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Incompressible Navier-Stokes equations in abstract Banach spaces

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Abstract

In this paper we present some existence and uniqueness theorems of global-in-time solutions to the Cauchy problem for the incompressible Navier-Stokes system with external forces and small initial conditions in abstract Besov type spaces. These results are compared with the analogous ones for the regularized Navier-Stokes system where the higher order viscosity term is added. The main result of the paper says that solutions of the regularized problem converge, as $t \to \infty$, toward suitable solutions of the classical Navier-Stokes equations.

1 Introduction

The Navier-Stokes equations describing the motion of an incompressible fluid in $I\!R^n$ $n \geq 2$, in presence of external forces are written as follows

$$v_t - \Delta v + (v \cdot \nabla)v + \nabla p = F(x,t) \tag{1.1}$$
$$\nabla \cdot v = 0 \tag{1.2}$$
$$v(0) = v_0. \tag{1.3}$$

Here, $v(x,t) = (v_1(x,t), ..., v_n(x,t))$ is the unknown velocity of the fluid at the point $x \in I\!R^n$ and time $t \geq 0$, and $p = p(x,t)$ is the unknown pressure. We assume that the Cauchy datum $v_0(x)$ is an n-dimensional real-valued vector function satisfying the compatibility condition $\nabla \cdot v_0 = 0$. Moreover, we postulate that the external force $F(x,t)$ arises from a potential $V(x,t)$ in such a way that

$$F = \nabla \cdot V \quad \text{i.e.} \quad F_j = \sum_{k=1}^{n} \partial_{x_k} V_{kj}, \quad j = 1, 2, ..., n. \tag{1.4}$$

The Navier-Stokes equations (1.1)–(1.3) are the starting point for most numerical simulations of turbulent fluids and very challenging mathematical problems arise from these equations. However, they are not always implemented precisely as stated in (1.1)–(1.3).

As pointed out by S. Tourville in [34, 35] "From the numerical point of view, one major reason for this deviation is that the number of degrees of freedom required for the direct numerical simulation increases with the complexity of the flow, characterised by the Reynolds number. In an effort to overcome such obstacles, researchers have considered modifications to the equations which allow the computation of more turbulent flows. For example, in such models the operator $-\Delta$, responsible for dissipating energy from the system, is replaced by a higher order dissipation mechanism which damps the high wave numbers more selectively. The operator $(-\Delta)^{\ell/2}$ for some $\ell > 2$ is a typical choice. The basic goal behind such modifications is to allow a smaller effective kinematic viscosity constant and thus a higher effective Reynolds number (see G.L. Browning and H.O. Kreiss 1989 [6], B. Fornberg 1977 [14], J.C. McWilliams, 1984 [23]). The Reynolds number of a flow can be thought of as a ratio of the energy of the nonlinear term $(v \cdot \nabla)v$ to the energy of the linear term $-\Delta$. Increasing the order of dissipation will decrease the strength of the linear term as can be seen on the Fourier transform side. [..] Therefore the presence of hyperdissipation results in a decrease of the energy of the linear term, hence an increase in the effective Reynolds number."

Another reason for introducing a hyperviscosity $(-\Delta)^{\ell/2}$, $\ell > 2$, comes from the remark that the Navier-Stokes equations are based on the assumption of Newtonian flows. For non Newtonian fluids, one may introduce hyperviscosity as in the papers of J.-L. Lions [27]–[29] or nonlinear viscosity as in the work of O. Ladyzhenskaya [26].

From the theoretical point of view, the uniqueness of the solution to eqs. (1.1)–(1.3) is one of the outstanding questions in hydrodynamics. According to M. Shinbrot [32] : "Instant fame awaits the person who answers it. (Especially if the answer is negative!). If it should turn out that solutions are not unique, that fact would qualify as a paradox and would justify complicating the model [...] There are two obvious ways in which the model can be changed. The first one is to allow the reduced stress tensor to depend on higher derivatives of the velocity than the first [...]. The second is to allow the reduced stress tensor to depend nonlinearly on the derivatives of the velocity. A piquant aspect is lent to the whole problem by the fact that in *either* case the solution can be shown to be unique ! For the first see O. Ladyzhenskaya [26], for the second see S. Kaniel [17]."

Before describing the system of equations we are interesting in, we should remind the reader that – as observed by H. Okamoto during the "International conference on Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics" – the modified Navier-Stokes equations where a hyperviscosity term $(-\Delta)^{\ell/2}$ replaces the Laplacian $-\Delta$ (e.g. [34, 35, 21]) are the most studied ones from the numerical point

Another interesting related model was considered by S. Kaniel in [17]. Here the Laplacian is replaced by a gradient of some special polynomial of the first derivatives of the velocity. If the order of that polynomial is 1, we obtain the classical Navier-Stokes system but, to get uniqueness, Kaniel needs to assume that polynomial is at least of degree 3.

Finally, in the pioneering papers by J.-L. Lions the Laplacian $-\Delta$ is replaced by the sum $-\Delta + (-\Delta)^{\ell/2}$, $\ell > 2$ (in a way that is reminiscent of a Taylor expansion), and precisely these modified Navier-Stokes equations will be considered in the present paper (see also [13]). For such a modified problem considered in a bounded domain, J.-L. Lions was able to prove (cf. [29, Chap. 1, Remarque 6.11]) the existence of a unique regular solution provided $\ell \geq (n+2)/2$. Hence, one can say that the hypperdissipative term $(-\Delta)^{\ell/2}$ in the equation smoothes out solutions for sufficiently large ℓ. In this paper our goal is precisely to show that, in the whole space $\mathbb{R}^n$, such a correction in the model disappears asymptotically as $t \to \infty$.

Let us now be more precise. We introduce the projection $\mathbb{P}$ of $(L^2(\mathbb{R}^n))^n$ onto the subspace $\mathbb{P}(L^2(\mathbb{R}^n))^n$ of the solenoidal vector fields (i.e. those characterized by the divergence condition (1.2)). It is known that $\mathbb{P}$ is a pseudodifferential operator of order 0. In fact, it can be written as a combination of the Riesz transforms R_j with symbols $\xi_j/|\xi|$,

$$\mathbb{P}(v_1, ..., v_n) = (v_1 - R_1\sigma, ..., v_n - R_n\sigma),$$

where $\sigma = R_1 v_1 + ... + R_n v_n$. Using $\mathbb{P}$ one can convert (1.1)–(1.4) into the evolution equation

$$v_t = \Delta v + \mathbb{P}\nabla(v \otimes v) + \mathbb{P}\nabla \cdot V. \tag{1.5}$$

Here, we replace $(v \cdot \nabla)v$ by $\nabla \cdot (v \otimes v) = \sum_{i=1}^{n} (v_i v)_{x_i}$ to avoid problems with the definition of products of distributions. This can be made because $\nabla \cdot v = 0$.

Now, given a Banach space E a solution $v(t)$ of (1.1)–(1.3) (or of (1.5), (1.3)) will be interpreted as an E-valued mapping defined on $[0, T)$. It is known, since the pioneering work of T. Kato and H. Fujita [20], that if Δ is the infinitesimal generator of a C_0-semigroup $S(t)$ on E, then sufficiently regular solutions of the Cauchy problem (1.1)–(1.3) satisfy the integral equation

$$\begin{aligned} v(t) &= S(t)v_0 - \int_0^t \mathbb{P}\nabla S(t-\tau) \cdot (v \otimes v)(\tau)\, d\tau \\ &\quad + \int_0^t \mathbb{P}\nabla S(t-\tau) \cdot V(\tau)\, d\tau. \end{aligned} \tag{1.6}$$

Here, $S(t)$ denotes the heat semigroup and the integrals with respect to τ in (1.6) are understood in the sense of Bochner. We skip the discussion, in which sense solutions of (1.6) solve equation (1.1) in a general context, because a deeper discussion of this problem may be found in [15, 16, 20, 22, 33, 38]. Throughout the remainder of this paper, by solutions to (1.1)–(1.3) we always mean solutions to (1.6).

Here, we would like to emphasize that, in our case, the heat semigroup $S(t) : E \to E$ will be a bounded operator for any $t \geq 0$. Moreover, for every $v_0 \in E$, $S(t)v_0 \in L^\infty((0,T); E)$, but $S(t)v_0$ will tend to v_0 as $t \searrow 0$ in the sense of $\mathcal{S}'$ only. In other words, we do not require that $S(t)$ is a strongly continuous semigroup on E. This does not allow us to apply the scheme of the existence proof directly from e.g.

[1, 12, 19, 20, 38, 39]. Hence, our solutions will be constructed in the space $C_*([0,T];E)$ consisting of E-valued measurable functions which are bounded on $[0,T]$ in the norm of E and which take the initial value v_0 as $t \searrow 0$ in the sense of tempered distributions.

One of the goals of this report is to show that the methods developed for the classical Navier-Stokes system (1.1)–(1.3) are well-adapted for the Navier-Stokes system with the higher order viscosity term

$$u_t - \Delta u + (-\Delta)^{\ell/2}u + (u\cdot\nabla)u + \nabla q = \nabla\cdot W \tag{1.7}$$

$$\nabla\cdot u = 0 \tag{1.8}$$

$$u(0) = u_0. \tag{1.9}$$

Here, the pseudodifferential operator $(-\Delta)^{\ell/2}$ is defined via the Fourier transform as follows

$$(\widehat{(-\Delta)^{\ell/2}w})(\xi) = |\xi|^\ell \widehat{w}(\xi)$$

and $\ell > 2$ is our standing assumption.

The idea of constructing solutions to the both of the models is the following. We impose the conditions on the Banach space E (cf. Definitions 2.1 and 2.2) which guarantee that our Cauchy problems have global-in-time solutions in the space $C_*([0,T),E)$ for some $T > 0$ (see Remark 2.1, below). We show, however, that a scaling property $\|\cdot\|_E$ allows us to obtain, moreover, global-in-time solutions for suitably small initial data. To get such results we introduce a new Banach space of distributions which, roughly speaking, is a homogeneous Besov type space modelled on E. This approach allows us to get solutions for initial data less regular than those from E. In this abstract setting, we also study large-time behavior of constructed solutions. We find a simple condition (in terms of decay properties of the heat semigroup) which guarantees that solutions have the same asymptotic behavior as $t \to \infty$.

It is not surprising that, in such an abstract setting, the theories on the existence of global-in-time solutions (and their large time behavior) to the both of the models, (1.1)–(1.3) and (1.7)–(1.9), are completely analogous. Below, in Theorems 3.1, 3.2, 4.1, and 4.2, we state this more precisely. However, the main result (and, hopefully, the new contribution to the theory) of this paper consists in showing that solutions of the problem with hyperdissipation (i.e. (1.7)–(1.9)) converge, as $t \to \infty$, toward suitable solutions of the classical Navier-Stokes system (1.1)–(1.3) corresponding to the same initial conditions and external forces (cf. Theorem 5.1, below). In other words, the higher order term $(-\Delta)^{\ell/2}$ with $\ell > 2$ is asymptotically negligible for large t.

All the details concerning the proofs of the results announced here will appear in a forthcoming paper [13].

Notation. Here, the heat semigroup $S(t)$ is defined as the convolution with the heat kernel: $S(t)v_0 = G(t) * v_0$ where $G(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/(4t))$. The usual Lebesgue space is denoted by $L^p(I\!R^n)$. Given a Banach space E its norm is usually denoted by $\|\cdot\|_E$. $\mathcal{S}(I\!R^n)$ is the space of Schwartz rapidly decreasing functions and $\mathcal{S}'(I\!R^n)$ is its dual space, i.e. the space of tempered distributions. In order to simplify the notations, we shall write, for instance, $L^p, \mathcal{S}, \mathcal{S}'$ instead of $(L^p(I\!R^n))^n$, $(\mathcal{S}(I\!R^n))^n$, $(\mathcal{S}'(I\!R^n))^n$ as well as for any Banach space $\mathcal{X}$ whose elements are n-dimensional vectors defined in $I\!R^n$. Given $T \in (0,\infty]$ and a Banach space $\mathcal{X} \subset \mathcal{S}'$, $C_*([0,T];\mathcal{X})$ consists of measurable and essentially bounded mappings $v : [0,T] \to \mathcal{X}$, such that $v(t) \to v(0)$ as $t \searrow 0$ in the topology of $\mathcal{S}'$.

2 Abstract Banach spaces

2.1 Adequate Banach space.

This is a usual situation when one deals with a semilinear evolution equation

$$u_t = \Delta u + J(u)$$

in a given Banach space E that the nonlinear mapping is "singular" in E, but $S(t)J$ is locally Lipschitz on E for each $t > 0$. There are several papers where this property was used to construct local-in-time solutions. Let us mention here the remarkable papers of T. Kato and H. Fujita [20], Y. Giga [16], Y. Meyer [24], M. Taylor [33], F.B. Weissler [38, 39], as well as the concept of well-suited spaces for the Navier-Stokes system introduced in [8, 10].

Our definition of an adequate space to the problem (1.1)–(1.3) (given below) is a variation of ideas contained in these papers. However, in our case, we shall also use it to construct global-in-time solutions for suitably small initial data. These ideas were previously introduced in [18].

Definition 2.1 *The Banach space $(E, \|\cdot\|_E)$ is said to be functional and translation invariant if the following three conditions are satisfied:*

i. *$\mathcal{S} \subset E \subset \mathcal{S}'$ and the both inclusions are continuous.*

ii. *For every $f \in E$, the mapping $\tau : \mathbb{R}^n \mapsto E$ defined on functions as $\tau_y f(x) = f(x+y)$ is measurable in the sense of Bochner with respect to the Lebesgue measure on $\mathbb{R}^n$.*

iii. *The norm $\|\cdot\|_E$ on E is translation invariant, i.e.*

$$\text{for all} \quad f \in E \quad \text{and} \quad y \in \mathbb{R}^n, \quad \|\tau_y f\|_E = \|f\|_E.$$

Definition 2.2 *We call the space $(E, \|\cdot\|_E)$ adequate to the problem (1.1)–(1.3) if*

i. *it is a functional translation invariant Banach space;*

ii. *for all $f, g \in E$, the product $f \otimes g$ is well-defined as the tempered distribution, moreover, there exist $T_0 > 0$ and a positive function $\omega \in L^1(0, T_0)$ such that*

$$\|\mathbb{P}\nabla S(\tau) \cdot (f \otimes g)\|_E \le \omega(\tau)\|f\|_E\|g\|_E \tag{2.1}$$

for every $f, g \in E$ and $\tau \in (0, T_0)$.

The proof of (2.1) for the space $E = L^p$ bases on the Hölder inequality combined with the well-known estimates for the heat semigroup and its derivatives. Indeed, if we recall that the Riesz transforms R_j are bounded on L^p for any $p \in (1, +\infty)$, we get the inequality

$$\|\mathbb{P}\nabla S(\tau) \cdot (f \otimes g)\|_{L^p} \le C\tau^{-(n/p+1)/2}\|f\|_{L^p}\|g\|_{L^p},$$

which is valid for any $\tau > 0$, $f, g \in L^p$, a constant C, and $p \geq 2$. Now, the assumption $p > n$ guarantees the integrability at 0 of $\tau^{-(n/p+1)/2}$. These calculations are very simple in the case of the space L^∞ because, as observed by T. Miyakawa in [25, Lemma 2.1],

$$\|I\!P\nabla \cdot G(t)\|_{L^1} \leq Ct^{-1/2}$$

for all $t > 0$ and a constant C. Finally, since we are interested in an incompressible flow, we can summarize the reasoning above by saying that the Banach space $I\!PL^p = \{f \in L^p : \nabla \cdot f = 0\}$ is adequate to the Navier-Stokes system (1.1)-(1.3) for every $p \in (n, \infty]$.

We refer the reader to the paper [18] for other examples of Banach spaces adequate to (1.1)-(1.3). Moreover, the well-suited spaces introduced in [8, 10] are functional translation invariant Banach spaces in the sense of our Definition 2.1 having some additional properties. In particular, they satisfy a slightly stronger condition than (2.1), so they are also adequate spaces in the sense of Definition 2.2 (see [10, Lemma 2.1]). Several examples of the well-suited spaces for the Navier-Stokes system (1.1)-(1.3) are also contained in [2, 8, 10].

Remark 2.1 Here, it is worth of emphasizing that if E is a well-suited Banach space (or, more generally, adequate for the problem (1.1)-(1.3)) then for any initial datum $v_0 \in E$, $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|_E)$ and the unique "mild" solution to the Navier-Stokes equations in the space $C_*([0,T); E)$. Details are contained in [10, Theorem 2.1]. $\square$

In this report, we use Banach spaces with norms having additional scaling properties. In order to state this more precisely, given $f : I\!R^n \to I\!R^n$, we define the rescaled function

$$f_\lambda(x) = f(\lambda x) \tag{2.2}$$

for each $\lambda > 0$. We extend this definition for all $f \in \mathcal{S}'$ in the standard way.

Definition 2.3 *Let $(E, \|\cdot\|_E)$ be a Banach space, which can be imbedded continuously in $\mathcal{S}'$. The norm $\|\cdot\|_E$ is said to have the scaling degree equal to k, if $\|f_\lambda\|_E = \lambda^k \|f\|_E$ for each $f \in E$ such that $f_\lambda \in E$ and for all $\lambda > 0$.*

It is evident that the usual norms of the spaces L^p (the Lebesgue space), L^p_w (the weak L^p-space or the Marcinkiewicz space), $L^{p,q}$ (the Lorenz space), $\mathcal{M}^p_q$ (the homogeneous Morrey space) have the scaling degree equal to $-n/p$ (more details on these spaces can be found e.g. in [18]). On the other hand, the standard norm in the homogeneous Sobolev space $\dot{H}^s = \{f \in \mathcal{S}' : |\xi|^s \hat{f}(\xi) \in L^2\}$ has scaling degree $s - n/2$.

Remark 2.2 In our considerations below, we systematically assume that the norms of Banach spaces have the scaling degree equals to some $k \in (-1, 0)$. Since the space L^p is our model example, to simplify the exposition, we shall assume that $k = -n/p$ with $p > n$. In this work, Banach spaces endowed with norms having this property will be usually denoted by E_p. $\square$

2.2 Besov type Banach space

Let us fix a Banach space $E \subset \mathcal{S}'$ and introduce a new space of distributions denoted by BE^α which, loosely speaking, is a homogeneous Besov space modelled on E. The definition we are going to introduce will be an important tool in the next sections, where global-in-time solutions will be constructed (for suitably small initial data) in $C_*([0,\infty); BE^\alpha)$.

Definition 2.4 *Let $\alpha \geq 0$. Given a Banach space E imbedded continuously in $\mathcal{S}'$, we define*

$$BE^\alpha = \{f \in \mathcal{S}' \, : \, \|f\|_{BE^\alpha} \equiv \sup_{t>0} t^{\alpha/2}\|S(t)f\|_E < \infty\}.$$

Let $E = L^p$ for a moment. It follows immediately from the estimates of the heat semigroup

$$\|S(t)f\|_{L^p} \leq C(p,q)t^{-n(1/q-1/p)/2}\|f\|_{L^q}$$

for each $1 \leq q \leq p \leq \infty$, that $L^q \subset BE_p^\alpha$ with $\alpha = n(1/q - 1/p)$. It is easy to obtain the analogous conclusions for the Marcinkiewicz, Lorentz, or Morrey spaces applying appropriate estimates of the heat semigroup mentioned in [18, Section 3].

Put again $E = L^p$ in Definition 2.4. In this case, the norm $\|\cdot\|_{BE^\alpha}$ is equivalent to the standard norm of the homogeneous Besov space $\dot{B}^{-\alpha}_{p,\infty}$ introduced via a dyadic decomposition (for a proof see [7, 8]). Moreover, since elements of $\dot{B}^{-\alpha}_{p,\infty}$ can be realized as tempered distributions and the imbedding $\dot{B}^{-\alpha}_{p,\infty} \subset \mathcal{S}'$ is continuous, it follows that $BE^\alpha = \dot{B}^{-\alpha}_{p,\infty}$. The standard references on the homogeneous Besov spaces are [5, 36, 37]. For constructions of equivalent norms on $\dot{B}^{\gamma}_{p,q}$, we refer the reader to [7] and to the references given there.

Remark 2.3 If E has a norm with scaling degree k, then $\|\cdot\|_{BE^\alpha}$ has degree $k-\alpha$. Indeed, first we observe that for any $f \in \mathcal{S}'$ and $\lambda > 0$,

$$S(t)f_\lambda = \big(S(\lambda^2 t)f\big)_\lambda. \tag{2.3}$$

Hence, the scaling property of the norm on E implies

$$\|f_\lambda\|_{BE^\alpha} = \sup_{t>0} t^{\alpha/2}\|S(t)f_\lambda\|_E = \lambda^{k-\alpha} \sup_{\lambda^2 t>0} (\lambda^2 t)^{\alpha/2}\|S(\lambda^2 t)f\|_E = \lambda^{k-\alpha}\|f\|_{BE^\alpha}.$$

□

Remark 2.4 To obtain global-in-time solutions to the semilinear heat equation and to the Navier-Stokes equations, Kozono and Yamazaki [22] used new function spaces, constructed in the same way as the Besov spaces, based on the Morrey spaces in place of the standard L^p. It seems to be reasonable to expect that, analogously as in the previous example, the norm $\|\cdot\|_{BE_p^\alpha}$ with $E_p = \mathcal{M}_q^p$ is equivalent to the norm introduced in [22, Def. 2.3]. □

3 Global-in-time solutions to the Navier-Stokes system

The goal of this section is to present two theorems – on the existence of the global-in-time solutions to (1.1)–(1.3) as well as on the large time behavior of these solutions.

Theorem 3.1 *Assume that E_p is the Banach space adequate to the problem (1.1)–(1.3) which norm has the order if scaling equal to $-n/p$ with $p>n$. There exists $\varepsilon>0$ such that for each $v_0 \in BE_p^{1-n/p}$ and $V(t)\in E_p$ satisfying*

$$\|v_0\|_{BE_p^{1-n/p}} + \sup_{t>0} t^{1-n/(2p)}\|V(t)\|_{E_p} < \varepsilon$$

the Cauchy problem (1.1)–(1.3) has a solution $v(x,t)$ in the space

$$(3.1)\; \mathcal{X} \equiv C_*([0,\infty) : BE_p^{1-n/p}) \cap \{v : (0,\infty) \to E_p : \sup_{t>0} t^{(1-n/p)/2}\|v(t)\|_{E_p} < \infty\}.$$

This is the unique solution satisfying the condition $\sup_{t>0} t^{(1-n/p)/2}\|v(t)\|_{E_p} \le 2\varepsilon$.

Theorem 3.2 *Let the assumptions from Theorem 3.1 remain valid. Assume that v and $\tilde{v}$ are two solutions of (1.1)–(1.3) constructed in Theorem 3.1 corresponding to the initial data $v_0, \tilde{v}_0 \in BE_p^{1-n/p}$ and $V(t), \tilde{V}(t) \in E_p$, respectively. Suppose that*

$$\lim_{t\to\infty} t^{(1-n/p)/2}\|S(t)(v_0-\tilde{v}_0)\|_{E_p} = 0 \tag{3.2}$$

and

$$\lim_{t\to\infty} t^{1-n/(2p)}\|V(t)-\tilde{V}(t)\|_{E_p} = 0. \tag{3.3}$$

Choosing $\varepsilon>0$ in Theorem 3.1 sufficiently small, we have

$$\lim_{t\to\infty} t^{(1-n/p)/2}\|v(\cdot,t)-\tilde{v}(\cdot,t)\|_{E_p} = 0.$$

Corollary 3.1 *Under the assumptions of Theorem 3.2, we also have*

$$\lim_{t\to\infty} \|v(t)-\tilde{v}(t)\|_{BE_p^{1-n/p}} = 0.$$

We obtain global-in-time solutions to the Cauchy problem (1.1)–(1.3) using the standard reasoning involving the integral equation

$$\begin{aligned} v(t) &= S(t)v_0 - \int_0^t I\!P\nabla S(t-\tau)\cdot(v\otimes v)(\tau)\,d\tau \\ &\quad + \int_0^t I\!P\nabla S(t-\tau)\cdot V(\tau)\,d\tau. \end{aligned} \tag{3.4}$$

Here we use T. Kato's favourite argument, say the classical Picard fixed point approach, which is based on the following lemma (for the proof, see [8]).

Lemma 3.1 *Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$ and $B:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ a bilinear operator, such that for a constant $\eta>0$ and any $x_1,x_2\in\mathcal{X}$, we have $\|B(x_1,x_2)\| \le \eta\|x_1\|\|x_2\|$. Then for every $y\in\mathcal{X}$ such that $4\eta\|y\|<1$ the equation $x=y+B(x,x)$ has a solution in $\mathcal{X}$. In particular the solution is such that $\|x\|\le 2\|y\|$ and it is the only one such that $\|x\|<\frac{1}{2\eta}$.*

For the proof of Theorem 3.1, we equip the space $\mathcal{X}$, defined in (3.1), with the norm

$$\|v\|_{\mathcal{X}} = \max\{\sup_{t>0} \|v(t)\|_{BE_p^{1-n/p}},\ \sup_{t>0} t^{(1-n/p)/2}\|u(t)\|_{E_p}\}$$

and we construct the solution $v \in \mathcal{X}$ by Lemma 3.1 applied to the equation $v = y_0 + B(v,v)$ with

$$y_0 = S(t)v_0 + \int_0^t I\!P\nabla S(t-\tau)\cdot V(\tau)\,d\tau \tag{3.5}$$

and the bilinear form

$$B(u,v) = -\int_0^t I\!P\nabla S(t-\tau)\cdot(u(\tau)\otimes v(\tau))\,d\tau.$$

Here, to conclude the proof, we need several inequalities involving the heat semigroup acting on Banach spaces adequate to the problem (1.1)-(1.3). Moreover, let us note that all estimates of the bilinear form $B(\cdot,\cdot)$ are derived directly from inequality (2.1) combined with the scaling property of the norm in E_p. These details will be gathered in [13].

The following lemma plays an important role in the proof of Theorem 3.2.

Lemma 3.2 *Let $w \in L^1(0,1)$, $w \geq 0$, and $\int_0^1 w(x)\,dx < 1$. Assume that f and g are two nonnegative, bounded functions such that*

$$f(t) \leq g(t) + \int_0^1 w(\tau)f(\tau t)\,d\tau. \tag{3.6}$$

Then $\lim_{t\to\infty} g(t) = 0$ implies $\lim_{t\to\infty} f(t) = 0$.

We refer the reader to [18] for the elementary proof of this lemma. Now, to show Theorem 3.2, we apply Lemma 3.2 with $f(t) = t^{(1-n/p)/2}\|v(t)-\tilde{v}(t)\|_{E_p}$ and

$$g(t) = t^{(1-n/p)/2}\|S(t)(v_0-\tilde{v}_0)\|_{E_p} + t^{(1-n/p)/2}\left\|\int_0^t I\!P\nabla S(t-\tau)\cdot(V(\tau)-\tilde{V}(\tau))\,d\tau\right\|_{E_p}.$$

Here, estimates which appear in the proof of Theorem 3.1 play again the crucial role in our reasoning.

4 Regularized Navier-Stokes system

Now, let us formulate two theorems on the global-in-time existence of solutions to the Navier-Stokes system with the hyperdissipation (1.7)-(1.9) as well as on their large time behavior.

Theorem 4.1 *Under the assumptions and the notation of Theorem 3.1, there exists $\varepsilon_1 \in (0,\varepsilon]$ such that for each $u_0 \in BE_p^{1-n/p}$ and $W(t) \in E_p$ satisfying*

$$\|u_0\|_{BE_p^{1-n/p}} + \sup_{t>0} t^{1-n/(2p)}\|W(t)\|_{E_p} < \varepsilon_1$$

the Cauchy problem (1.7)-(1.9) has a solution $u(x,t)$ in the space $\mathcal{X}$ defined in (3.1). This is the unique solution satisfying the condition $\sup_{t>0} t^{(1-n/p)/2}\|u(t)\|_{E_p} \leq 2\varepsilon$.

Theorem 4.2 *Let the assumptions from Theorem 4.1 hold. Assume that u and $\tilde{u}$ are two solutions of (1.7)-(1.9) constructed in Theorem 4.1 corresponding to the initial data $u_0, \tilde{u}_0 \in BE_p^{1-n/p}$ and $W(t), \tilde{W}(t) \in E_p$, respectively. Suppose that*

$$\lim_{t\to\infty} t^{(1-n/p)/2}\|S(t)(u_0 - \tilde{u}_0)\|_{E_p} = 0 \tag{4.1}$$

and

$$\lim_{t\to\infty} t^{1-n/(2p)}\|W(t) - \tilde{W}(t)\|_{E_p} = 0.$$

Choosing $\varepsilon_1 > 0$ in Theorem 4.1 sufficiently small, we have

$$\lim_{t\to\infty} t^{(1-n/p)/2}\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{E_p} = 0.$$

In the case of the regularized system (1.7)–(1.9), the counterpart of the integral equation (3.4) has the following form

$$\begin{aligned} u(t) &= S_\ell(t)S(t)u_0 - \int_0^t I\!P\nabla S_\ell(t-\tau)S(t-\tau)\cdot(u\otimes u)(\tau)\,d\tau \\ &\quad + \int_0^t I\!P\nabla S_\ell(t-\tau)S(t-\tau)\cdot W(\tau)\,d\tau. \end{aligned} \tag{4.2}$$

In the formula above, the semigroup generated by the operator $(-\Delta)^{\ell/2}$ is denoted by $S_\ell(t)$, and it is given by the convolution with the kernel

$$G_\ell(x,t) = \int_{I\!R^n} e^{-t|\xi|^\ell + ix\cdot\xi}\,d\xi.$$

Note that $G_2(x,t)$ corresponds to the Gauss-Weierstrass kernel $G(x,t)$. Moreover, let us recall that the kernel

$$G_\ell(x,t) = \int_{I\!R^n} e^{-t|\xi|^\ell + ix\cdot\xi}\,d\xi = t^{-n/\ell}G(x/t^{1/\ell}, 1)$$

is integrable for every $\ell > 0$ and all $t > 0$. Here, the self-similar form of G_ℓ implies that

$$\|G_\ell(\cdot,t)\|_{L^1(I\!R^n)} = \|G_\ell(\cdot,1)\|_{L^1(I\!R^n)} \geq 1.$$

Hence, if E is a functional translation invariant Banach space (cf. Definition 2.1), we have

$$\|S_\ell(t)S(t)f\|_E \leq C\|S(t)f\|_E \tag{4.3}$$

for all $f \in E$, $t > 0$, and a constant $C \geq 1$ independent of t (see [13], for details). This implies that all estimates needed in the proof of Theorems 3.1 and 3.2 remain true, if we replace the heat semigroup $S(t)$ by $S_\ell(t)S(t)$. One should remember, however, that constants in all inequalities may increase in such a new setting. Finally, if we recall that solutions to the regularized Navier-Stokes system (1.7)-(1.9) satisfy the integral equation (4.2), we obtain immediately that the proofs of Theorems 4.1 and 4.2 are completely analogous to their counterparts from Section 3.

Remark 4.5 Let us emphasize that the argument described above implies that every Banach space adequate to the Navier-Stokes system (1.1)-(1.3) is also adequate to the system with hyperdissipation (1.7)-(1.9). In other words, if inequality (2.1) holds true for the heat semigroup $S(t)$ and a functional Banach space E, it is also true for $S(\tau)$ replaced by $S_\ell(\tau)S(\tau)$ and $\omega(\tau)$ replaced by $C\omega(\tau)$. □

5 Asymptotic equivalence of both models

Now, we would like to compare solutions to the models (1.1)–(1.3) and (1.7)–(1.9). Our main result says that, as $t \to \infty$, solutions to the regularized system (1.7)–(1.9) converge towards solutions of the classical Navier-Stokes equations with the same initial data and external forces.

Theorem 5.1 *Let the assumptions from Theorems 3.1 and 4.1 hold true. Assume that $n\ell/2 < p$. Let $v_0 \in BE_p^{1-n/p}$ and $V(t) \in E_p$ satisfy*

$$\|v_0\|_{BE_p^{1-n/p}} + \sup_{t>0} t^{1-n/(2p)} \|V(t)\|_{E_p} < \varepsilon_1,$$

where ε_1 is taken from Theorem 4.1. Denote by $v(x,t)$ and $u(x,t)$ the unique solutions to (1.1)-(1.3) and to (1.7)-(1.9), respectively, the both corresponding to the initial datum v_0 and the external potential $V(t)$. Then

$$\lim_{t\to\infty} t^{(1-n/p)/2} \|v(\cdot,t) - u(\cdot,t)\|_{E_p} = 0, \tag{5.1}$$

provided ε and ε_1 from Theorems 3.1 and 4.1 are sufficiently small.

The crucial lemma in the proof of Theorem 5.1 says that the semigroup generated by the operator $\Delta - (-\Delta)^{\ell/2}$ can be well-approximated in $L^1(\mathbb{R}^n)$ by the heat semigroup $S(t)$.

Lemma 5.1 *Let $\ell > 0$. There exists a constant C independent of t such that*

$$\|G_\ell(t) * G(t/2) - G(t/2)\|_{L^1(\mathbb{R}^n)} \le Ct^{-(1/2-1/\ell)}$$

for all $t > 0$.

Using this lemma we are able to derive an integral inequality of the form (3.6) for the function $f(t) = t^{(1-n/p)/2}\|v(\cdot,t) - u(\cdot,t)\|_{E_p}$. It is important in computations that the function v and u satisfy the integral equations (3.4) and (4.2), respectively. Finally, Lemma 3.2 completes the proof.

6 Self-similar solutions

To explain better the asymptotic results contained in Theorems 3.2, 4.2, and 5.1, we apply them to *self-similar solutions.* In this section, to simplify the exposition, we assume that there are no external forces in our models, i.e. $V \equiv W \equiv 0$.

Note first that if a function v solves (1.1) then for each $\lambda > 0$ the rescaled function $\Lambda v(x,t) \equiv \lambda v(\lambda x, \lambda^2 t)$ is also a solution of (1.1). The solutions satisfying the scaling invariance property $\Lambda v \equiv v$, for any $\lambda > 0$, are called *forward self-similar solutions.* Obviously, they are global in time. It is expected that they describe large-time behavior of general solutions, because if the limit relation $\lim_{\lambda\to\infty} \lambda v(\lambda x, \lambda^2 t) = U(x,t)$ exists in an appropriate sense, then $t^{1/2}v(xt^{1/2}, 1) \to U(x,1)$ as $t \to \infty$, and U satisfies the invariance property $\Lambda U \equiv U$. Hence, U is a self-similar solution and

$$U(x,t) = t^{-1/2}U(xt^{-1/2}, 1). \tag{6.1}$$

Let us observe that if $v_0(x) = \lim_{t\to 0} t^{-1/2}U(xt^{-1/2}, 1)$ exists, then v_0 is necessarily homogeneous of degree -1. On the other hand, a self-similar solution to (1.1)-(1.3) can be obtained directly from Theorem 3.1, taking v_0, homogeneous of degree -1 and suitably small. Recall that such a reasoning for the Navier-Stokes equations appears in [3, 4, 8, 9, 11, 30, 31, 33]. Moreover, this argument applies without changes to more general equations (cf. [18]). The following corollary is a direct consequence of that reasoning.

Corollary 6.1 *Under the assumption of Theorem 3.1, if v_0 is homogeneous of degree -1, then the solution of (1.1)-(1.3) has the form $v(x,t) = t^{-1/2}U(xt^{-1/2})$, i.e. v is self-similar.*

Now, by Theorem 3.2, every self-similar solution $t^{-1/2}U(xt^{-1/2})$ describes the large time behavior of those general solutions whose initial data v_0 satisfy (3.2), where $\tilde{v}_0(x) = \lim_{t\to 0} t^{-1/2}U(xt^{-1/2})$ in the sense of distributions. For a deeper discussion of condition (3.2) with $E_p \in \{L^p, L^p_w\}$, we refer the reader to [4, 30, 31], where the large time behavior of solutions to the Navier-Stokes system (1.1)-(1.2) in the three-dimensional case was described by self-similar solutions.

The regularized Navier-Stokes system (1.7)-(1.9) is not invariant under rescaling $\lambda u(\lambda x, \lambda^2 t)$, hence, it has no self-similar solutions of the form (6.1). However, there is a large class of solutions to (1.7)-(1.9) for which the large time behavior is self-similar. This is the immediate consequence of Theorems 3.2, 4.2, and 5.1, and it is worth of stating more precisely.

Corollary 6.2 *Let the assumptions of Theorem 5.1 hold true. Assume that $v_0 \in BE_p^{1-n/p}$ is homogeneous of degree -1. Denote by $U(x,t)$ the self-similar solution to (1.1)-(1.3) with v_0 as the initial datum and $V \equiv 0$. Denote by $u(x,t)$ the solution to the regularized system (1.7)-(1.9) corresponding to $u_0 \in BE_p^{1-n/p}$ and $W(t) \in E_p$. Suppose that*

$$\lim_{t\to\infty} t^{(1-n/p)/2} \|S(t)(u_0 - v_0)\|_{E_p} = 0$$

and

$$\lim_{t\to\infty} t^{1-n/(2p)} \|W(t)\|_{E_p} = 0.$$

Choosing $\varepsilon > 0$ and $\varepsilon_1 > 0$ in Theorems 3.1 and 4.1 sufficiently small, we have

$$\lim_{t\to\infty} t^{(1-n/p)/2} \|u(\cdot,t) - t^{-1/2}U(\cdot t^{-1/2})\|_{E_p} = 0. \tag{6.2}$$

As the conclusion, let as emphasize that, in the asymptotic relation (6.2), the self-similar solutions to the Navier-Stokes system describe the large time behavior of some solutions to the regularized system (1.7)-(1.9). This is one of the reasons why we call the term $(-\Delta)^{\ell/2}$ asymptotically negligible as $t \to \infty$ for $\ell > 2$.

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