle corresponds to the boundary of a co-core of the 2-handle $h$. Once ignore them). The diagram describes a 4-manifold obtained by attaching a 2-handle $h$ to $N_2$. All we have to do is to verify that $(M \setminus \text{int}N(K)) \cup h^\perp \cong M \setminus \text{int}N(K \| P_0)$, where we use the notation $h^\perp$ for the piece $h$ since we switch the core and the co-core. See Figure 4(1) and the proof of Theorem 4.1 in [KSTY] for the goal.
By the calculus in Figure 7, we have the attaching circle of $h^\perp$ in $\partial(M\setminus \text{int}N(K)) \cong S^1 \times S^2$ and the framing: it is the thin circle in the diagram. (If one care orientation of the diagram, it would be better take the mirror image.) We have the lemma. □

References


YAMADA Yuichi
yyyamada@matha.e-one.uec.ac.jp
Dept. of Systems Engineering
The Univ. of Electro-Communications
1-5-1, Chofugaoka, Chofu,
Tokyo, 182-8585, JAPAN