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On the Well-Posedness of the Euler Equations in the Besov and the Triebel-Lizorkin Spaces

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Abstract

We prove the local in time unique existence and the blow-up criterion of solutions in the Besov($B^s_{p,q}$) and the Triebel-Lizorkin spaces($F^s_{p,q}$) for the Euler equations of inviscid incompressible fluid flows in $\mathbb{R}^n$, $n=2,3$. We consider both the super-critical($s>n/p+1$) and the critical($s=n/p+1$) cases. For the 2-D Euler equations we obtain the global persistence of the initial data regularity characterized by these spaces. In order to prove these results we establish the logarithmic inequality of the Beale-Kato-Majda type, the Moser type of inequality as well as the commutator estimate in these function spaces. The key methods of the proof of these estimates are the Littlewood-Paley decomposition and the paradifferential calculus by J.M. Bony.

1 Introduction and Main Results

We are concerned on the Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^n$, $n=2,3$.

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p,\quad (x,t)\in\mathbb{R}^n\times(0,\infty) \tag{1.1}$$

$$\text{div }v=0,\quad (x,t)\in\mathbb{R}^n\times(0,\infty) \tag{1.2}$$

$$v(x,0)=v_0(x),\quad x\in\mathbb{R}^n \tag{1.3}$$

where $v = (v_1, \cdots, v_n)$, $v_j = v_j(x,t)$, $j = 1, \cdots, n$, is the velocity of the fluid flows, $p = p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity satisfying div $v_0 = 0$. Given $v_0 \in H^m(\mathbb{R}^n)$, $m > \frac{n}{2} + 1$, Kato proved the local in time existence and uniqueness of solution in the class $C([0,T]; H^m(\mathbb{R}^n))$, where $T = T(\|v_0\|_{H^m})$[15]. This local existence of solutions in $\mathbb{R}^n$ has been extended to the fractional order Sobolev space $L^s_p(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$, $s > n/p + 1$, by Kato and Ponce[17], using new commutator estimate in this space. On the other hand, Lichtenstein[22] established local existence in the Hölder space $C^{1,\gamma}(\mathbb{R}^n)$. Later, Kato refined the proof and extended the local existence ressult to the Hölder space in a bounded domain[16]. (See [11] for another local existence proof in $\mathbb{R}^n$.) One of the most outstanding open problems in the mathematical fluid mechanics is to prove the global in time continuation of the local solution, or to find an initial data $v_0 \in H^m(\mathbb{R}^n)$ such that the associated local solution blows up in finite time for $n = 3$. Of the significant achievements in this direction is the Beale-Kato-Majda criterion[3] for the finite time blow-up of solutions, which states

$$\limsup_{t \nearrow T_*} \|v(t)\|_{H^m} = \infty \tag{1.4}$$

if and only if

$$\int_0^{T_*} \|\omega(s)\|_{L^\infty} ds = \infty, \tag{1.5}$$

where ω =curl v is the vorticity of the flows. Recently this criterion has been refined by Kozono and Taniuch[20], replacing the L^∞ norm of vorticity by the BMO norm for the vorticity, and $H^m(\mathbb{R}^n)$ by $W^{s,p}(\mathbb{R}^n)$ for the velosity. (We recall the continuous imbedding relation, $L^\infty \hookrightarrow BMO$. See e.g. [26] for detailed description of the space BMO.) In [2](See also [10].) Bahouri and Dehman also obtained the blow-up criterion for the local solution in the Hölder space. For the Euler equations in $n = 2$ it is well-known that the above local solutions can be continued beyond any finite time, and one of the main questions in this case is persistence problem of the regularity of initial data. Kato and Ponce proved the persistence of the fractional Sobolev space regularity for the super critical Sobolev space initial data, i.e $v_0 \in W^{s,p}(\mathbb{R}^2)$ with $s > 2/p + 1$[17],[18]. In the current paper we study the initial value problem for the initial data belonging to the Besov and the Triebel-Lizorkin spaces, namely and obtain a finite time blow-up criterion in these spaces as well as the local in time existence of solutions. We recall, in particular, that the Triebel-Lizorkin space is a unification of most of the classical function spaces used in the partial differential equations; we just note here $W^{s,p}(\mathbb{R}^n) = F^s_{p,2}$, and $C^s(\mathbb{R}^n) = F^s_{\infty,\infty}$ for $s > 0$ (See e.g. [27],[28]). Moreover, our criterion is sharper than the Beale-Kato-Majda's[3] and the Kozono-Taniuch's result[20], in the sense that the BMO norm of vorticity is replaced by $\dot{B}^0_{\infty,\infty} = \dot{F}^0_{\infty,\infty}$ norm, which is even weaker than the BMO norm (namely, $BMO = \dot{F}^0_{\infty,2} \hookrightarrow \dot{F}^0_{\infty,\infty}$. As a corollary of our criterion we prove the global in time existence and persistence of the Besov and the Triebel-Lizorkin space regularity in the 2-dimensional Euler equations for $v_0 \in F^s_{p,q}$, $s > 2/p + 1$. This result obviously includes the result by Kato and Ponce in [18]. We also obtain the global persistence of initial data in the 2-D Euler equations for the critical Triebel-Lizorkin spaces. This result is not immediate from the blow-up criterion for the critical spaces, and we need new estimate on the voticities of 2-D Euler flows, which is based on the logarithmic type of composite mapping estimate(See Proposition 3.2 below.), which is a generalization of the previous estimate of Vishik[30]. The followings are our main theorems.

Theorem 1.1 (Super-Critical Besov and Triebel-Lizorkin Spaces)

(i) *Local in time existence: Let $s > n/p + 1$ with $p, q \in [1, \infty]$ Suppose $v_0 \in B^s_{p,q}$, (resp. $F^s_{p,q}$) satisfying div $v_0 = 0$, is given. Then, there exists $T = T(\|v_0\|_{B^s_{p,q}})$(resp. $T = T(\|v_0\|_{F^s_{p,q}})$)such that a unique solution $v \in C([0,T]; B^s_{p,q})$ (resp. $v \in C([0,T]; F^s_{p,q})$) of the system (1.1)-(1.3) exists.*

(ii) *Blow-up criterion: Let s, p, q, v_0 be given as in the above. Then, the local in time solution $v \in C([0,T]; B^s_{p,q})$ (resp. $v \in C([0,T]; F^s_{p,q})$) constructed in (i) blows up at $T_* > T$ in $B^s_{p,q}$, namely*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{B^s_{p,q}} = \infty \tag{1.6}$$

(resp. $\limsup_{t \nearrow T_*} \|v(t)\|_{F^s_{p,q}} = \infty$) if and only if

$$\int_0^{T_*} \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty. \tag{1.7}$$

(resp. $\int_0^{T_*} \|\omega(t)\|_{\dot{F}^0_{\infty,\infty}} dt = \infty$)

Remark 1.1 Since $F^s_{p,2} = W^{s,p}(\mathbb{R}^n)$, the local existence in Theorem 1.1(i) includes the corresponding result by Kato and Ponce in [18],[19]. Also, since $F^s_{\infty,\infty} = C^s(\mathbb{R}^n)$, the Hölder space, it extends the result by Chemin in [11].

Remark 1.2 From the continuous imbeddings, $L^\infty \hookrightarrow BMO \hookrightarrow \dot{F}^0_{\infty,\infty}$, we find that Theorem 1.1(ii) improves the original Beale-Kato-Majda criterion[3], and its refined version by Kozono and Taniuchi[19]. On the other hand, the result of blow-up criterion in the Hölder space by Bahouri and Dehman[2] corresponds to an extreme case of Theorem 1.1(ii). We mention that in [21] Kozono-Ogawa-Taniuch obtained similar blow-up criterion as described in Theorem 1.1 but using the standard Sobolev norm in (1.6).

Remark 1.3 In the 2-D Euler equations the above blow-up criterion, combined with the global presevation of $\|\omega(t)\|_{L^\infty}$(See e.g. [29] for existence

and uniqueness of weak solution satisying this conservation of voticity.) implies the global persistence of initial data regularity for $v_0 \in B^s_{p,q}$(or $F^s_{p,q}$), $s > 2/p + 1$. We thus recover the results in [17].
The following result is on the study of the similar problems to the above, but in the critical Besov spaces. We note first that the space $B^s_{p,1}$ with $s = n/p$ is "barely" imbedded in $L^\infty(\mathbb{R}^n)$.

Theorem 1.2 (Critical Besov spaces for the n-D Euler)

(i) Local in time existence: *Let $s = n/p + 1$ with $p \in [1, \infty]$. Suppose $v_0 \in B^s_{p,1}$, satisfying div $v_0 = 0$, is given. Then, there exists $T = T(\|v_0\|_{B^s_{p,1}})$ such that a unique solution $v \in C([0,T]; B^s_{p,1})$ of the system (1.1)-(1.3) exists.*

(ii) Blow-up criterion: *Let s, p, q, v_0 be given as in the above. Then, the local in time solution $v \in C([0,T]; B^s_{p,1})$ constructed in (i) blows up at $T_* > T$ in $B^s_{p,1}$, namely*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{B^s_{p,1}} = \infty \tag{1.8}$$

if and only if

$$\int_0^{T_*} \|\omega(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty. \tag{1.9}$$

Next we present our results on the study of Euler equations in the critical Tribel-Lizorkin spaces. We first note that the following ("bare") imbedding relations, which are easy to establish.

$$F^n_{1,q} \hookrightarrow B^s_{p,1} \hookrightarrow B^0_{\infty,1} \hookrightarrow F^0_{\infty,1} \hookrightarrow L^\infty. \tag{1.10}$$

Theorem 1.3 (Critical Triebel-Lizorkin spaces for the n-D Euler)

(i) Local in time existence: *Let $q \in [1, \infty]$ Suppose $v_0 \in F^{n+1}_{1,q}$, satisfying div $v_0 = 0$, is given. Then, there exists $T = T(\|v_0\|_{F^{n+1}_{1,q}})$ such that a unique solution $v \in C([0,T]; F^{n+1}_{1,q})$ of the system (1.1)-(1.3) exists.*

(ii) Blow-up criterion: *Let s, p, q, v_0 be given as in the above. Then, the local in time solution $v \in C([0,T]; F^{n+1}_{1,q})$ constructed in (i) blows up at $T_* > T$ in $F^{n+1}_{1,q}$, namely*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{F^{n+1}_{1,q}} = \infty \tag{1.11}$$

if and only if

$$\int_0^{T_*} \|\omega(t)\|_{\dot{F}^0_{\infty,1}} dt = \infty. \tag{1.12}$$

The study of global existence problem of the 2-D Euler equations for the critical Besov space is studied originally by Vishik in [30],[31]. Here we present our result on the similar problem for the critical Triebel-Lizorkin space.

Theorem 1.4 (Critical Triebel-Lizorkin spaces for the 2-D Euler) *Let $q \in [1, \infty]$, and let $v_0 \in F^3_{1,q}$, satisfying div $v_0 = 0$, be given. Then, there exists a unique solution $v \in C([0,\infty); F^3_{1,q})$ to the system (1.1)-(1.3) with $n = 2$. Moreover, the solution satisfies the following global in time estimate.*

$$\|\omega(t)\|_{F^2_{1,q}} \leq \|\omega_0\|_{F^2_{1,q}} \exp\left[C \exp\{C(1+\|\omega_0\|_{F^2_{1,q}})t\}\right], \tag{1.13}$$

for all $t \geq 0$ in both of the cases.

We outline the key steps of proofs of Theorem 1.1-1.4. The details of the proofs are in [7]-[9]. Our study is concentrated on the incompressible Euler equations. We mention that the study of the incompressible Navier-Stokes equations in the critical Besov spaces, where the "criticality" is different from ours, was done extensively by Cannone and his collaborators(See [5],[6], and the references therein.).

2 Function Spaces

We first set our notations, and recall definitions on the Besov spaces and the Triebel-Lizorkin spaces. We follow [27] and [28]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$ its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying $\mathrm{Supp}\,\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2\}$, and $\hat{\varphi}(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$. Setting $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$ (In other words, $\varphi_j(x) = 2^{jn}\varphi(2^j x)$.), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j\in\mathbf{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j\geq k+1} \hat{\varphi}_j(\xi).$$

In particular we set $\hat{S}_{-1}(\xi) = \hat{\Phi}(\xi)$. We observe

$$\mathrm{Supp}\ \hat{\varphi}_j \cap \mathrm{Supp}\ \hat{\varphi}_{j'} = \emptyset \text{ if } |j - j'| \geq 2.$$

Let $s \in \mathbb{R}$, $p, q \in [0, \infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f = \varphi_j * f$, and then the homogeneous Besov norm $\|f\|_{\dot{B}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left[\sum_{-\infty}^{\infty} 2^{jqs} \|\varphi_j * f\|_{L^p}^q\right]^{\frac{1}{q}} & \text{if } q \in [1, \infty) \\ \sup_j \left[2^{js} \|\varphi_j * f\|_{L^p}\right] & \text{if } q = \infty \end{cases}.$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is a semi-normed space with the semi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s > 0$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}.$$

The inhomogeneous Besov space is a Banach space equipped with the norm, $\|\cdot\|_{B^s_{p,q}}$. The homogeneous Tribel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jqs} |\Delta_j f(\cdot)|^q\right)^{\frac{1}{q}} \right\|_{L^p} & \text{if } q \in [1, \infty) \\ \left\| \sup_{j \in \mathbb{Z}} (2^{js} |\Delta_j f(\cdot)|) \right\|_{L^p} & \text{if } q = \infty \end{cases}.$$

The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is a semi-normed space with the semi-norm given by $\|\cdot\|_{\dot{F}^s_{p,q}}$. For $s > 0$, $(p, q) \in (1, \infty) \times [1, \infty]$ we define the inhomogeneous Triebel-Lizorkin space norm $\|f\|_{F^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{F^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$

The inhomogeneous Triebel-Lizorkin space is a Banach space equipped with the norm, $\|\cdot\|_{F^s_{p,q}}$.

3 Key Estimates and Outline of the Proofs

The main ingredients of the proof of our main theorems are the followings.

(i) Moser type of inequalities in the Besov and Triebel-Lizorkin spaces

(ii) Commutator type of estimates

(iii) Beale-Kato-Majda type of inequalities

(iv) Composition mapping estimate(for critical spaces)

On the other hand, the basic tools used in the proof of the above estimates are Bony's paraproduct formula, Young's inequality, Minkowski's inequality, and Berstein's inequality.

(i) Moser type of inequalities:

Lemma 3.1 *Let $s > 0$, $(p,q) \in [1,\infty]^2$, then there exists a constant C such that the following inequalities hold.*

$$\|fg\|_{\dot{F}^s_{p,q}} \le C(\|f\|_{L^{p_1}}\|g\|_{\dot{F}^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{\dot{F}^s_{r_2,q}}),$$

$$\|fg\|_{F^s_{p,q}} \le C(\|f\|_{L^{p_1}}\|g\|_{F^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{F^s_{r_2,q}}).$$

Similarly for the Besov space norm. Here $p_1, r_1 \in [1,\infty]$ such that $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$.

The proof uses Bony's formula[4] for paraproduct of two functions is

$$fg = T_f g + T_g f + R(f,g),$$

where we set

$$T_f g = \sum_j S_{j-2} f \Delta_j g, \quad T_g f = \sum_j S_{j-2} g \Delta_j f,$$

and

$$R(f,g) = \sum_{|i-j|\le 1} \Delta_i f \Delta_j g.$$

In the case of the Besov norm estimate we use Young's inequality for the convolution and the Hölder inequalty to estimate each term. For the Triebel-Lizorkin norm estimate we use the following *mixed type of the Minkowski inequality*:

$$\begin{aligned}\left\| \left(\sum_{j\in\mathbf{Z}} \left[\int_{\mathbb{R}^n} |f_j(z,\cdot)| dz \right]^q \right)^{\frac{1}{q}} \right\|_{L^p} &\le \left\| \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbf{Z}} |f_j(z,\cdot)|^q \right)^{\frac{1}{q}} dz \right\|_{L^p} \\ &\le \int_{\mathbb{R}^n} \left\| \left(\sum_{j\in\mathbf{Z}} |f_j(z,\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p} dz,\end{aligned}$$

for $(p,q) \in [1,\infty]^2$ and the vector Maximal inequality due to Fefferman-Stein[12]:

$$\left\| \left(\sum_j |Mf_j(\cdot)|^q \right)^{1/q} \right\|_{L^p} \le C \left\| \left(\sum_j |f_j(\cdot)|^q \right)^{1/q} \right\|_{L^p},$$

where the maximal function Mf is defined by

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mathrm{vol}\{B(x,r)\}} \int_{B(x,r)} |f(y)| dy.$$

(ii) Commutator type of estimate:

Lemma 3.2 *Let $\omega = curl v$, $s > 0$, $(p,q) \in [1,\infty]^2$, then*

$$\left\| \left(\sum_{j\in\mathbb{Z}} 2^{jqs} \left| [(S_{j-2}v\cdot\nabla)\Delta_j\omega - \Delta_j((v\cdot\nabla)\omega)] \right|^q \right)^{\frac{1}{q}} \right\|_{L^p} \le C\|\nabla v\|_{L^\infty}\|\omega\|_{\dot{F}^s_{p,q}}.$$

Similarly for the Besov space norm.

The proof of this lemma also uses Bony's paraproduct formula. Using that formula, we decompose

$$\begin{aligned}(S_{j-2}v\cdot\nabla)\Delta_j\omega_k - \Delta_j((v\cdot\nabla)\omega_k &= -\sum_{i=1}^n \Delta_j T_{\partial_i\omega_k} v_i \\ &+ \sum_{i=1}^n [T_{v_i}\partial_i, \Delta_j]\omega_k - \sum_{i=1}^n T_{v_i - S_{j-2}v_i}\partial_i\Delta_j\omega_k \\ &- \sum_{i=1}^n \{\Delta_j R(v_i, \partial_i\omega_k) - R(S_{j-2}v_i, \Delta_j\partial_i\omega_k)\}\end{aligned}$$

which was originally used by Bahouri and Chemin[1]. Now, we estimate each term by using the Minkowski inequality and the vector maximal inequality. We also use the following two well-known results.

$$\|D^k f\|_{L^p} \sim 2^{jk}\|f\|_{L^p},$$

if Supp $\hat{f} \subset \{2^{j-2} \le |\xi| < 2^j\}$ which is called Bernstein's Lemma[10] for the Besov space estimate. On the other hand for the Triebel-Lizorkin space estimate we use

$$\|f\|_{\dot{F}^{s+k}_{p,q}} \sim \|D^k f\|_{\dot{F}^s_{p,q}}.$$

(See e.g. [13].) Now we outline the proof of a priori estimate leading to prove the local existence and the blow-up criterion. We consider the following vorticity formulation of the Euler equations

$$\frac{\partial\omega}{\partial t} + (v\cdot\nabla)\omega = \omega\cdot\nabla v, \quad (x,t)\in\mathbb{R}^3\times(0,\infty), \tag{3.1}$$
$$v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y,t)}{|x-y|^3}dy$$

for $n = 3$. In the case of $n = 2$ the vorticity formulation for $\omega = \partial_{x_1}v_2 - \partial_{x_2}v_1$ is

$$\frac{\partial\omega}{\partial t} + (v\cdot\nabla)\omega = 0, \quad (x,t)\in\mathbb{R}^2\times(0,\infty), \tag{3.2}$$

$$v(x) = K*\omega, \qquad K(x) = \frac{1}{2\pi|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

where $*$ denotes the convolution operation in $\mathbb{R}^2$,

$$(f * g)(x) = \int_{\mathbb{R}^2} f(x-y)g(y)dy.$$

Taking operation, Δ_j on the both sides of (3.1), we have

$$\partial_t \Delta_j \omega + (S_{j-2}v \cdot \nabla)\Delta_j \omega = (S_{j-2}v \cdot \nabla)\Delta_j \omega - \Delta_j((v \cdot \nabla)\omega).$$

Next, consider particle trajectory mapping $\{X_j(\alpha, t)\}$ defined by

$$\begin{aligned} \frac{\partial}{\partial t} X_j(\alpha, t) &= (S_{j-2}v)(X_j(\alpha, t), t) \\ X_j(\alpha, 0) &= \alpha. \end{aligned}$$

We note that div $S_{j-2}v = 0$ implies each $\alpha \mapsto X_j(\alpha, t)$ is a volume preserving mapping. Integrating along the trajectories, we obtain

$$\begin{aligned} |\Delta_j \omega(X_j(\alpha, t), t)| &\le |\Delta_j \omega_0(\alpha)| + \int_0^t |\Delta_j((\omega \cdot \nabla)v)(X_j(\alpha, \tau), \tau)| d\tau \\ &\quad + \int_0^t |[(S_{j-2}v \cdot \nabla)\Delta_j \omega - \Delta_j((v \cdot \nabla)\omega)](X_j(\alpha, \tau), \tau)|\, d\tau. \end{aligned}$$

Multiplying both sides by 2^{js}, and taking l^q norm, we deduce by the Minkowski inequality:

$$\begin{aligned} \left(\sum_{j \in \mathbf{Z}} 2^{jqs} |\Delta_j \omega(X_j(\alpha, t), t)|^q\right)^{\frac{1}{q}} &\le \left(\sum_{j \in \mathbf{Z}} 2^{jqs} |\Delta_j \omega_0(\alpha)|^q\right)^{\frac{1}{q}} \\ &+ \int_0^t \left(\sum_{j \in \mathbf{Z}} 2^{jqs} |\Delta_j((\omega \cdot \nabla)v)(X_j(\alpha, \tau), \tau)|^q\right)^{\frac{1}{q}} d\tau \\ &+ \int_0^t \left(\sum_{j \in \mathbf{Z}} 2^{jqs} |[(S_{j-2}v \cdot \nabla)\Delta_j \omega - \Delta_j((v \cdot \nabla)\omega)](X_j(\alpha, \tau), \tau)|^q\right)^{\frac{1}{q}} d\tau. \end{aligned}$$

Next ,we take $L^p(\mathbb{R}^n)$ norm of the both sides, then thanks to the fact that $\alpha \mapsto X_j(\alpha, t)$ is volume preserving, we obtain again by the Minkowski inequality

$$\begin{aligned} \|\omega(t)\|_{\dot{F}^s_{p,q}} &\le \|\omega_0\|_{\dot{F}^s_{p,q}} + \int_0^t \|((\omega \cdot \nabla)v)(\tau)\|_{\dot{F}^s_{p,q}} d\tau \\ &\quad + \int_0^t \left\| \left(\sum_{j \in \mathbf{Z}} 2^{jqs} |[(S_{j-2}v \cdot \nabla)\Delta_j \omega - \Delta_j((v \cdot \nabla)\omega)]|^q\right)^{\frac{1}{q}} \right\|_{L^p} d\tau. \end{aligned}$$

We substitute the Moser type of inequality and the commutator estimmate established in Lemma 3.1 and 3.2 respectively to obtain the homogeneous space inequality:

$$\|v(t)\|_{\dot{F}^s_{p,q}} \leq \|v_0\|_{\dot{F}^s_{p,q}} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|v(\tau)\|_{\dot{F}^s_{p,q}} d\tau.$$

Combing this with the following easy estimate,

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^p} d\tau,$$

we get the inhomogeneous space estimate:

$$\|\omega(t)\|_{F^s_{p,q}} \leq \|\omega_0\|_{F^s_{p,q}} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{F^s_{p,q}} d\tau. \tag{3.3}$$

For construction of local solution we proceed to

$$\|\omega(t)\|_{F^s_{p,q}} \leq \|\omega_0\|_{F^s_{p,q}} + C\int_0^t \|\omega(\tau)\|^2_{F^s_{p,q}} d\tau.$$

Defining $X^s_T := C([0,T]; F^s_{p,q})$, we have

$$\|\omega\|_{X^s_T} \leq \|\omega_0\|_{F^s_{p,q}} + CT\|\omega\|^2_{X^s_T}. \tag{3.4}$$

Now the local existence results by applying the contraction mapping principle to (3.4).

(iii) BKM type of inequalities:

Proposition 3.1 *Let $s > n/p$ with $p \in [1,\infty]$, $q \in [1,\infty)$, then there exists a constant C such that the following inequality holds.*

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{\dot{F}^0_{\infty,\infty}}(\log^+ \|f\|_{F^s_{p,q}} + 1)).$$

Similarly for the Besov space norm.

Remark 3.1 We mention that in [25] Ogawa recently obtained an inequality of the above type, but sharper than ours. Unfortunately, however, application of his inequality to our problem blow-up criterion does not improve our result in Theorem 1.1.

The proof of the above proposition uses the Littlewood-Paley decomposition and the standard optimization of parameter argument. This BKM type of inequality implies

$$\|\nabla v\|_{L^\infty} \leq C(1 + \|\omega\|_{\dot{F}^0_{\infty,\infty}}(\log^+ \|\omega\|_{F^s_{p,q}} + 1)), \tag{3.5}$$

where $s > n/p$. We substitute this into (3.2) to obtain

$$\|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} + C\int_0^t (\|\omega(s)\|_{\dot{F}^0_{\infty,\infty}} + 1)\left(\log^+\|v(s)\|_{F^s_{p,q}} + 1\right)\|\omega(s)\|_{F^s_{p,q}} ds$$

Then, we finally have by Gronwall's lemma

$$\|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} \exp\left[C \exp[C\int_0^t (\|\omega(s)\|_{\dot{F}^0_{\infty,\infty}} + 1)ds]\right].$$

This proves the blow-up criterion in Theorem 1.1. The proof for the case of Besov space is similar.
In the case of critical spaces the BKM type of inequality is not available, and instead we use the inequality(See (1.10).)

$$\|\nabla v\|_{L^\infty} \le C\|\nabla v\|_{\dot{F}^0_{\infty,1}} \le C\|\omega\|_{\dot{F}^0_{\infty,1}}, \tag{3.6}$$

where the second inequality follows from the fact that the corresponding singular integral operator in bounded from $\dot{F}^0_{\infty,1}$ into itself(See [14],[26]). We substitute (3.6) into (3.3) to obtain the inequality

$$\|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} \exp\left[C\int_0^t \|\omega(s)\|_{\dot{F}^0_{\infty,1}} ds\right].$$

This provides us the blow-up criterion in Theorem 1.3. In the case of critical Besov spaces we use instead

$$\|\nabla v\|_{L^\infty} \le C\|\omega\|_{\dot{B}^0_{\infty,1}}. \tag{3.7}$$

(iv) Composition mapping estimate and its application to 2-D Euler equations:

Proposition 3.2 *Let $(p,q) \in [1,\infty]\times[1,\infty]$. Suppose g be a volume preserving bi-Lipshitz homeomorphism. Let $f \in \dot{F}^0_{p,q}$(resp. $\dot{B}^0_{p,q}$). Then, $f\circ g^{-1} \in \dot{F}^0_{p,q}$(resp. $\dot{B}^0_{p,q}$), and there exists a constant C such that the following inequality holds.*

$$\|f\circ g^{-1}\|_{\dot{F}^0_{p,q}} \le C\left(1 + \log(\|g\|_{Lip}\|g^{-1}\|_{Lip})\right)\|f\|_{\dot{F}^0_{p,q}},$$

(similar for inhomogeneous spaces) where

$$\|g\|_{Lip} = \sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|}.$$

We note that Proposition 3.2 is a generalization of Vishik's result for $f \in B^0_{\infty,1}$ in [30]. The proof consists of the Littlewood-Paley decomposition, the Minkowski inequality, and the optimization of parameter argument. We now describe how to apply the above proposition to obtain Theorem 1.4. Let us consider the mapping $\alpha \mapsto X(\alpha, t)$ defined by solution of the ordinary differential equations

$$\begin{aligned} \frac{\partial}{\partial t} X(\alpha,t) &= v(X(\alpha,t),t), \quad \text{for} \quad t>0 \\ X(\alpha,0) &= \alpha. \end{aligned}$$

In terms of $X(\alpha,t)$ we can represent solution of (3.2) by

$$\omega(X(\alpha,t),t) = \omega_0(\alpha),$$

and thus

$$\omega(x,t) = \omega_0(X^{-1}(x,t)),$$

where $X(\cdot,t)$ is bi-Lipshitz, and volume preserving. he previous proposition implies

$$\|\omega(t)\|_{\dot F^0_{\infty,1}} \le C\|\omega_0\|_{\dot F^0_{\infty,1}}(1+\log(\|X(t)\|_{Lip}\|X^{-1}(t)\|_{Lip}).$$

Taking derivative with respect to α, we obtain

$$\|X(\cdot,t)\|_{Lip} \le 1 + \int_0^t \|\nabla v(X(\alpha,\tau),\tau)\|_{L^\infty}\|X(\cdot,\tau)\|_{Lip}\, d\tau.$$

By Gronwall's lemma

$$\begin{aligned} \|X(\cdot,t)\|_{Lip} &\le \exp\left[\int_0^t \|\nabla v(X(\cdot,\tau),\tau)\|_{L^\infty} d\tau\right] \\ &\le \exp\left[\int_0^t \|\nabla v(X(\cdot,\tau),\tau)\|_{\dot F^0_{\infty,1}} d\tau\right] \\ &\le \exp\left[C\int_0^t \|\omega(X(\cdot,\tau),\tau)\|_{\dot F^0_{\infty,1}} d\tau\right] \\ &\le \exp\left[C\|\omega_0\|_{\dot F^0_{\infty,1}}\int_0^t (1+\log(\|X(\cdot,\tau)\|_{Lip}\|X^{-1}(\cdot,\tau)\|_{Lip})\, d\tau\right] \end{aligned}$$

By definition $X^{-1}(\cdot,t) = X(\cdot,-t)$, and by similar argument as the above

$$\|X^{-1}(t)\|_{Lip} \le \exp\left[C\|\omega_0\|_{\dot F^0_{\infty,1}}\int_0^t (1+\log(\|X(\cdot,\tau)\|_{Lip}\|X^{-1}(\cdot,\tau)\|_{Lip}))\, d\tau\right].$$

Combining the above two results, we obtain

$$\begin{aligned} &\|X(\cdot,t)\|_{Lip}\|X^{-1}(\cdot,t)\|_{Lip} \\ &\le \exp\left[C\|\omega_0\|_{\dot F^0_{\infty,1}}\int_0^t (1+\log(\|X(\cdot,\tau)\|_{Lip}\|X^{-1}(\cdot,\tau)\|_{Lip}))\, d\tau\right] \quad (3.8) \end{aligned}$$

We take logarithm of (3.8) to have

$$
\begin{aligned}
&1+\log(\|X(\cdot,t)\|_{Lip}\|X^{-1}(\cdot,t)\|_{Lip}) \\
&\quad \le \; 1+C\|\omega_0\|_{\dot F^0_{\infty,1}}\int_0^t \left(1+\log(\|X(\cdot,\tau)\|_{Lip}\|X^{-1}(\cdot,\tau)\|_{Lip})\right) d\tau.
\end{aligned}
$$

By Gronwall's lemma

$$1+\log(\|X(\cdot,t)\|_{Lip}\|X^{-1}(\cdot,t)\|_{Lip}) \le \exp\left[C\|\omega_0\|_{\dot F^0_{\infty,1}}t\right].$$

Thus we have

$$\|\omega(t)\|_{\dot F^0_{\infty,1}} \le \|\omega_0\|_{\dot F^0_{\infty,1}} \exp\left[C\|\omega_0\|_{\dot F^0_{\infty,1}}t\right].$$

Combining all of these, we finally have

$$
\begin{aligned}
\|\omega(t)\|_{F^2_{1,q}} &\le \|\omega_0\|_{F^2_{1,q}} \exp\left[C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right] \\
&\le \|\omega_0\|_{F^2_{1,q}} \exp\left[C\int_0^t \|\nabla v(\tau)\|_{\dot F^0_{\infty,1}} d\tau\right] \\
&\le \|\omega_0\|_{F^2_{1,q}} \exp\left[C\int_0^t \|\omega(\tau)\|_{\dot F^0_{\infty,1}} d\tau\right] \\
&\le \|\omega_0\|_{F^2_{1,q}} \exp\left[C\|\omega_0\|_{\dot F^0_{\infty,1}}\int_0^t \exp(C_1\|\omega_0\|_{\dot F^0_{\infty,1}}\tau) d\tau\right] \\
&\le \|\omega_0\|_{F^2_{1,q}} \exp\left[\frac{C}{C_1}\exp(C_1\|\omega_0\|_{\dot F^0_{\infty,1}}t)\right] \\
&\le \|\omega_0\|_{F^2_{1,q}} \exp\left[C_2\exp(C_1\|\omega_0\|_{F^2_{1,q}}t)\right],
\end{aligned}
$$

where we used the imbedding $F^2_{1,q} \hookrightarrow \dot F^0_{\infty,1}$(See (1.10)) in the last inequality. This is the global estimate of Theorem 1.4.

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