Asymptotic behaviour and net force for the Navier-Stokes flows in exterior domains (Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics)

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Asymptotic behaviour and net force for the Navier-Stokes flows in exterior domains

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To the memory of Professor Tosio Kato

1 Introduction.

Let \( \Omega \subset \mathbb{R}^n (n \geq 3) \) be an exterior domain, i.e., a domain having a compact complement \( \mathbb{R}^n \setminus \Omega \) with the smooth boundary \( \partial \Omega \). Consider the initial-boundary value problem of the Navier-Stokes equations in \( \Omega \times (0, \infty) \):

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p &= 0 \quad \text{in} \ x \in \Omega, \ 0 < t < \infty, \\
\text{div} u &= 0 \quad \text{in} \ x \in \Omega, \ 0 < t < \infty, \\
u &= 0 \quad \text{on} \ \partial \Omega, \quad u(x,t) \to 0 \quad \text{as} \ |x| \to \infty, \\
u|_{t=0} &= a,
\end{align*}
\]

(N-S)

where \( u = u(x,t) = (u_1(x,t), \ldots, u_n(x,t)) \) and \( p = p(x,t) \) denote the unknown velocity vector and the pressure of the fluid at the point \( (x,t) \in \Omega \times (0, \infty) \), while \( a = a(x) = (a_1(x), \ldots, a_n(x)) \) is the given initial velocity vector.

The global existence of strong solutions \( u \) to (N-S) for small data \( a \) had been investigated by many authors, Fujita-Kato [8], Solonnikov [25], Heywood [13], Giga-Miyakawa [11] and Kato [15]. In exterior domains, Iwashita [14] proved the most remarkable result together with the asymptotic behaviour. In [14], it turns out that for small \( a \in L^n(\Omega) \cap L^s(\Omega) \) with \( 1 < s \leq n \) there exists a unique strong solution \( u \) with the following decay property

\[
\begin{align*}
\|u(t)\|_{L^r(\Omega)} &= O(t^{-\frac{n}{2}\left(\frac{1}{s}-\frac{1}{r}\right)}), \quad s \leq r \leq \infty, \\
\|\nabla u(t)\|_{L^r(\Omega)} &= O(t^{-\frac{n}{2}\left(\frac{1}{\epsilon}-\frac{1}{r}\right)-\frac{1}{2}}), \quad s \leq r \leq n
\end{align*}
\]

as \( t \to \infty \). The first purpose of this article is to consider whether or not it is possible to take \( s = 1 \) in (1.1). Our problem is motivated by the fundamental question on the energy decay of solutions which was proposed by Leray [20]. For every \( a \in L^2(\Omega) \), there exists at least one weak solution \( u \) to (N-S). In his famous paper [20], he had asked whether every weak solution does satisfy

\[
\|u(t)\|_{L^2(\Omega)} \to 0 \quad \text{as} \ t \to \infty.
\]
After 50 years of Leary's proposal, Masuda [21] and Kato [15] independently gave a positive answer to his question for all weak solutions \( u \) satisfying the energy inequality of the strong form. Then much effort had been made to obtain the decay rate of \( \|u(t)\|_{L^2(\Omega)} \) as \( t \to \infty \). At the present, the best rate is given by Borchers-Miyakawa [3] who proved that if

\[
\|e^{-tA}a\|_{L^2(\Omega)} = O(t^{-\alpha}) \quad \text{as } t \to \infty \quad (A; \text{the Stokes operator}),
\]

then there holds

\[
\|u(t)\|_{L^2(\Omega)} = \begin{cases} O(t^{-\alpha}) & \text{for } 0 < \alpha \leq n/4, \\ O(t^{-\frac{n}{2}}) & \text{for } n/4 < \alpha < \infty \end{cases}
\]

as \( t \to \infty \). It should be noted that the decay rate \( t^{-n/4} \) can be obtained formally by taking \( r = 2 \) and \( s = 1 \) in (1.1). We shall show that if the initial data \( a \in L^1(\Omega) \cap L^n(\Omega) \) with certain regularity, then every strong solution \( u \) of (N-S) with (1.1) for \( s \) sufficiently close to 1 decays like

\[
\|u(t)\|_{L^r(\Omega)} = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all } 1 < r < \infty
\]

as \( t \to \infty \).

The second purpose of this article is to consider whether the above decay rate \( t^{-\frac{n}{2}(1-\frac{1}{r})} \) is optimal in the norm of \( L^r(\Omega) \). In the whole space \( \mathbb{R}^n \), Wiegner [28] showed that there exists a weak solution \( u \) such that

\[
\|u(t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{n+2}{4}}) \quad \text{as } t \to \infty.
\]

It was proven by Schonbek [23], [24] that this decay rate \( t^{-\frac{n+2}{4}} \) is optimal in \( L^2(\mathbb{R}^n) \). In exterior domains \( \Omega \), however, we shall show that the strong solution \( u \) decays like

\[
\|u(t)\|_{L^r(\Omega)} = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } 1 < r < \infty
\]

as \( t \to \infty \) if and only if

\[
\int_0^\infty \int_{\partial\Omega} T[u, p](y, t) \cdot \nu \, dS_y \, dt = 0,
\]

where \( T[u, p] = \{ \partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij}p \}_{i,j=1,\ldots,n} \) denotes the stress tensor and \( \nu = (\nu_1, \ldots, \nu_n) \) is the unit outward normal to \( \partial\Omega \). This implies that the faster decay rate than \( t^{-\frac{n}{2}(1-\frac{1}{r})} \) in \( L^r(\Omega) \) of the velocity causes necessarily physical restriction on the net force exerted by the fluid to the obstacle. As a result, from a physical point of view, the decay rate like (1.2) seems to be optimal.

\section{Results.}

Before stating our results, we first introduce some function spaces. Let \( C^\infty_{0,\sigma}(\Omega) \) denote the set of all \( C^\infty \) vector functions \( \phi = (\phi_1, \ldots, \phi_n) \) with compact support in \( \Omega \), such that \( \mathrm{div} \phi = 0 \). \( L^r_\sigma(\Omega) \) is the closure of \( C^\infty_{0,\sigma}(\Omega) \) with respect to the \( L^r \)-norm \( \| \cdot \| = \| \cdot \|_{L^r(\Omega)} \); \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( L^r(\Omega) \) and \( L^{r'}(\Omega) \), where \( 1/r + 1/r' = 1 \). \( L^r(\Omega) \) stands for the
usual (vector-valued) $L^r$-space over $\Omega$, $1 \leq r \leq \infty$. It is known that for $1 < r < \infty$, $L^r_\sigma(\Omega)$ is characterized as

$$L^r_\sigma(\Omega) = \{ u \in L^r(\Omega); \text{div } u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial \Omega \text{ in the sense } W^{1-1/r',r'}(\partial \Omega)^* \}$$

and that there holds the Helmholtz decomposition

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus G^r(\Omega) \quad \text{(direct sum), } 1 < r < \infty,$$

where $G^r(\Omega) = \{ \nabla p \in L^r(\Omega); p \in L^r_{loc}(\overline{\Omega}) \}$. We denote by $P_r$ the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ along $G^r(\Omega)$. Then the Stokes operator $A_r$ is defined by

$$A_r = -P_r$, for $u \in W^{2,r}(\Omega) \cap L^r_{\sigma}(\Omega)|u_{\partial \Omega} = 0 \text{ in the sense } W^{1-1/r',r'}(\partial \Omega)^*.$$}

It is proved by Giga [10] and Giga-Sohr [12] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{ e^{-tA_r} \}_{t \geq 0}$ of class $C^\infty$ in $L^r_{\sigma}(\Omega)$ for $1 < r < \infty$. Hence one can define the fractional power $A_r^\alpha$ for $0 \leq \alpha \leq 1$.

The class of solutions which we consider is as follows.

**Definition.** Let $1 < s \leq n$ and let $a \in L^s_{\sigma}(\Omega) \cap L^n_{\sigma}(\Omega)$. A measurable function $u$ on $\Omega \times (0, \infty)$ is called a strong solution of (N-S) in the class $CL^s_s(0, \infty)$ if

(i) $u \in C([0, \infty); L^s_{\sigma}(\Omega) \cap L^n_{\sigma}(\Omega))$;

(ii) $Au, \partial u/\partial t \in C((0, \infty); L^s_{\sigma}(\Omega))$;

(iii) 

(N-S') \quad \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + Au + P(u \cdot \nabla u) = 0 \text{ in } L^n_{\sigma}(\Omega), \ 0 < t < \infty, \\
u(0) = a,
\end{array} \right.$

**Remarks.** 1. It was shown by Kato [15] and Iwashita [14] that for $1 < s \leq n$ there is a constant $\lambda(s,n)$ such that for every $a \in L^s_{\sigma}(\Omega) \cap L^n_{\sigma}(\Omega)$ with $||a||_n \leq \lambda$, there exists a unique strong solution $u$ of (N-S) in the class $CL_s(0, \infty)$. Moreover, such a solution satisfies (1.1).

2. Every strong solution $u$ in the class $CL^s_s(0, \infty)$ satisfies (N-S') also in $L^s_{\sigma}(\Omega)$ and there holds

$$\frac{\partial |\alpha| u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \frac{\partial u}{\partial t} \in C(\overline{\Omega} \times (0, \infty))$$

for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Moreover, there exists a unique (up to an additive function of $t$) scalar function $p \in C^1(\Omega \times (0, \infty))$ with

(2.1) \quad $\nabla p \in C((0, \infty); L^s(\Omega) \cap L^n(\Omega))$

such that the pair $\{u, p\}$ satisfies (N-S) in the classical sense. We call such $p$ the pressure associated with $u$. 
3. If $1 < s < n$, by (2.1) and the Sobolev embedding ([12, Corollary 2.2]), we may take $p$ as $p \in C((0, \infty); L^{ns/(n-s)}(\Omega))$.

Throughout this paper, we impose the following assumption on the initial data.

**Assumption.** For some $\frac{n}{n-2} < q_* < \infty$ and $\varepsilon > 0$ the initial data $a$ satisfies

$$a \in L^1(\Omega) \cap L^n(\Omega) \cap D(A_q^\varepsilon).$$

Our first result on the decay property of strong solutions now reads:

**Theorem 1.** Let $a$ be as in the Assumption. Suppose that $u$ is the strong solution of (N-S) in the class $CL_s(0, \infty)$ with (1.1) for $1 < s < \min \left\{ \frac{n}{n-1}, \frac{2n}{n+2} \right\}$. Then $u(t)$ decays like

$$||u(t)||_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all} \quad 1 < r < \infty.$$

**Remarks.**

1. Iwashita [14] showed the existence of the strong solution $u$ in the class $CL_s(0, \infty)$ with (1.1) for $a \in L^s_\sigma(\Omega) \cap L^n_\sigma(\Omega)$ with $1 < s \leq n$ provided $||a||_n$ is small. Concerning the linear Stokes flows for $s = 1$, the author [19] proved

$$||e^{-tA}a||_r \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})}(||a||_1 + ||a||_{q_*} + ||A^\varepsilon a||_{q_*}), \quad 1 < r \leq \infty,$$

$$||\nabla e^{-tA}a||_r \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}(||a||_1 + ||a||_{q_*} + ||A^\varepsilon a||_{q_*}), \quad 1 \leq r \leq n,$$

for all $t > 1$ and for all $a$ as in the Assumption.

2. In (2.2), we do not know whether $r = 1$ is possible; the author [18] showed that $u \in C([0, \infty); L^1(\Omega))$ with its associated pressure $p \in C((0, \infty); L^{\frac{n}{n-1}}(\Omega))$ if and only if there holds

$$\int_{\partial \Omega} T[u, p](y, t) \cdot \nu dS_y = 0, \quad \text{for all} \quad 0 < t < \infty,$$

where $T[u, p] = \{ \partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij} p \}_{i,j=1,\cdots,n}$ denotes the stress tensor and $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outward normal to $\partial \Omega$. Hence, it seems to be difficult to take $r = 1$ in (2.2) for all $a$ satisfying the Assumption.

We next investigate the faster decay than (2.2):

**Theorem 2.** Let $a$ be as in the Assumption. Suppose that $u$ is the strong solution as in Theorem 1. If

$$||u(t)||_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some} \quad 1 < r \leq \infty$$

as $t \to \infty$, then there holds

$$\int_0^\infty dt \int_{\partial \Omega} T[u, p](y, t) \cdot \nu dS_y = 0.$$
Conversely, if (2.7) holds, then we have

\begin{equation}
\|u(t)\|_{r} = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all } 1 < r \leq \infty
\end{equation}

as \( t \to \infty \).

**Remarks.**

1. In case \( \Omega = \mathbb{R}^{n} \), the situation is quite different. Wiegner [28] showed existence of a weak solution \( u \) of (N-S) with the property that

\[ \|u(t)\|_{L^{2}(\mathbb{R}^{n})} = O(t^{-\frac{n}{4}-\frac{1}{2}}) \quad \text{as } t \to \infty. \]

Schonbek [23], [24] and Miyakawa-Schonbek [22] proved that there exist an initial data \( a \in L^{1}(\mathbb{R}^{n}) \cap L_{\sigma}^{2}(\mathbb{R}^{n}) \) and a weak solution \( u \) of (N-S) such that

\[ \|u(t)\|_{L^{2}(\mathbb{R}^{n})} \geq Ct^{-\frac{n}{4}-\frac{1}{2}} \quad \text{for large } t. \]

Fujigaki-Miyakawa [6] proved that there exist an initial data \( a \in L^{1}(\mathbb{R}^{n}) \cap L_{\sigma}^{n}(\mathbb{R}^{n}) \) and a strong solution \( u \) of (N-S) such that

\[ \|u(t)\|_{L^{r}(\mathbb{R}^{n})} \geq Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} \quad \text{for large } t. \]

2. In case \( \Omega = \mathbb{R}_{+}^{n} \) (half space), based on the Ukai's formula [27] for \( e^{-tA}a \), faster decay rates than in \( \mathbb{R}^{n} \) were obtained by Bae-Choe [1], Bae [2] and Fujigaki-Miyakawa [7].

3. The net force plays an important role also for the spatial decay at infinity of the solutions to the stationary problem in \( \Omega \subset \mathbb{R}^{3} \):

\begin{align*}
\begin{cases}
-\Delta w + w \cdot \nabla w + \nabla p = \text{div} F, & \text{in } x \in \Omega \\
\text{div } w = 0 & \text{in } x \in \Omega, \quad \text{in } x \in \Omega \\
w = 0 & \text{on } \partial \Omega, \quad w(x) \to w^{\infty} \text{ as } |x| \to \infty,
\end{cases}
\end{align*}

where \( F = F(x) = \{F_{ij}(x)\}_{i,j=1,2,3} \) denotes the given \( 3 \times 3 \) tensor, while \( w^{\infty} = (w_{1}^{\infty}, w_{2}^{\infty}, w_{3}^{\infty}) \) is the prescribed constant vector in \( \mathbb{R}^{3} \). Finn [4], [5] treated the case when \( F \equiv 0, w^{\infty} \neq 0 \). Introducing the notion of physically reasonable solution \( w \) of (E), i.e.,

\[ |w(x) - w^{\infty}| = O(|x|^{-\frac{1}{2}-\varepsilon}) \quad (\varepsilon > 0) \quad \text{as } |x| \to \infty, \]

he proved that

\[ |w(x) - w^{\infty}| = o(|x|^{-1}) \quad \text{as } |x| \to \infty, \]

if and only if there holds

\[ \int_{\partial \Omega} T[w, p](y) \cdot \nu dS_{y} = 0. \]

Kozono-Sohr-Yamazaki [17] considered the case when \( F \neq 0, w^{\infty} = 0 \). They dealt with the \( D \)-solution \( w \), i.e.,

\[ \int_{\Omega} |
\nabla w(x)|^{2} dx < \infty \]

and showed that \( w \in L^{3}(\Omega) \) if and only if

\[ \int_{\partial \Omega} (T[w, p](y) + F(y)) \cdot \nu dS_{y} = 0. \]
3 Outline of the proof of the theorems.  

In this section, we shall give a sketch of the proof of Theorems 1 and 2. Let us first recall the fundamental tensor \( \{E_{ij}(x,t)\}_{i,j=1,\ldots,n} \) to the linear Stokes system defined by

\[
E_{ij}(x,t) = \Gamma(x,t)\delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j}(\Gamma(\cdot,t) * G)(x), \quad i,j = 1,\ldots,n,
\]

where

\[
\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad G(x) = \frac{1}{n(n-2)\omega_n} |x|^{2-n}, \quad \omega_n = \text{vol.}(S^{n-1}).
\]

We have the following representation formula of the strong solution.

**Lemma 3.1 (Representation formula)** Let \( a \) be as in the Assumption. The strong solution \( u(t) \) to \((N-S)\) in the class \( CL_s(0,\infty)\) for \( 1 < s \leq n \) can be represented as

\[
u_{i}(x, t) = \int_{\Omega} \Gamma(x-y, t)a_{i}(y)dy \\
+ \int_{0}^{t} d\tau \int_{\partial\Omega} \sum_{j,k=1}^{n} E_{ij}(x-y, t-\tau)T_{jk}[u,p](y, \tau)\nu_{k}(y)dS_{y} \\
+ \int_{0}^{t} d\tau \int_{\Omega} \sum_{j,k=1}^{n} \frac{\partial}{\partial y_{k}}E_{ij}(x-y, t-\tau)u_{k}\cdot u_{j}(y, \tau)dy
\]

(3.1)

\[
\equiv U_{i}(x, t) + V_{i}(x, t) + W_{i}(x, t), \quad i = 1,\ldots,n
\]

for all \((x, t) \in \Omega \times (0,\infty)\).

To make use of this representation formula, we need to investigate behaviour of the boundary integral

\[
\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_{y} \quad \text{for all} \quad t \in (0,\infty).
\]

**Lemma 3.2** Let \( a \) be as in the Assumption. Let \( q \equiv nq_*/(n+q_*) \).

(i) Every strong solution \( u \) of \((N-S)\) in the class \( CL_s(0,\infty)\) for \( 1 < s \leq n \) and its associated pressure \( p \) satisfy

\[
\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_{y} \leq Ct^{\alpha-1} \quad \text{for all} \quad 0 < t \leq 1
\]

with \( \alpha \equiv (\frac{1-1/q}{1-1/q_*})\epsilon \), where \( C = C(n,q_*,\epsilon) \).

(ii) Let \( u \) be a strong solution of \((N-S)\) in the class \( CL_s(0,\infty)\) for \( 1 < s \leq \min \left\{ \frac{n}{n-1}, \frac{2n}{n+2} \right\} \) with the decay property (1.1). For every \( l \) with \( 1 < s \leq l < n \), \( u \) and its associated pressure \( p \) are subject to the estimate

\[
\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_{y} \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{l})-\frac{1}{2}} \quad \text{for all} \quad 1 < t < \infty,
\]

where \( C = C(n,s,l) \).
For the proof we need the trace theorem and the following estimate by Kozono-Ogawa [16]
\[
\|\nabla^2 u\|_s \leq C(\|Au\|_s + \|\nabla u\|_s), \quad 1 < s < \infty
\]
for all \( u \in D(A_s) \) together with the decay property
\[
\|Au(t)\|_l = O(t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{l})-1}), \quad s \leq l < \infty, \quad \text{as } t \to \infty.
\]

**Proof of Theorem 1:**

By Lemma 3.1, we may estimate \( U(t) \), \( V(t) \) and \( W(t) \) in \( L^r \), respectively. First, let us consider the case \( 1 < r < n/(n-1) \).

Recall
\[
U_i(x,t) = \int_{\Omega} \Gamma(x-y,t)a_i(y)dy, \quad i = 1, \cdots, n
\]

Since
\[
\int_{\Omega} |a(x)|dx < \infty \quad \text{with } \text{div } a = 0 \text{ in } \Omega, \quad a \cdot \nu = 0 \quad \text{on } \partial \Omega,
\]
there holds
\[
\int_{\Omega} a_i(y)dy = 0, \quad i = 1, \cdots, n.
\]

Hence we have by the Hausdorff-Young inequality that
\[
\|U(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \to \infty. \tag{3.2}
\]

To deal with
\[
V_i(x,t) = \sum_{j,k=1}^{n} \int_{0}^{t} d\tau \int_{\partial\Omega} E_{ij}(x-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad i = 1, \cdots, n,
\]
we need to notice that \( \{E_{ij}\}_{i,j=1,\cdots,n} \) can be expressed as
\[
E_{ij}(\cdot,t) = (\delta_{ij} + R_{\dot{i}} R_{j}) \Gamma(\cdot,t), \quad i,j = 1, \cdots, n, \tag{3.3}
\]
where \( R_i = \frac{\partial}{\partial x_i} (-\Delta)^{-\frac{1}{2}} \), \( i = 1, \cdots, n \) denote the Riesz transforms. Since \( R_i : L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n) \) is bounded, we have
\[
\|\partial_x^m \partial_t^k E_{ij}(\cdot,t)\|_r \leq C t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{m}{2}-k}, \quad m, k = 0, 1, \forall t > 0, \tag{3.4}
\]
which yields
\[
\|V(t)\|_r \leq \sum_{i,j,k=1}^{n} \int_{0}^{t} d\tau \int_{\partial\Omega} \|E_{ij}(\cdot-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)\|_r dS_y
\]
\[
\leq \sum_{i,j,k=1}^{n} \int_{0}^{t} d\tau \int_{\partial\Omega} |T_{jk}[u,p](y,\tau)\nu_k(y)||E_{ij}(\cdot-y,t-\tau)||_r dS_y
\]
\[
\leq C \int_{0}^{t} (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})} \left( \int_{\partial\Omega} (|\nabla u(y,\tau)| + |p(y,\tau)|) dS_y \right) d\tau. \tag{3.5}
\]
Applying Lemma 3.2 to the estimate of the R.H.S., we obtain

\[ \|V(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \to \infty. \]  

Finally, we treat the third term

\[ W_i(x, t) = \int_0^t d\tau \int_\Omega \sum_{j,k=1}^n \frac{\partial}{\partial y_k} E_{ij}(x - y, t - \tau) u_k \cdot u_j(y, \tau) dy, \quad i = 1, \ldots, n \]

By (3.4) and the Housdorff-Young inequality, we have

\[ \|W(t)\|_r \leq \int_0^t \|\nabla E(\cdot, t - \tau)\|_r \|u \otimes u(\tau)\|_1 d\tau \leq C \int_0^t (t - \tau)^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} \|u(\tau)\|_2^2 d\tau. \]

Since \( \|u(t)\|_2 \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{2})} \) (see (1.1)), we obtain from the above estimate

\[ \|W(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \quad \text{as } t \to \infty. \]

Notice that \(-\frac{n}{2}(1-\frac{1}{r}) - \frac{1}{2} > -1 \iff r < n/(n-1)\). Then by (3.2), (3.6) and (3.7), we have the desired decay for \( \|u(t)\|_r \) provided \( 1 < r < n/(n-1) \).

In case \( n/(n-1) \leq r < \infty \), some skillful technique by duality is necessary. Here we omit the detail. This proves Theorem 1.

**Proof of Theorem 2:**

Without loss of generality, we may assume that

\[ \|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } r \text{ with } 1 < r < n/(n-1). \]

as \( t \to \infty \). Indeed, if (2.6) holds for some \( n/(n-1) \leq r \leq \infty \), then by choosing \( 1 < r_0 < r_1 < n/(n-1) \) and \( 0 < \theta < 1 \) with \( 1/r_1 = (1-\theta)/r_0 + \theta/r \), we have

\[ \|u(t)\|_{r_1} \leq \|u(t)\|_r^{1-\theta} \|u(t)\|_{r_0}^\theta \]

\[ = O(t^{-\frac{n}{2}\left(1-\frac{1}{r_0}\right)(1-\theta)}) \cdot o(t^{-\frac{n}{2}(1-\frac{1}{r})\theta}) \]

\[ = o(t^{-\frac{n}{2}(1-\frac{1}{r_1})}) \]

as \( t \to \infty \), which yields (3.8).

By Lemma 3.1, we have similarly to (3.1) that

\[ u_i(x, t) = U_i(x, t) + \tilde{V}_i(x, t) + W_i(x, t) + \sum_{j,k=1}^n E_{ij}(x, t) \int_0^t d\tau \int_{\partial \Omega} T_{jk}[u, p](y, \tau) u_k(y) dS_y, \]

\[ i = 1, \ldots, n, \]
for all \((x, t) \in \Omega \times (0, \infty)\), where

\[ U_i(x, t) = \int_{\Omega} \Gamma(x - y, t)a_i(y)dy, \]

\[ \tilde{V}_i(x, t) = \sum_{j,k=1}^{n} \int_{0}^{t} d\tau \int_{\partial\Omega} \{E_{ij}(x-y, t-\tau) - E_{ij}(x, t)\} T_{jk}[u, p](y, \tau)\nu_k(y)dS_y, \]

\[ W_i(x, t) = \int_{0}^{t} d\tau \int_{\Omega} \sum_{j,k=1}^{n} \frac{\partial}{\partial y_k}E_{ij}(x-y, t-\tau)u_k \cdot u_j(y, \tau)dy \]

for \(i = 1, \cdots, n\). Since \(1 < r < n/(n-1)\), we have by (3.2) and (3.7) that

\[ \|U(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \]

\[ \|W(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \]

as \(t \to \infty\). Using the expression

\[ \tilde{V}_i(x, t) \]

\[ = \sum_{j,k=1}^{n} \int_{0}^{t} d\tau \int_{\partial\Omega} \left( \int_{0}^{1} \frac{d}{d\theta}E_{ij}(x-\theta y, t-\theta\tau)d\theta \right) T_{jk}[u, p](y, \tau)\nu_k(y)dS_y \]

\[ + \sum_{j,k=1}^{n} \int_{0}^{t} d\tau \int_{\partial\Omega} \left( \int_{0}^{1} \nabla E_{ij}(x-\theta y, t-\theta\tau) \cdot (-y)d\theta \right) T_{jk}[u, p](y, \tau)\nu_k(y)dS_y, \]

we can show, with the aid of some technical calculation, that

(3.11) \[ \|\tilde{V}(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \] as \(t \to \infty\).

On the other hand, there holds

\[ \liminf_{t \to \infty} t^{\frac{n}{2}(1-\frac{1}{r})} \left\| \sum_{j=1}^{n} E_{ij}(\cdot, t) \int_{0}^{t} f_j(\tau)d\tau \right\|_r \]

\[ \geq \left( \int_{y \in \mathbb{R}^n} \left\| \sum_{j=1}^{n} E_{ij}(y, 1) \int_{0}^{\infty} f_j(\tau)d\tau \right\|_r \right)^{\frac{1}{r}}, \quad i = 1, \cdots, n \]

where

\[ f_j(\tau) = \int_{\partial\Omega} \sum_{k=1}^{n} T_{jk}[u, p](y, \tau)\nu_k(y)dS_y, \quad j = 1, \cdots, n. \]

Now, assume that

\[ \|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \] as \(t \to \infty\).
Then it follows from (3.10), (3.11) and (3.12) that

\[
\sum_{j=1}^{n} E_{ij}(y, 1) \int_{0}^{\infty} f_{j}(\tau) d\tau = 0, \quad i = 1, \ldots, n, \text{ for all } y \in \mathbb{R}^{n}.
\]

Since \( \hat{E}_{ij}(\xi, 1) = (\delta_{ij} - \frac{\xi_{i} \xi_{j}}{|\xi|^{2}}) e^{-|\xi|^{2}} \), \( i, j = 1, \ldots, n \), we have by (3.13) that

\[
\sum_{j=1}^{n} (\delta_{ij} - \omega_{i} \omega_{j}) \int_{0}^{\infty} f_{j}(\tau) d\tau = 0, \quad i = 1, \ldots, n
\]

for all \( \omega = (\omega_{1}, \ldots, \omega_{n}) \in \mathbb{R}^{n} \) with \( |\omega| = 1 \). Obviously, we conclude that

\[
\int_{0}^{\infty} f_{1}(\tau) d\tau = \cdots = \int_{0}^{\infty} f_{n}(\tau) d\tau = 0,
\]

which implies

\[
\int_{0}^{\infty} d\tau \int_{\partial \Omega} \sum_{k=1}^{n} T_{jk}[u, p](y, \tau) \nu_{k}(y) dS_{y} = 0, \quad j = 1, \ldots, n.
\]

This shows (2.7).

Conversely, if (2.7) holds, then we have by (3.9), (3.10) and (3.11) that

\[
||u(t)||_{r} \leq ||U(t)||_{r} + ||\tilde{V}(t)||_{r} + ||W(t)||_{r} + \sum_{i,j=1}^{n} ||E_{ij}(\cdot, t)||_{r} \left| \int_{0}^{t} f_{j}(\tau) d\tau \right| = o(t^{-\frac{n}{2}(1-\frac{1}{r})})
\]

for all \( 1 < r < n/(n-1) \) as \( t \to \infty \). By the same technique as before, we get (2.8). This proves Theorem 2.

References

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