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ON A STABILITY THEOREM OF THE NAVIER-STOKES EQUATION IN A THREE DIMENSIONAL EXTERIOR DOMAIN

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1. INTRODUCTION

The motion of non-stationary flow of an incompressible viscous fluid past an isolated rigid body is formulated by the following initial boundary value problem of the Navier-Stokes equation :

$$
\left\{\begin{array}{l}
\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,\infty)\times\Omega, \\
\quad \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{a}, \\
\quad \lim_{|x|\to\infty} \mathbf{u}(t,x) = \mathbf{u}_\infty
\end{array}\right. \tag{1.1}
$$

Here, Ω is an exterior domain in $\mathbb{R}^3$ identified with the region filled by a viscous incompressible fluid; $\partial\Omega$ denotes the boundary of Ω which is assumed to be a smooth and compact hypersurface ; $\mathbf{u} = {}^t(u_1, u_2, u_3)$ (${}^tM$ meaning the transposed M) and $\mathfrak{p}$ denote the unknown 3 dim. velocity vector and pressure, respectively, while $\mathbf{f} = {}^t(f_1, f_2, f_3)$ and $\mathbf{a} = {}^t(a_1, a_2, a_3)$ denote the given external force and intital velocity, respectively; $\mathbf{u}_\infty$ is a given constant velocity vector at infinity. Here and hereafter, we use the standard notation in the vector analysis. For example, we put

$$
\Delta\mathbf{u} = {}^t(\Delta u_1, \Delta u_2, \Delta u_3), \;\; \Delta u_j = \sum_{k=1}^3 \frac{\partial^2 u_j}{\partial x_k^2}, \;\; \nabla = {}^t(\partial_1, \partial_2, \partial_3), \;\; \partial_k = \frac{\partial}{\partial x_k},
$$

$$
(\mathbf{u}\cdot\nabla)\mathbf{v} = {}^t((\mathbf{u}\cdot\nabla)v_1, (\mathbf{u}\cdot\nabla)v_2, (\mathbf{u}\cdot\nabla)v_3), \;\; (\mathbf{u}\cdot\nabla)v_j = \sum_{k=1}^3 u_k\partial_k v_j,
$$

$$
\nabla\cdot\mathbf{u} = \operatorname{div}\mathbf{u} = \sum_{k=1}^3 \partial_k u_k, \;\; \mathbf{u}\otimes\mathbf{v} = \begin{pmatrix} u_1v_1, u_2v_1, u_3v_1 \\ u_1v_2, u_2v_2, u_3v_2 \\ u_1v_3, u_2v_3, u_3v_3 \end{pmatrix}
$$

$$
\nabla\cdot F = \begin{pmatrix} \sum_{k=1}^3 \partial_k f_{1k} \\ \sum_{k=1}^3 \partial_k f_{2k} \\ \sum_{k=1}^3 \partial_k f_{3k} \end{pmatrix}, \quad F = \begin{pmatrix} f_{11}, f_{12}, f_{13} \\ f_{21}, f_{22}, f_{23} \\ f_{31}, f_{32}, f_{33} \end{pmatrix}.
$$

Putting $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$, (1.1) is reduced to the following equation :

$$
(1.2) \qquad \begin{cases} \mathbf{v}_t - \Delta\mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla\mathfrak{p} = \mathbf{f}, \ \ \nabla \cdot \mathbf{v} = 0 \ \text{ in } (0,\infty) \times \Omega, \\ \qquad \mathbf{v}|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \mathbf{v}|_{t=0} = \mathbf{a} - \mathbf{u}_\infty, \\ \displaystyle\lim_{|x|\to\infty} \mathbf{v}(t,x) = 0. \end{cases}
$$

In this note, we consider the case where the external force $\mathbf{f}$ is independent of time variable t, namely $\mathbf{f} = \mathbf{f}(x)$. We will discuss the problem from the point of the stability of stationary solutions. When the external force is independent of time, we expect that the flow becomes stable asymptotically in time because of the viscousity. Therefore, we also consider the stationary problem corresponding to (1.2) which is given by the following formulas :

$$
(1.3) \qquad \begin{cases} -\Delta\mathbf{w} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla\pi = \mathbf{f}, \ \ \nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega, \\ \qquad \mathbf{w}|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \displaystyle\lim_{|x|\to\infty} \mathbf{w}(x) = 0. \end{cases}
$$

Concerning (1.2), Leray [34, 35] and Hopf [23] proved the existence of square-integrable weak solutions for an arbitrary square-integrable initial velocity, whose uniqueness is a still unknown and challenging problem. Leray [34, 35] proved the existence of a smooth steady solution with a finite Dirichlet integral. But, the solutions obtained by Leray and Hopf did not provide much qualitative information. In particular, nothing was proven about the asymptotic structure of the wake behind the body $\mathcal{O} = \mathbb{R}^3 - \Omega$. This is a topic of great interest in itself. Finn [9] to [14] studied (1.3) within the class of solutions, termed by him physically reasonable, which tend to a limit at infinity like $|x|^{-1/2-\epsilon}$ for some $\epsilon > 0$. For small data he proved both existence and uniqueness whithin this class. In fact, his solutions satisfy the following estimate :

$$
(1.4) \qquad |\mathbf{w}(x)| \leqq C|x|^{-1} \quad \text{as } |x| \to \infty \text{ and } \nabla\mathbf{w} \in L_3(\Omega)
$$

where C is a constant. Furthermore, his solutions exhibit paraboloidal wake region behind the body $\mathcal{O}$. Rerated topics were also discussed in Fujita [15] and Ladyzhenskaia [33].

Finn has conjectured [14] that for sufficiently small data physically reasonable solutions are attainable. Namely, if we put $\mathbf{v}(t,x) = \mathbf{w}(x) + \mathbf{z}(t,x)$ and $\mathfrak{p}(t,x) = \pi(x) + \tau(t,x)$ in (1.2), (1.2) is reduced to the following equation :

$$
(1.5) \qquad \begin{cases} \mathbf{z}_t - \Delta\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{z} + \nabla\tau = 0, \\ \qquad \nabla \cdot \mathbf{z} = 0 \quad \text{in } (0,\infty) \times \Omega, \\ \qquad \mathbf{z}|_{\partial\Omega} = 0, \quad \mathbf{z}|_{t=0} = \mathbf{b} = \mathbf{a} - \mathbf{u}_\infty - \mathbf{w}, \\ \displaystyle\lim_{|x|\to\infty} \mathbf{z}(t,x) = 0. \end{cases}
$$

Then, the attainable problem is to find a solution $\mathbf{z}(t,x)$ of (1.5) such that $\mathbf{z}(t,x) \to 0$, that is $\mathbf{v}(t,x) - \mathbf{w}(x) \to 0$ as $t \to \infty$. This is called a stability problem.

The stability problem was first solved by Heywood [20, 21] in the L_2 framework. Roughly speaking, he proved that if the L_2-norm of $\mathbf{b}(x)$ is very small and if $C < 1/2$, C being the constant in (1.4), then there exists a unique solution $\mathbf{z}(t,x)$ of (1.5) satisfying the convergence property :

$$\int_\Omega |\nabla(\mathbf{u}(t,x)-\mathbf{w}(x))|^2\,dx \to 0 \quad \text{and} \quad \int_{\substack{x\in\Omega\\ |x|\leqq R}} |\mathbf{u}(t,x)-\mathbf{w}(x)|\,dx \to 0$$

as $t \to \infty$ where R is any positive number. His result was sharpened, in particular with respect to the rate of the convergence, by Masuda [37], Heywood himself [22], Miyakawa [38] and Maremonti [36] (cf. further references cited therein). But, as Finn showed in [11], if $\mathbf{w}(x)$ is a physically reasonable solution and if the forced exerted to the body $\mathcal{O}$ by the flow does not vanish, then $\mathbf{w}(x)$ is not square-integrable over Ω. Therefore, it is natural to ask the question :

(Q) Seek a solution of the problem (1.5) which belongs to the same function class as $\mathbf{w}(x)$ belongs to for each time section.

In this direction, Kato [25] solved the problem (1.1) in the L_n - framework when $\Omega = \mathbb{R}^n$ ($n \geqq 2$), $\mathbf{u}_\infty = 0$, $\mathbf{f} = 0$ and the L_n norm of $\mathbf{a}$ is very small. He employed various L_p norms and L_p - L_q estimates for the semigroup generated by the Stokes operator. His method gives us a simple but strong tool in proving globally in time existence theorem of small and smooth solutions for the non-linear equations of the parabolic type. Iwashita [24] and Dan and Shibata [6, 7] extended Kato's result to the case where $\Omega \neq \mathbb{R}^n$ ($n \geq 2$), $\mathbf{u}_\infty$, $\mathbf{f} = 0$ and the L_n norm of $\mathbf{a}$ is very small.

In this note, in §2 we consider the case where $\Omega \neq \mathbb{R}^3$, $\mathbf{u}_\infty \neq 0$ but $|\mathbf{u}_\infty|$ small enough, $\mathbf{f} \not\equiv 0$ and a certain norm of $\mathbf{f}$ and the L_3 norm of $\mathbf{b}$ are small enough. And we shall give an answer to the question (Q).

Recently, when $\mathbf{u}_\infty = 0$ and $\Omega \subset \mathbb{R}^n$ ($n \geqq 3$), Borchers and Miyakawa [4, 5], Kozono and Yamazaki [31, 32] and Yamazaki [48] proved the stability of non-trivial physically reasonable solutions by the small weak L_n perturbation. Namely, they proved that if L_n weak norm of $\mathbf{b}$ is very small, then (1.5) admits a unique solution $\mathbf{z}(t,x)$ which converges to $\mathbf{w}(x)$ as $t \to \infty$ in the L_n weak space with suitable rate of convergence. Since the physically reasonable solution of (1.3) belong to L_n weak space when $\mathbf{u}_\infty = 0$, the question (Q) was answered in the case where $\mathbf{u}_\infty = 0$. In § 3 and § 4, we extend this result to the case where $\mathbf{u}_\infty \neq 0$, focusing on the uniformity with respect to $\mathbf{u}_\infty$. Moreover, we consider a convergence problem when $|\mathbf{u}_\infty| \to 0$.

2. Existence of Stationary Solution I

In order to describe the wake region, we introduce the Oseen weight function :

$$s_{\mathbf{u}_\infty}(x) = |x| - x\cdot \mathbf{u}_\infty/|\mathbf{u}_\infty|.$$

The following result was proved by Shibata [45, Theorem 1.1] and it tells us an unique existence of small solutions to (1.3) which provides a qualitative information about the asymptotic structure of the wake behind the body $\mathcal{O}$ in terms of $s_{\mathbf{u}_\infty}$.

Theorem 2.1. *Let $3 < p < \infty$ and let δ and beta be any numbers such that $0 < \delta < 1/4$ and $0 < \delta < \beta < 1 - \delta$. Let $\mathbf{f} \in L_\infty(\Omega)$. Then, there exists a constant ϵ, $0 < \epsilon \leqq 1$, depending on p, δ and β but independent of $\mathbf{u}_\infty$ such that if $0 < |\mathbf{u}_\infty| \leqq \epsilon$ and $<< \mathbf{f} >>_{2\delta} \leqq \epsilon|\mathbf{u}_\infty|^{\beta+\delta}$, then the problem (1.3) admits solution $\mathbf{w}$ and π possessing the estimate :*

$$\|\mathbf{w}\|_{W_p^2(\Omega)} + |||\mathbf{w}|||_\delta + \|\pi\|_{W_p^1(\Omega)} \leqq |\mathbf{u}_\infty|^\beta, \tag{2.1}$$

where

$$\begin{aligned}
<< \mathbf{f} >>_{2\delta} &= \sup_{x\in\Omega}(1+|x|)^{5/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+2\delta}|\mathbf{f}(x)|,\\
|||\mathbf{w}|||_\delta &= \sup_{x\in\Omega}(1+|x|)(1+s_{\mathbf{u}_\infty}(x))^{\delta}|\mathbf{w}(x)|\\
&\quad + \sup_{x\in\Omega}(1+|x|)^{3/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+\delta}|\nabla\mathbf{w}(x)|
\end{aligned}$$

Remark. The estimate (2.1) represents the wake region behind $\mathcal{O}$. By (2.1) we see easily that $\mathbf{w} \in L_3(\Omega)$ and $\nabla\mathbf{w} \in L_{3/2}(\Omega)$. On the other hand, as we will state with references in §4, in the case where $\mathbf{u}_\infty = 0$, $\mathbf{w} \notin L_3(\Omega)$ but $\in L_{3,\infty}(\Omega)$ and $\nabla\mathbf{w} \notin L_{3/2}(\Omega)$ but $\nabla\mathbf{w} \in L_{3/2,\infty}(\Omega)$, where $L_{p,\infty}$ means the Lorents space defined in § 4, below. In fact, when $\mathbf{u}_\infty = 0$, $\mathbf{w}(x) \approx C|x|^{-1}$ and $\nabla\mathbf{w}(x) \approx C'|x|^{-2}$ as $|x| \to \infty$ with suitable constants C and C'. On the other hand, when $\mathbf{u}_\infty \neq 0$ by (2.1) we see easily that

$$\|\mathbf{w}\|_{L_3} \leqq \left[2\pi\int_0^\infty \frac{dr}{(1+r)^3 r^\delta}\int_0^\pi \frac{\sin\theta d\theta}{(1-\cos\theta)^\delta}\right]^{1/3} |\mathbf{u}_\infty|^\beta,$$

$$\|\nabla\mathbf{w}\|_{L_{3/2}} \leqq \left[2\pi\int_0^\infty \frac{dr}{(1+r)^{9/4} r^{(3+\delta)/4}}\int_0^\pi \frac{\sin\theta d\theta}{(1-\cos\theta)^{(3+\delta)/4}}\right]^{2/3} |\mathbf{u}_\infty|^\beta.$$

In order to prove Theorem 2.1, we have to investigate the estimate for solutions to the following linear Oseen equation :

$$-\Delta\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f},\;\; \nabla\cdot\mathbf{u} = 0 \;\text{ in } \Omega \text{ and } \mathbf{u}|_{\partial\Omega} = 0. \tag{2.2}$$

In [45, Theorem 4.1], we proved the following theorem.

Theorem 2.2. *Let $3 < p < \infty$ and $0 < \delta < 1/4$. Let $<< \cdot >>_{2\delta}$ and $|||\cdot|||_\delta$ be the same as in* Theorem 2.1. *Assume that $0 < |\mathbf{u}_\infty| \leqq 1$. If $<< \mathbf{f} >>_{2\delta} < \infty$, then the problem* (2.2) *admits a unique solution $(\mathbf{u},\mathfrak{p}) \in W_p^2(\Omega)^3 \times W_p^1(\Omega)$ having the estimate :*

$$\|\mathbf{u}\|_{W_p^2(\Omega)} + \|\mathfrak{p}\|_{W_p^1(\Omega)} + |||\mathbf{u}|||_\delta \leqq C_{p,\delta}|\mathbf{u}_\infty|^{-\delta} << \mathbf{f} >>_{2\delta}.$$

Since we can construct a function $\mathbf{d}$ satisfying the properties : $\mathbf{d} \in C_0^\infty(\mathbb{R}^3)^3$, $\nabla\cdot\mathbf{d} = 0$ in Ω, $\mathbf{d}|_{\partial\Omega} = -\mathbf{u}_\infty$ and $|\partial_x^\alpha\mathbf{d}| \leq C_\alpha|\mathbf{u}_\infty|$ for any α, putting $\mathbf{w} = \mathbf{d} + \mathbf{z}$, (1.3) is reduced to the equation :

$$\begin{aligned}
-\Delta\mathbf{z} + (\mathbf{u}_\infty\cdot\nabla)\mathbf{z} + (\mathbf{d}\cdot\nabla)\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{d} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla\pi&\\
= \mathbf{f} + \Delta\mathbf{d} - (\mathbf{d}\cdot\nabla)\mathbf{d} \text{ and } \nabla\cdot\mathbf{z} = 0 \text{ in } \Omega&,\\
\mathbf{z}|_{\partial\Omega} = 0.&
\end{aligned} \tag{2.3}$$

Then, given $\mathbf{y}$, let $\mathbf{z}$ be a solution to the linear Oseen equation :

$$\begin{aligned}&-\Delta\mathbf{z}+(\mathbf{u}_\infty\cdot\nabla)\mathbf{z}+\nabla\pi\\&\quad=\mathbf{f}+\Delta\mathbf{d}-(\mathbf{d}\cdot\nabla)\mathbf{d}-(\mathbf{d}\cdot\nabla)\mathbf{y}-(\mathbf{y}\cdot\nabla)\mathbf{d}-(\mathbf{y}\cdot\nabla)\mathbf{y}\quad\text{in }\Omega,\\&\nabla\cdot\mathbf{z}=0\quad\text{in }\Omega,\quad \mathbf{z}|_{\partial\Omega}=0.\end{aligned}$$

And if we consider the map $G:\mathbf{y}\to\mathbf{z}$, then by using Theorem 2.2, we can easily show that G is a contraction map in a suitable underlying space under a smallness assumption on $|\mathbf{u}_\infty|$. The fixed point of G gives a solution to (1.3). In this way, we can show Theorem 2.1 by Theorem 2.2.

In order to prove Theorem 2.2, the essential part is to estimate the convolution operator with the Oseen fundamental solution $E(\mathbf{u}_\infty)=(E_{jk}(\mathbf{u}_\infty))$ (cf. Oseen [43]) which is given by the following formula :

$$E_{jk}(\mathbf{u}_\infty(x)=(\delta_{jk}\Delta-\partial_j\partial_k)\Xi(\sigma)(x), \tag{2.4}$$
$$\Xi(\sigma)(x)=\frac{1}{8\pi\sigma}\int_0^{\sigma s_{\mathbf{u}_\infty}(x)}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha,\quad \sigma=|\mathbf{u}_\infty|/2\neq0$$

In fact let us consider the Oseen equation :

$$-\Delta\mathbf{w}+(\mathbf{u}_\infty\cdot\nabla)\mathbf{w}+\nabla\pi=\mathbf{g},\quad \nabla\cdot\mathbf{w}=0\quad\text{in }\mathbb{R}^3.$$

Then, the solution $\mathbf{w}$ is given by the formula : $\mathbf{w}=E(\mathbf{u}_\infty)*\mathbf{g}$, where $*$ is the convolution. Since

$$|E_{jk}(\mathbf{u}_\infty(x)|\leqq\frac{C_\delta}{(\sigma s_{\mathbf{u}_\infty}(x))^\delta|x|}, \tag{2.5}$$
$$|\nabla E_{jk}(\mathbf{u}_\infty(x)|\leqq\frac{C_\delta}{(\sigma s_{\mathbf{u}_\infty}(x))^\delta s_{\mathbf{u}_\infty}(x)^{1/2}|x|^{3/2}},$$
$$|\nabla E_{jk}(\mathbf{u}_\infty(x)|\leqq\frac{C_\delta}{(\sigma s_{\mathbf{u}_\infty}(x))^\delta}\left[\frac{\sigma^{1/2}}{|x|^{3/2}}+\frac{1}{|x|^2}\right],$$

we have the following theorem which was proved by [45 ,Lemma 4.3].

Theorem 2.3. *Let $0<\delta<1/4$. Let $g\in L_\infty(\mathbb{R}^3)^3$ and assume that*

$$\sup_{x\in\mathbb{R}^3}(1+|x|)^{5/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+2\delta}|\mathbf{g}(x)|<\infty.$$

Then, for $|x|\geqq1$ we have the relations :

$$|E(\mathbf{u}_\infty)*\mathbf{g}(x)|\leqq C_\delta|\mathbf{u}_\infty|^{-\delta}(1+s_{\mathbf{u}_\infty}(x))^{-\delta}|x|^{-1},$$
$$|\nabla E(\mathbf{u}_\infty)*\mathbf{g}(x)|\leqq C_\delta|\mathbf{u}_\infty|^{-\delta}(1+s_{\mathbf{u}_\infty}(x))^{-(1/2+\delta)}|x|^{-3/2},$$

Remark. The more general estimation for the convolution operator with the Oseen fundamental solutions was given by Farwig [8], where he refined the argument due to Finn [9, 10, 12, 13, 14]. A proof given in [45] is completely different from [8] for the gradient estimate.

By Theorem 2.3 and a compact perturbation argument, we can prove Theorem 2.2. A detailed proof was given in [45, §3]. This completes a rough sketch of a proof of Theorem

3. Stability Theorem I

In this section, we will discuss an unique existence theorem of globally in times solutions to (1.5) according to Shibata [45]. As a corresponding linear problem to (1.5), we consider the non-stationary linear Oseen equation :

$$(3.1) \qquad \begin{gathered} \mathbf{v}_t - \Delta \mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v} + \mathfrak{p} = 0, \;\; \nabla \cdot \mathbf{v} = 0 \;\; \text{in } (0,\infty)\times\Omega, \\ \mathbf{v}|_{\partial\Omega} = 0, \quad \mathbf{v}|_{t=0} = \mathbf{b}. \end{gathered}$$

Put

$$\begin{aligned} J_p &= \text{the completion in } L_p(\Omega)^3 \text{ of the set } \{\mathbf{u} \in C_0^\infty(\Omega)^3 \mid \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\}, \\ G_p &= \{\nabla\pi \mid \pi \in \hat{W}_p^1(\Omega)\}, \quad \hat{W}_p^1(\Omega) = \{\pi \in L_{p,loc}(\Omega) \mid \nabla\pi \in L_p(\Omega)^3\}. \end{aligned}$$

According to Fujiwara and Morimoto [16] and Miyakawa [38] (cf. Galdi [17, III]), the Banach space $L_p(\Omega)^3$ admits the Helmholtz decomposition :

$$L_p(\Omega)^3 = J_p \oplus G_p.$$

Let P be a continuous projection from $L_p(\Omega)^3$ to J_p along G_p. Applying P to (3.1), we have the Oseen evolution equation :

$$\mathbf{v}_t + \mathbb{O}(\mathbf{u}_\infty)\mathbf{v} = 0, \;\; \mathbf{v}|_{t=0} = \mathbf{b}$$

where $\mathbb{O}(\mathbf{u}_\infty) = P(-\Delta + (\mathbf{u}_\infty\cdot\nabla))$ with domain:

$$\mathcal{D}(\mathbb{O}(\mathbf{u}_\infty)) = \{\mathbf{v} \in J_p \mid \mathbf{v} \in W_p^2(\Omega)^3, \;\; \mathbf{v}|_{\partial\Omega} = 0\}.$$

Miyakawa [38] proved that $\mathbb{O}(\mathbf{u}_\infty)$ generates an analytic semigroup $\{T_{\mathbf{u}_\infty}(t)\}_{t\geqq 0}$. Applying P to (1.5), we have

$$(3.2) \qquad \mathbf{z}_t + \mathbb{O}(\mathbf{u}_\infty)\mathbf{z} + P\{L_{\mathbf{w}}\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z}\} = 0, \;\; \mathbf{z}|_{t=0} = \mathbf{b},$$

where

$$L_{\mathbf{w}}\mathbf{z} = (\mathbf{w}\cdot\nabla)\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{w}.$$

According to Kato [25], instead of (3.2) we consider the following integral equation :

$$(3.3) \qquad \mathbf{z}(t) = T_{\mathbf{u}_\infty}(t)\mathbf{b} - \int_0^t T_{\mathbf{u}_\infty}(t-s)P\{L_{\mathbf{w}}\mathbf{z}(s) + (\mathbf{z}(s)\cdot\nabla)\mathbf{z}(s)\}\,ds.$$

Shibata [45] proved the following theorem which is an answer to (Q).

Theorem 3.1. *Let* $3 < p < \infty$ *and let* δ *and* β *be the same as in* Theorem 2.1. *In addition, we assume that* $0 < \delta < 1/6$. *Let* $\mathbf{f} \in L_\infty(\Omega)$ *and* $\mathbf{b} \in J_3$. *Then, there exists an* $\epsilon > 0$, $0 < \epsilon \leqq 1$, *depending only on p,* β *and* δ *essentially such that if* $0 < |\mathbf{u}_\infty| \leqq \epsilon$, $<< \mathbf{f} >>_{2\delta} \leqq \epsilon |\mathbf{u}_\infty|^{\beta+\delta}$ *and* $\|\mathbf{b}\|_{L_3(\Omega)} \leqq \epsilon$, *then the problem* (3.3) *admits a unique solution* $\mathbf{z} \in BC([0,\infty), J_p)$ *possessing the following properties :*

$$
\begin{aligned}
&[\mathbf{z}]_{3,0,t} + [\mathbf{z}]_{\infty,1/2-3/(2p),t} + [\nabla \mathbf{z}]_{3,1/2,t} \leqq \sqrt{\epsilon}, \\
&\lim_{t\to 0+} \left[\|\mathbf{z}(t,\cdot) - \mathbf{b}\|_{L_3(\Omega)} + [\mathbf{z}]_{p,1/2-3/(2p),t} + [\nabla \mathbf{z}]_{3.1/2,t} \right] = 0.
\end{aligned}
\tag{3.4}
$$

Here and hereafter, we put

$$
[\mathbf{z}]_{p,\rho,t} = \sup_{0<s<t} s^\rho \|\mathbf{z}(s,\cdot)\|_{L_p(\Omega)}.
$$

Moreover, we have the relations :

$$
\begin{aligned}
[\mathbf{z}]_{q,1/2-3/(2q),t} &\leqq C_q \left(\epsilon + \epsilon^{1/2+\beta} \right), \quad p < q < \infty, \\
\|\mathbf{z}(t,\cdot)\|_{L_\infty} &\leqq C_m \left(\epsilon + \epsilon^{1/2+\beta} \right) t^{-1/2},
\end{aligned}
$$

for any $t \geqq 1$ *where* m *is a number such that* $3 < m < p$.

When $\mathbf{f} = 0$, the solution to (3.3) converges to the soluiton to the integral equation corresponding to the case where $\mathbf{u}_\infty = 0$ when $|\mathbf{u}_\infty| \to 0$. In order to state the theorem more precisely, we formulate the problem. Let us consider the Navier-Stokes equation with $\mathbf{u}_\infty = 0$ and $\mathbf{f} = 0$:

$$
\begin{aligned}
\mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla \mathfrak{p} = 0, \quad \nabla\cdot\mathbf{y} = 0 \quad &\text{in } (0,\infty)\times\Omega, \\
\mathbf{y}|_{\partial\Omega} = 0, \quad \mathbf{y}|_{t=0} = \mathbf{b}.&
\end{aligned}
\tag{3.5}
$$

Put $\mathbf{A} = P(-\Delta)$ with domain : $\mathcal{D}(\mathbf{A}) = \mathcal{D}(\mathbb{O}(\mathbf{u}_\infty))$. Applying P to (3.5), we have

$$
\mathbf{y}_t + \mathbf{A}\mathbf{y} + P(\mathbf{y}\cdot\nabla)\mathbf{y} = 0, \quad \mathbf{y}|_{t=0} = \mathbf{b}.
$$

Let $\{T(t)\}_{t\geq 0}$ be an analytic semigroup generated by $\mathbf{A}$. Then, instead of (3.5), we have the integral equation :

$$
\mathbf{y}(t) = T(t)\mathbf{b} - \int_0^t T(t-s)P(\mathbf{y}(s)\cdot\nabla)\mathbf{y}(s)\,ds.
\tag{3.6}
$$

A unique existence theorem of globally in time solution to (3.6) was proved by Iwashita [24]. Concerning the convergence of solutions of (3.3) to solutions of (3.6) as $|\mathbf{u}_\infty| \to 0$, we have the following theorem.

Theorem 3.2. *Let* $\mathbf{f} = 0$. *Let* $0 < \beta < 1$ *and let* $\mathbf{b}$ *be an initial velocity. Then, there exists an* ϵ, $0 < \epsilon \leqq 1$, *depending on* β *but independent of* $\mathbf{u}_\infty$ *and* $\mathbf{b}$ *such that if* $0 < |\mathbf{u}_\infty| \leqq \epsilon$, $\mathbf{b} \in J_3$ *and* $\|\mathbf{b}\|_{L_3} \leqq \epsilon$, *then* (3.3) *admits a unique solution* $\mathbf{z}(t,x)$ *such that* $\mathbf{z}(t,x) \in BC([0,\infty), J_3)$ *and* $\mathbf{z}$ *has the estimate* (3.4). *Moreover, if* $\mathbf{y} \in BC([0,\infty), J_3)$ *be a solution to* (3.6), *then we have the following convergence property :*

$$\|\mathbf{z}(t,\cdot) - \mathbf{y}(t,\cdot)\|_{L_q(\Omega)} \leqq C_q \left(t^{-(1/2-3/(2q))} + t^{3/2q}\right) |\mathbf{u}_\infty|^\beta, \ \ 3 \leqq q < \infty,$$

$$\|\mathbf{z}(t,\cdot) - \mathbf{y}(t,\cdot)\|_{L_\infty(\Omega)} \leqq C_m \left(t^{(1-3/2m)} + 1\right) \|\mathbf{u}_\infty|^\beta,$$

$$\|\nabla(\mathbf{z}(t,\cdot) - \mathbf{y}(t,\cdot))\|_{L_3(\Omega)} \leqq C \left(t^{-1/2} + 1\right) \|\mathbf{u}_\infty|^\beta$$

for any $t > 0$ *where* m *is a constant* > 3.

Now, we will give a rough sketch of a proof of Theorem 3.1 according to [45, § 5]. We will show the following assertion only in this note :

Assertion. *There exists an* $\epsilon > 0$ *such that if* $\mathbf{w}$ *and* $\mathbf{b} \in J_3$ *satisfy the condition :* $\|\mathbf{b}\|_{L_3(\Omega)} + |||\mathbf{w}|||_\delta \leqq \epsilon$, *then* (3.6) *admits a unique solution* $\mathbf{y} \in BC([0,\infty), J_3)$ *satisfying the estimate :*

$$\|\mathbf{y}(t)\|_{L_3(\Omega)} \leqq C\epsilon, \ \ \|\mathbf{y}(t)\|_{L_p(\Omega)} \leqq C\epsilon t^{-(1/2-3/(2p))}, \ \ \|\nabla \mathbf{y}(t)\|_{L_3(\Omega)} \leqq C\epsilon t^{-1/2},$$

for any $t > 0$ *with some constant* $C > 0$.

Our proof is based on the following two theorems.

Estimate of Oseen semigroup I. *Let* $|\mathbf{u}_\infty| \leqq M$. *Then, for* $t \geqq 1$ *we have the following estimate :*

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_q(\Omega)} \leqq C_{M,p,q} t^{-\nu} \|\mathbf{a}\|_{L_p(\Omega)}, \ \ \nu = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \ \ 1 < p \leqq q \leqq \infty,$$

$$\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_q(\Omega)} \leqq C_{M,p,q} t^{-(\nu+k/2)} \|\mathbf{a}\|_{L_p(\Omega)}, \ \ 1 < p \leqq q < \infty,$$

Moreover, for $0 < t \leqq 1$ *we have*

$$\|\nabla^k T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_q(\Omega)} \leqq C_{M,k,p,q} t^{-(\nu+k/2)} \|\mathbf{a}\|_{L_p(\Omega)}, \ \ 1 < p \leqq q < \infty.$$

The estimate of Oseen semigroup I was proved by Kobayashi and Shibata [26].

Hardy type inequality. *Let* $0 \leqq \alpha \leqq 1/3$ *and put* $d_\alpha(x) = s_{\mathbf{u}_\infty}(x)^\alpha |x|^{1-\alpha} \log|x|$. *Then, we have*

$$\|v/d_\alpha\|_{L_3(\Omega)} \leqq C_\alpha \|\nabla v\|_{L_3(\Omega)}, \ \ v \in W_3^1(\Omega) \ \ \text{with } v|_{\partial\Omega} = 0.$$

This kind of Hardy type inequality was proved by Shibata [45]. The integral equation (3.6) is solved by contraction mapping principle. Therefore, the essential part is to estimate the integral of the right-hand side of (3.6). Put

$$A(t) = \int_0^t T_{\mathbf{u}_\infty}(t-s) P L_{\mathbf{w}} \mathbf{y}(s)\, ds, \ \ B(t) = \int_0^t T_{\mathbf{u}_\infty}(t-s) P(\mathbf{y}(s) \cdot \nabla)\mathbf{y}(s)\, ds.$$

Let $\mathbf{y}(t) \in BC([0,\infty), J_3)$ satisfy the condition : $\mathbf{y}(t) \in W_3^1(\Omega)^3$ and $\mathbf{y}(t)|_{\partial\Omega} = 0$ for ar $t > 0$. Recall that $L_{\mathbf{w}}\mathbf{y}(s) = (\mathbf{w}\cdot\nabla)\mathbf{y}(s) + (\mathbf{y}(s)\cdot\nabla)\mathbf{w}$. To estimate A, we use the followir relations :

$$
\begin{aligned}
\|P(\mathbf{w}\cdot\nabla)\mathbf{y}(s)\|_{L_{3/2}(\Omega)} &\leqq C\|\mathbf{w}\|_{L_3(\Omega)}\|\nabla\mathbf{y}(s)\|_{L_3(\Omega)} \leqq C|||\mathbf{w}|||_\delta\|\nabla\mathbf{y}(s)\|_{L_3(\Omega)}\\
\|P(\mathbf{y}(s)\cdot\nabla)\mathbf{w}\|_{L_{3/2}(\Omega)} &\leqq C\|d_\alpha\nabla\mathbf{w}\|_{L_3(\Omega)}\|\mathbf{y}(s)/d_\alpha\|_{L_3} \leqq C|||\mathbf{w}|||_\delta\|\nabla\mathbf{y}(s)\|_{L_3(\Omega)},\\
\|P(\mathbf{w}\cdot\nabla)\mathbf{y}(s)\|_{L_3(\Omega)} &\leqq C\|\mathbf{w}\|_{L_\infty(\Omega)}\|\nabla\mathbf{y}(s)\|_{L_3(\Omega)} \leqq C|||\mathbf{w}|||_\delta\|\nabla\mathbf{y}(s)\|_{L_3(\Omega)},\\
\|P(\mathbf{y}(s)\cdot\nabla)\mathbf{w}\|_{L_3(\Omega)} &\leqq C\|d_\alpha\mathbf{w}\|_{L_\infty(\Omega)}\|\mathbf{y}(s)/d_\alpha\|_{L_3(\Omega)} \leqq C|||\mathbf{w}|||_\delta\|\nabla\mathbf{y}(s)\|_{L_3(\Omega)}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\|A(t)\|_{L_3(\Omega)} &\leqq C|||\mathbf{w}|||_\delta\int_0^t (t-s)^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{3})}s^{-\frac{1}{2}}\,ds[\nabla\mathbf{y}]_{3,1/2,t}\\
&\leqq CB(1/2,1/2)|||\mathbf{w}|||_\delta[\nabla\mathbf{y}]_{3,1/2,t}\\
\|A(t)\|_{L_p(\Omega)} &\leqq C|||\mathbf{w}|||_\delta\int_0^t (t-s)^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})}s^{-\frac{1}{2}}\,ds[\nabla\mathbf{y}]_{3,1/2,t}\\
&\leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})}B(3/(2p),1/2)|||\mathbf{w}|||_\delta[\nabla\mathbf{y}]_{3,1/2,t},\\
\|\nabla A(t)\|_{L_3(\Omega)} &\leqq C|||\mathbf{w}|||_\delta\left\{\int_0^{t-1}(t-s)^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{3})-\frac{1}{2}}s^{-\frac{1}{2}}\,ds\right.\\
&\qquad\left.+\int_{t-1}^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\,dx\right\}[\nabla\mathbf{y}]_{3,1/2,t}\\
&\leqq Ct^{-\frac{1}{2}}|||\mathbf{w}|||_\delta[\nabla\mathbf{y}]_{2,1/2,t},
\end{aligned}
$$

where $B(a,b)$ means the beta function. In order to estimate $B(t)$, we fix q such as $1/q$ $1/p + 1/3$ and we use the estimate :

$$
\|P(\mathbf{y}_1(s)\cdot\nabla)\mathbf{y}_2(s)\|_{L_q(\Omega)} \leqq C\|\mathbf{y}_1(s)\|_{L_p(\Omega)}\|\nabla\mathbf{y}_2(s)\|_{L_3(\Omega)}.
$$

Then, we have

$$
\begin{aligned}
\|B(t)\|_{L_3(\Omega)} &\leqq C\int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{3}+\frac{1}{p}-\frac{1}{3})}s^{-(\frac{1}{2}-\frac{3}{2p})}s^{-\frac{1}{2}}\,ds[\mathbf{y}]_{p,\mu,t}[\nabla\mathbf{y}]_{3,1/2,t}\\
&\leqq CB(1-3/(2p),3/(2p))[\mathbf{y}]_{p,\mu,t}[\nabla\mathbf{y}]_{3.1/2,t},\quad \mu=\frac{1}{2}-\frac{3}{2p},\\
\|B(t)\|_{L_p(\Omega)} &\leqq C\int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{3}+\frac{1}{p}-\frac{1}{p})}s^{-(\frac{1}{2}-\frac{3}{2p})}s^{-\frac{1}{2}}\,ds[\mathbf{y}]_{p,\mu,t}[\nabla\mathbf{y}]_{3,1/2,t}\\
&\leqq CB(1/2,3/(2p))t^{-(\frac{1}{2}-\frac{3}{2p})}[\mathbf{y}]_{p,\mu,t}[\nabla\mathbf{y}]_{3,1/2,t},\\
\|\nabla B(t)\|_{L_3(\Omega)} &\leqq C\int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{3}+\frac{1}{p}-\frac{1}{3})-\frac{1}{2}}s^{-(\frac{1}{2}-\frac{3}{2p})}s^{-\frac{1}{2}}\,ds[\mathbf{y}]_{p,\mu,t}[\nabla\mathbf{y}]_{3,1/2,t}\\
&\leqq CB(1/2-3/(2p),3/(2p))t^{-\frac{1}{2}}[\mathbf{y}]_{p,\mu,t}[\nabla\mathbf{y}]_{3,1/2,t}.
\end{aligned}
$$

From these estimations, we see easily that the map $\mathbf{y}(t) \mapsto \mathbf{z}(t)$:

$$\mathbf{z}(t) = T_{\mathbf{u}_\infty}(t)\mathbf{b} - \int_0^t T_{\mathbf{u}_\infty}(t-s)P[L_{\mathbf{w}}\mathbf{y}(s) + (\mathbf{y}(s)\cdot\nabla)\mathbf{y}(s)]\,ds,$$

is contraction, provided that $\|\mathbf{b}\|_{L_3(\Omega)}$ and $|||\mathbf{w}|||_\delta$ are small enough. This completes the proof of Assertion. Further estimations in Theorem 3.1 is also obtained by using Kato's argument [25]. This is rough sketch of a proof of Theorem 3.1 by using Kato's method, further developed in combination with the L_p-L_q estimate of Oseen semigroup and Hardy type inequality.

4. Uniform estimate of Stationary Solutions with respect to $\mathbf{u}_\infty$ near 0

In this section and next section, we consider the convergence problem as $|\mathbf{u}_\infty| \to 0$ when an external force $\mathbf{f} \not\equiv 0$. In this section and next section, we assume that the external force is given by potential only, namely,

$$\mathbf{f} = \nabla \cdot F$$

with some potential force F. The difficulty arises from the fact that the solution $\mathbf{w}$ of (1.3) with $\mathbf{u}_\infty = 0$, even if it is small enough, does not belong to the space $L_3(\Omega)$ in general, contrary to the case $\mathbf{u}_\infty \neq 0$ as already mensionted in the last part of § 1. In fact, Borchers and Miyakawa [5, Theorem 2.4], Kozono and Sohr [29, Theorem C] and Kozono, Sohr and Yamazaki [30, Theorem 2, (1)] showed that the solution $\mathbf{w}$ of (1.3) with $\mathbf{u}_\infty = 0$ belong to $L_3(\Omega)$ only in very restricted situations. More detailed references are found in Kozono and Yamazaki [31, 32]. It follows that one cannot find the limit of the solution $\mathbf{w}$ in the space $L_3(\Omega)$ in general as $|\mathbf{u}_\infty| \to 0$.

On the other hand, the problem (1.3) is considered by many authors in the $\mathbf{u}_\infty = 0$ case. Novotny and Padula [41, 42] and Borchers and Miyakawa [4, 5] proved the following assertion : If $|F(x)| \leqq c|x|^{-2}$ holds with sufficiently small c, then there exists a unique solution $\mathbf{w}$ of (1.3) such that $|\mathbf{w}(x)| \leqq C|x|^{-1}$ and that $|\nabla\mathbf{w}(x)| \leqq C|x|^{-2}$. Furthermore, Nazarov and Pileckas [39, 40] obtained the asymptotic expansion of the solution, the principal term in which is homogeneous of order -1. Hence the solution $\mathbf{w}$ does not belong to $L_3(\Omega)$ in general, but belongs to the weak-L_3 space $L_{3,\infty}(\Omega)$,which is slightly larger than $L_3(\Omega)$. Similarly, the derivative $\nabla\mathbf{w}$ belongs to $L_{3/2,\infty}(\Omega)$ but not to $L_{3/2}(\Omega)$ unlike the $\mathbf{u}_\infty \neq 0$ case.

Later on, by introducing the weak - L_p spaces and modifying the L_p - theory and duality argument of Kozono and Sohr [27, 28] for $n \geqq 4$ accordingly, Kozono and Yamazaki [31] gave a sufficient condition on the external force for the problem (1.3) to have a unique small solution $\mathbf{w} \in L_{n,\infty}(\Omega)$ satisfying $\nabla\mathbf{w} \in L_{n/2,\infty}(\Omega)$ in the case $\mathbf{u}_\infty = 0$ when $n \geqq 3$. In this note, we will state an extension of Kozono-Yamazaki to the case $\mathbf{u}_\infty \neq 0$ only when $n = 3$. The argument due to Kozono and Yamazaki [31] is based on the homogeneity of the Stokes operator and hence is not applicable to our situation here. Instead we construct the parametrices of the stationary Oseen equation in exterior domains from the fundamental solution on the whole space by way of the standard cut-off procedure. Our method is similar to that of Shibata [45] , but in order to treat external forces with little regularity as in Kozono and Sohr [28], we have to construct two parametrices on two different function spaces. We can prove that our argument holds in the case $n \geqq 4$ as well with little extra effort, cf. Shibata and Yamazaki [46, 47] and Yamazaki [49].

In order to state our main results precisely, first of all we introduce the definition of the Lorenz spaces $L_{p,q}(\Omega)$ for $1 \leqq p < \infty$ as follows:

$$f \in L_{p,q}(G) \overset{def}{\Longleftrightarrow} \begin{cases} \|f\|_{L_{p,q}(G)} = \left\{ \int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t} \right\}^{1/q} & 1 \leqq q < \infty, ; \\ \|f\|_{L_{p,\infty}(G)} = \sup_{\sigma > 0} \sigma m(\sigma, f)^{1/p} < \infty & q = \infty, \end{cases}$$

where

$$f^*(t) = \inf\{\sigma > 0 \mid m(\sigma, f) \leqq t\}\,; \;\; m(\sigma, f) = |\{x \in G \mid |f(x)| > \sigma\}|$$

and $|\cdot|$ denotes the Lebesgue measure.

Note that under the assumption : $\nabla \cdot \mathbf{w} = 0$ we have

$$(\mathbf{w} \cdot \nabla)\mathbf{w} = \nabla \cdot (\mathbf{w} \otimes \mathbf{w}).$$

Below, we say that $(\mathbf{w}, \pi)$ is a solution of (1.3) if $(\mathbf{w}, \pi)$ satisfy the following formulas:

$$(\nabla \mathbf{w}, \nabla \varphi) + ((\mathbf{u}_\infty \cdot \nabla)\mathbf{w}, \varphi) - (\mathbf{w} \otimes \mathbf{w}, \nabla \varphi) - (\pi, \nabla \cdot \varphi) = -(F, \nabla \varphi)$$

for any $\varphi = {}^t(\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\Omega)^3$, and

$$\nabla \cdot \mathbf{w} = 0 \;\; \text{in } \Omega, \quad \mathbf{w}|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \lim_{|x| \to \infty} \mathbf{w}(x) = 0,$$

where

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(x) \cdot \mathbf{v}(x)\, dx, \;\; (F, G) = \sum_{j,k=1}^{3} \int_\Omega F_{jk}(x) G_{jk}(x)\, dx$$

for two 3×3 matrix functions F and G.

The following theorem is our main result in this section which is proved by Shibata and Yamazaki [47].

Theorem 4.1. (1)(Existence) *There eixsts an $\epsilon > 0$ such that if $F = (F_{jk})$, $F_{jk} \in L_{3/2,\infty}(\Omega)$ and*

$$\sum_{j,k=1}^{3} \|F_{jk}\|_{L_{3/2,\infty}(\Omega)} + |\mathbf{u}_\infty| \leqq \epsilon,$$

then the problem (1.3) *admits a solution $(\mathbf{w}, \pi) \in L_{3,\infty}(\Omega)^3 \times L_{3/2,\infty}(\Omega)$ such that $\nabla \mathbf{w} \in L_{3/2,\infty}(\Omega)^{3\times 3}$, and moreover*

$$\|\nabla \mathbf{w}\|_{L_{3/2,\infty}(\Omega)} + \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} + \|\pi\|_{L_{3/2,\infty}(\Omega)} \leqq C\epsilon$$

where C is independent of F, $\mathbf{w}$, π, ϵ and $\mathbf{u}_\infty$.

(2) (Uniqueness) *There exists an* $\epsilon' > 0$ *such that if* $(\mathbf{w}_j, \pi_j)$, $j = 1, 2$, *are solutions of* (1.3) *with the same external force* $\mathbf{f}$ *such that* $\mathbf{w}_j \in L_{3,\infty}(\Omega)$, $\nabla \mathbf{w}_j \in L_{3/2,\infty}(\Omega)$, $\pi_j \in L_{3/2,\infty}(\Omega)$ *and moreover*

$$\|\mathbf{w}_j\|_{L_{3,\infty}(\Omega)} \leqq \epsilon'$$

then $\mathbf{w}_1 = \mathbf{w}_2$ *and* $\pi_1 = \pi_2$.

Since we have the uniform estimate of solutions $\mathbf{w}$ of (1.3) with respect to $\mathbf{u}_\infty$, if we fix the external force $\mathbf{f} = \nabla \cdot F$, then when $|\mathbf{u}_\infty| \to 0$ the solution of (1.3) in the $\mathbf{u}_\infty \neq 0$ case converges to the solution of (1.3) with $\mathbf{u}_\infty = 0$ constructed by Kozono and Yamazaki [31] in the weak $*$ $L_{3,\infty}$ norm. But, this convergence is not in the strong $L_{3,\infty}$ norm. In fact, since from the discussion in §2 we know that the solution of (1.3) in the $\mathbf{u}_\infty \neq 0$ case belongs to L_3, if we have the strong convergence in the $L_{3,\infty}$ - norm, then the limit function must belong to $L_3(\Omega)$. But, as we already stated, in general it does not hold, so that we can not have the strong convergence in general. This fact was discussed in [47, § 4].

Now, we shall give a sketch of a proof of Theorem 4.1 below. The linearized equation corresponding to (1.3) is the following Oseen equation in Ω :

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla \pi = \nabla \cdot F, \ \ \nabla \cdot \mathbf{u} \quad \text{in } \Omega, \\ \qquad \mathbf{u}|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

As already mentioned, since the Oseen equation has the first order term $\mathbf{u}_\infty \cdot \nabla$, Kozono and Sohr method developed in [28] does not seem to match with the Oseen equation. We used a compact perturbation method, the idea of which goes back to Shibata [44]. Namely, combining the unique existence and estimates of solutions in the whole space case and in the bounded domain case by using the cut-off techique, we reduce the problem to the Fredholm type equation on the right hand side. And then, the sharp uniqueness theorem for the Oseen equation in Ω implies the invertibility of this Fredholm equation. Since we have to keep the divergence free condition, we use Bogovski-Pileckas lemma ([2, 3] and also [17, 24]). While we have proved a linear theorem with very general exponents p and q in [47], here we only state the following theorem in order to explain our basical idea.

Linear Theorem. *Let* $3/2 \leqq p < 3$ *and* $F = (F_{i,j})$ (3×3 *matrix*) *with* $F_{ij} \in L_{p,\infty}(\Omega)$. *Then, there exists an* $\epsilon > 0$ *independent of* F *such that if* $|\mathbf{u}_\infty| \leqq \epsilon$, *then* (4.1) *admits a unique solution* $(\mathbf{u}, \pi) \in L_{3p/(3-p),\infty}(\Omega)^3 \times L_{p,\infty}(\Omega)$ *with* $\nabla \mathbf{u} \in L_{p,\infty}(\Omega)^{3\times 3}$.

Moreover, there exists a constant C *independent of* $\mathbf{u}_\infty$, F, $\mathbf{u}$ *and* π *such that*

$$\|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\Omega)} + \|\nabla \mathbf{u}\|_{L_{p,\infty}(\Omega)} + \|\pi\|_{L_{p,\infty}(\Omega)} \leqq C\|F\|_{L_{p,\infty}(\Omega)}. \tag{4.2}$$

Now, we explain how to solve (1.3) by using Linear Theorem. As was already stated in §2, first we construct a vector of $C_0^\infty(\mathbb{R}^3)$ functions $\mathbf{d}(x)$ satisfying the properties :

$$\nabla \cdot \mathbf{d}(x) = 0, \quad \mathbf{d}(x)|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \mathbf{d}(x) = 0 \quad (|x| \geqq {}^{\exists}R),$$
$$|\partial_x^\alpha \mathbf{d}(x)| \leqq C_\alpha |\mathbf{u}_\infty| \quad \forall \alpha.$$

Such a vector-valued function is easily constructed by using the Bolovski lemma. Put $\mathbf{u} = \mathbf{d} + \mathbf{z}$ and then (1.3) is reduced to (2.3). As the underlying space, we put

$$\mathcal{I}_\sigma = \{(\mathbf{u}, \pi) \in L_{3,\infty}(\Omega)^3 \times L_{3/2,\infty}(\Omega) \mid \nabla \mathbf{u} \in L_{3/2,\infty}(\Omega)^{3\times 3},\ \mathbf{u}|_{\partial\Omega} = 0,\ \nabla \cdot \mathbf{u} = 0$$
$$\|\mathbf{u}\|_{L_{3,\infty}(\Omega)} + \|\nabla \mathbf{u}\|_{L_{3/2,\infty}(\Omega)} + \|\nabla \pi\|_{L_{3/2,\infty}(\Omega)} \leqq \sigma\},$$

because the exponent p for which the assertions that $\mathbf{w} \in L_{3p/(3-p)}(\Omega)$ implies $\mathbf{w} \otimes \mathbf{w} \in L_p(\Omega)$ and that $\nabla \mathbf{w} \in L_p(\Omega)$ imples $\mathbf{w} \in L_{3p/(3-p)}(\Omega)$ is equal to 3/2 only. By using Linear Theorem and the contraction mapping principle, we can prove the existence of solutions to (1.3) in $\mathcal{I}_\sigma$ immediately under suitable choice of small positive number σ.

From now on, we give

A Sketch of Our Proof of Linear Theorem. *1st step : Analysis of solutions in* $\mathbb{R}^3$. By Fourier transform we can write a solution $(\mathbf{u}, \pi)$ to the equation in the whole space :

$$(-\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla))\mathbf{u} + \nabla \pi = \nabla \cdot F, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3$$

by the following form :

$$\mathbf{u}(x) = E_{\mathbf{u}_\infty} * (\nabla \cdot F)(x) = \mathcal{F}^{-1}\left[\sum_{j=1}^{3} \frac{i\xi_j}{|\xi|^2 + i\mathbf{u}_\infty \cdot \xi}\left(\hat{F}_j(\xi) - \frac{\xi(\xi \cdot \hat{F}_j(\xi))}{|\xi|^2}\right)\right](x),$$
$$\pi(x) = \Pi * (\nabla \cdot F)(x) = \mathcal{F}^{-1}\left[\sum_{j=1}^{3} \frac{\xi_j(\xi \cdot \hat{F}_j(\xi))}{|\xi|^2}\right](x).$$

Since

$$\left|\xi^\alpha \left(\frac{\partial}{\partial \xi}\right)^\alpha (|\xi|^2 + i|\mathbf{u}_\infty|\xi_1)^{-1}\right| \leqq C_\alpha ||\xi|^2 + i|\mathbf{u}_\infty|\xi_1|^{-1} \quad \forall \alpha,$$

where C_α is independent of $|\mathbf{u}_\infty|$, by the orthogonal transformation in ξ and the Lizorkin theorem about the Fourier multiplier oprator we can see easily that

$$\|\mathbf{u}\|_{L_{3p/(3-p)}(\mathbf{R}^3)} + \|\nabla \mathbf{u}\|_{L_p(\mathbf{R}^3)} + \|\pi\|_{L_p(\mathbf{R}^3)} \leqq C_p \|F\|_{L_p(\mathbf{R}^3)}.$$

Since $L_{p,\infty} = (L_{p_1}, L_{p_2})_{\theta,\infty}$, $1/p = (1-\theta)/p_1 + \theta/p_2$ in the real interpolation sense, we have

(4.3) $$\|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\mathbf{R}^3)} + \|\nabla \mathbf{u}\|_{L_{p,\infty}(\mathbf{R}^3)} + \|\pi\|_{L_{p,\infty}(\mathbf{R}^3)} \leqq C_p \|F\|_{L_{p,\infty}(\mathbf{R}^3)}.$$

After cuting off the solutions, we have to handle with the following equation :

(4.4) $$-\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3,$$

where $\mathbf{f} \in L_{p,\infty}(\mathbb{R}^3)$ with supp $\mathbf{f} \subset B_b = \{x \in \mathbb{R}^3 \mid |x| < b\}$. Let $(E(\mathbf{u}_\infty)(x), P(x))$ denote the Oseen fundamental solution, and then the solution of (4.4) is given by the convolution formula : $\mathbf{u} = E(\mathbf{u}_\infty) * \mathbf{f}$ and $\pi = \Pi * \mathbf{f}$ where $E_{jk}(\mathbf{u}_\infty)$ is given by the formula (2.4) and

$$\Pi(x) = \frac{1}{4\pi}\frac{x}{|x|^3}, \quad x = {}^t(x_1, x_2, x_3).$$

Since

$$|E(\mathbf{u}_\infty)(x)| \leqq \frac{C}{|x|}, \quad |\nabla E(\mathbf{u}_\infty)| \leqq \begin{cases} \dfrac{C}{|x|^{3/2} s_{\mathbf{u}_\infty}(x)^{1/2}} & (\mathbf{u}_\infty \neq 0) \\ \dfrac{C}{|x|^2} & (\mathbf{u}_\infty = 0), \end{cases} \quad |\Pi(x)| \leqq \frac{C}{|x|^2}$$

as follows from (2.5) with $\delta = 0$ where C is independent of $\mathbf{u}_\infty$, we have

$$\|E(\mathbf{u}_\infty)\|_{L_{3,\infty}(\mathbb{R}^3)} \leqq C, \quad \|\nabla E(\mathbf{u}_\infty)\|_{L_{3/2,\infty}(\mathbb{R}^3)} \leqq C, \quad \|\Pi\|_{L_{3/2,\infty}(\mathbb{R}^3)} \leqq C,$$

where C is independent of $\mathbf{u}_\infty$. Therefore, by the generalized Young inequality we see that

$$\begin{aligned} \|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\mathbb{R}^3)} &\leqq \|E(\mathbf{u}_\infty)\|_{L_{3/2,\infty}(\mathbb{R}^3)}\|\mathbf{f}\|_{L_q(\mathbb{R}^3)} \leqq C_b\|\mathbf{f}\|_{L_{p,\infty}(\mathbb{R}^3)}, \\ \|\nabla\mathbf{u}\|_{L_{p,\infty}(\mathbb{R}^3)} &\leqq \|\nabla E(\mathbf{u}_\infty)\|_{L_{3/2,\infty}(\mathbb{R}^3)}\|\mathbf{f}\|_{L_q(\mathbb{R}^3)} \leqq C_b\|\mathbf{f}\|_{L_{p,\infty}(\mathbb{R}^3)}, \\ \|\pi\|_{L_{p,\infty}(\mathbb{R}^3)} &\leqq \|\Pi\|_{L_{3/2,\infty}(\mathbb{R}^3)}\|\mathbf{f}\|_{L_q(\mathbb{R}^3)} \leqq C_b\|\mathbf{f}\|_{L_{p,\infty}(\mathbb{R}^3)}, \end{aligned}$$

where $1 + (3-p)/3p = 1/3 + 1/q$, $1 + 1/p = 2/3 + 1/q$ and $1 \leqq q < p$. To obtain that $q \geqq 1$, we need the assumption : $p \geqq 3/2$. The restriction : $p < 3$ comes from the Sobolev inequality :

$$\|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\mathbb{R}^3)} \leqq C_p\|\nabla\mathbf{u}\|_{L_p(\mathbb{R}^3)}.$$

2nd step : Solutions in a bounded domain. Let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D. By interpolationg the well-known theorem concerning the Stokes equation and Oseen equation in a bounded domain, we have the following theorem.

Theorem. *Given $F = (F_{ij}) \in L_{p,\infty}(D)^{3\times 3}$, $F_0 \in L_{p,\infty}(D)$ and $c \in \mathbb{R}$, there exists a unique solution $(\mathbf{w}, \pi) \in W^1_{p,\infty}(D)^3 \times L_{p,\infty}(D)$ to the equation :*

$$\begin{aligned} &(\nabla\mathbf{w}, \nabla\varphi)_D + ((\mathbf{u}_\infty \cdot \nabla)\mathbf{w}, \varphi)_D - (\pi, \nabla\cdot\varphi)_D \\ &\qquad = (F, \nabla\varphi)_D + (F_0, \varphi)_D \quad \forall\,\varphi \in C_0^\infty(D), \end{aligned}$$

$$\int_D \pi\,dx = c, \quad \nabla\cdot\mathbf{w} = 0 \quad \text{in } \Omega, \quad \mathbf{w}|_{\partial\Omega} = 0.$$

Moreover, if $|\mathbf{u}_\infty| \leqq \sigma_0$ and $1 < p < 3$, then there exists a constant C depending on p, D and σ_0 such that

$$\|\mathbf{w}\|_{L_{3p/(3-p),\infty}(D)} + \|\nabla\mathbf{w}\|_{L_{p,\infty}(D)} + \|\pi\|_{L_{p,\infty}(D)} \leqq C\|(F, F_0)\|_{L_{p,\infty}(D)}$$

If $F = 0$, then $\mathbf{w} \in W^2_{p,\infty}(D)$, $\pi \in W^1_{p,\infty}(D)$ and

$$\|\mathbf{w}\|_{W^2_{p,\infty}(D)} + \|\pi\|_{W^1_{p,\infty}(D)} \leqq C\|F_0\|_{L_{p,\infty}(D)}.$$

Here ,

$$(\mathbf{u}, \mathbf{v})_D = \int_D \mathbf{u}(x) \cdot \mathbf{v}(x)\, dx, \ \ (F, G)_D = \sum_{j,k=1}^{3} \int_D F_{jk}(x) G_{jk}(x)\, dx$$

for any 3×3 matrices valued functions F and G.

For the latter purpose, we write the solution given in the above theorem as follows :

$$\mathbf{w} = \mathcal{L}(D, \mathbf{u}_\infty)[F, F_0, c], \quad \pi = \mathfrak{p}(D, \mathbf{u}_\infty)[F, F_0, c].$$

3rd step : Bogovskii - Pileckas Operator. Let $1 < p < \infty$ and let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D. Put

$$W^m_{p,\infty,0}(D) = \{u \in W^m_{p,\infty}(D) \mid \partial_x^\alpha u|_{\partial D} = 0 \quad (|\alpha| \leqq m-1)\},$$
$$\overset{o}{W}{}^m_{p,\infty,0}(D) = \{u \in W^m_{p,\infty,0}(D) \mid \int_D u\, dx = 0\}.$$

Interpolating the well-known Bogovskii - Pileckas lemma (cf. [17, III 3]), we can construct a linear operator $\mathbb{B} : \overset{o}{W}{}^m_{p,\infty,0}(D) \longrightarrow W^{m+1}_{p,\infty,0}(D)^3$ such that for $f \in \overset{o}{W}{}^m_{p,\infty,0}(D)$ we have $\nabla \cdot \mathbb{B}[f] = f$ in D and

$$\|\mathbb{B}[f]\|_{W^{m+1}_{p,\infty}(D)} \leqq C\|f\|_{W^m_{p,\infty}(D)}$$

where the constant C depends on m, p and D. Since $\mathbb{B}[f] \in W^{m+1}_{p,\infty,0}(D)^3$, we can extend $\mathbb{B}[f]$ to the whole space by 0 outside of D, and then $\mathbb{B}[f] \in W^{m+1}_{p,\infty}(\mathbb{R}^3)^3$, supp $\mathbb{B}[f] \subset D$, $\nabla \cdot \mathbb{B}[f] = f_0$ in $\mathbb{R}^3$ and

$$\|\mathbb{B}[f]\|_{W^{m+1}_{p,\infty}(\mathbb{R}^3)} \leqq C\|f\|_{W^m_{p,\infty}(D)}$$

where $f_0(x)$ also denotes the 0 extension of f to the whole space.

4th step : A Reduction to the Fredholm Type Equation. Devide solution to (4.1) into three parts as follows :

$$\mathbf{u} = \mathbf{v}_\infty + \mathbf{v}_0 + \mathbf{v}_c, \quad \pi = \pi_\infty + \pi_0 + \pi_c.$$

$\mathbf{v}_\infty$ and π_∞ are defined in the following manner. Let φ_∞ and ψ_∞ be functions in $C^\infty(\mathbb{R}^3)$ such that

$$\varphi_\infty = \begin{cases} 1 & |x| \geqq R \\ 0 & |x| \leqq R-1 \end{cases}, \qquad \psi_\infty = \begin{cases} 1 & |x| \geqq R-1 \\ 0 & |x| \leqq R-2 \end{cases}.$$

Note that $\psi_\infty = 1$ on supp φ_∞. Put

$$\mathbf{v}_\infty = \psi_\infty E_{\mathbf{u}_\infty}[\varphi_\infty F] - \mathbb{B}[\nabla\psi_\infty \cdot E_{\mathbf{u}_\infty}[\varphi_\infty F]], \quad \pi_\infty = \psi_\infty \Pi[\varphi_\infty F].$$

Put $\varphi_0 = 1 - \psi_\infty$ and let $\psi_0 \in C_0^\infty(\mathbb{R}^3)$ such that

$$\psi_0(x) = \begin{cases} 1 & |x| \leqq R \\ 0 & |x| \geqq R+1 \end{cases}, \qquad \psi_0(x) = 1 \quad \text{on supp } \varphi_0.$$

Take R so large that $B_{R-4} \supset \partial\Omega$. Put $D = \Omega_{R+2} = \Omega \cap B_{R+2}$, and therefore

$$\begin{aligned} v_0 &= \psi_0 \mathcal{L}(D, \mathbf{u}_\infty)[\varphi_0 F, 0, 0] - \mathbb{B}[\nabla\psi_0 \cdot \mathcal{L}(D, \mathbf{u}_\infty)[\varphi_0 F, 0, 0]], \\ \pi_0 &= \psi_0 \mathfrak{p}(D, \mathbf{u}_\infty)[\varphi_0 F, 0, 0]. \end{aligned}$$

Then, we arrive at the following equation to $(\mathbf{v}_c, \pi_c)$:

$$\text{(4.5)} \qquad \begin{cases} -\Delta \mathbf{v}_c + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v}_c + \nabla\pi_c = r(\mathbf{u}_\infty)[f], \quad \nabla \cdot \mathbf{v}_c = 0 \quad \text{in } \Omega, \\ \mathbf{v}_c|_{\partial\Omega} = 0 \end{cases}$$

where $r(\mathbf{u}_\infty)[F] \in L_{p,\infty}(\Omega)$, supp $r(\mathbf{u}_\infty)[F] \subset D = \{x \in \mathbb{R}^3 \mid R-2 \leqq |x| \leqq R+1\}$ and $\|r(\mathbf{u}_\infty)[F]\|_{L_{p,\infty}(\Omega)} \leqq \|F\|_{L_{p,\infty}(\Omega)}$ with some constant $C > 0$ independent of $\mathbf{u}_\infty$ whenever $|\mathbf{u}_\infty| \leqq \sigma_0$. From this point of view, we have to solve the following equation :

$$\text{(4.6)} \qquad \begin{cases} -\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla\pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = 0 \end{cases}$$

where $\mathbf{f} \in L_{p,\infty}(\Omega)$ and supp $\mathbf{f} \subset D = \{x \in \mathbb{R}^3 \mid R-2 \leqq |x| \leqq R+1\}$. The equation (4.6) is solved by the compact perturbation method. In fact, put

$$\begin{aligned} P(\mathbf{u}_\infty)\mathbf{f} &= (1-\varphi)E(\mathbf{u}_\infty) * \mathbf{f}^0 + \varphi\mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] \\ &\quad + \mathbb{B}[(\nabla\varphi) \cdot (E(\mathbf{u}_\infty) * \mathbf{f}^0)] - \mathbb{B}[(\nabla\varphi) \cdot \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]] \\ Q\mathbf{f} &= (1-\varphi)p * \mathbf{f}^0 + \varphi\mathfrak{p}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] \end{aligned}$$

where

$$c = \int_{\Omega_{R+2}} \pi * \mathbf{f}^0 \, dx, \quad \varphi(x) = \begin{cases} 1 & |x| \leqq R-2 \\ 0 & |x| \geqq R+1 \end{cases}, \quad \mathbf{f}^0(x) = \begin{cases} \mathbf{f}(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

and $\mathbf{f}|_{\Omega_{R+2}}$ is the restriction of $\mathbf{f}$ to Ω_{R+2}. $P(\mathbf{u}_\infty)\mathbf{f}$ and $Q\mathbf{f}$ satisfy the following equation :

$$\begin{aligned} (-\Delta + \mathbf{u}_\infty \cdot \nabla)P(\mathbf{u}_\infty)\mathbf{f} + \nabla(Q\mathbf{f}) = \mathbf{f} + S(\mathbf{u}_\infty)\mathbf{f}, \quad \nabla \cdot P(\mathbf{u}_\infty)\mathbf{f} = 0 \quad \text{in } \Omega. \\ P(\mathbf{u}_\infty)\mathbf{f}|_{\partial\Omega} = 0 \end{aligned}$$

where

$$\begin{aligned} S(\mathbf{u}_\infty)\mathbf{f} &= 2(\nabla\varphi) \cdot \nabla E(\mathbf{u}_\infty) * \mathbf{f}^0 + (\Delta\varphi)E(\mathbf{u}_\infty) * \mathbf{f}^0 + [(\mathbf{u}_\infty \nabla \cdot)\varphi]E(\mathbf{u}_\infty) * \mathbf{f}^0 \\ &+ 2(\nabla\varphi) \cdot \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] - (\Delta\varphi)\mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] - \\ &+ (\mathbf{u}_\infty \cdot \nabla)(\varphi\mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]) \\ &+ (-\Delta + \mathbf{u}_\infty \nabla \cdot)(\mathbb{B}[(\nabla\varphi) \cdot E(\mathbf{u}_\infty) * \mathbf{f}^0] - \mathbb{B}[(\nabla\varphi) \cdot \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]]) \\ &- (\nabla\varphi)(p * \mathbf{f}^0 - \mathfrak{p}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]). \end{aligned}$$

Since $S(\mathbf{u}_\infty)\mathbf{f} \in W^1_{p,\infty}(\Omega)$ and supp $S(\mathbf{u}_\infty)\mathbf{f} \subset D$, $S(\mathbf{u}_\infty)$ is a compact operator from $L_{p,\infty,D}(\Omega)$ into itself, where

$$L_{p,\infty,D}(\Omega) = \{\mathbf{f} \in L_{p,\infty}(\Omega)^3 \mid \text{supp } \mathbf{f} \subset D\}.$$

By using the representation formula of $E(\mathbf{u}_\infty) * \mathbf{f}^0$, we see easily that

$$\|S(\mathbf{u}_\infty) - S(0)\|_{\mathcal{L}(L_{p,\infty,D}(\Omega))} \leqq C|\mathbf{u}_\infty|^{1/2} \tag{4.7}$$

when $|\mathbf{u}_\infty| \leqq 1$, where $\mathcal{L}(L_{p,\infty,D}(\Omega))$ is the set of bounded linear operators from $L_{p,\infty,D}(\Omega)$ into itself.

Our uniqueness theorem is the following one.

Uniqueness Theorem. *Let $1 < p < \infty$. If $(\mathbf{u}, \pi) \in \mathcal{S}'(\Omega)^4 \cap (W^2_{p,loc}(\Omega)^3 \times W^1_{p,loc}(\Omega))$ satisfies the homogeneous equation :*

$$-\Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla\pi = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}|_{\partial\Omega} = 0$$

and the growth order condition :

$$\lim_{R\to\infty} R^{-3} \int_{R\leqq|x|\leqq 2R} |\mathbf{u}(x)| = 0, \quad \lim_{R\to\infty} R^{-3} \int_{R\leqq|x|\leqq 2R} |\pi(x)|\,dx = 0,$$

then $\mathbf{u} = 0$ and $\pi = 0$. Here, we put

$$\mathcal{S}'(\Omega) = \{u \mid \exists U \in \mathcal{S}' \text{such that } u = U \text{on } \Omega\}.$$

Remark. if $1 < p < 3$ and $\mathbf{u} \in L_{3p/(3-p),\infty}(\Omega)$, $\nabla\mathbf{u} \in L_{p,\infty}(\Omega)$ and $\pi \in L_{p,\infty}(\Omega)$, then $(\mathbf{u}, \pi)$ satisfies the growth order condition. But, in general the uniqueness does not hold for the exterior domain when $\mathbf{u} \in L_{p,loc}(\Omega)^3$ with $\nabla\mathbf{u} \in L_{p,\infty}(\Omega)^{3\times 3}$ and $p \geqq 3$.

By using the Fredholm alternative theorem for the $I+$ compact operator, we have the following lemma.

Key Lemma. *There exists an $\epsilon > 0$ such that if $|\mathbf{u}_\infty| \leqq \epsilon$, then the inverse operator $(I + S(\mathbf{u}_\infty))^{-1}$ of $I + S(\mathbf{u}_\infty)$ exists as a bounded operator from $L_{p,\infty,D}(\Omega)$ into itself. Moreover, we have*

$$\|(I + S(\mathbf{u}_\infty))^{-1}\|_{\mathcal{L}(L_{p,\infty,D}(\Omega))} \leqq C$$

where C is independent of $\mathbf{u}_\infty$ whenever $|\mathbf{u}_\infty| \leqq \epsilon$.

Proof. By (4.7), it is sufficient to show the lemma in the case where $\mathbf{u}_\infty = 0$. In view of Fredholm alternative theorem, it is sufficient to show the injectivity of $I + S(0)$. Therefore, we take $\mathbf{f} \in L_{p,\infty,D}(\Omega)$ such that $(I + S(0))\mathbf{f} = 0$. And, we will show that $\mathbf{f} = 0$. By the definition of $S(0)$ we have $-\Delta P(0)\mathbf{f} + \nabla Q\mathbf{f} = 0$ in Ω, $\nabla \cdot P(0)\mathbf{f} = 0$ in Ω and $P(0)\mathbf{f}|_{\partial\Omega} = 0$. By the uniqueness theorem, $P(0)\mathbf{f} = 0$ and $Q\mathbf{f} = 0$. And then, employing the argument due to Koboayashi and Shibata [26], Shibata [44, 45] and Iwashita [24], we see that $\mathbf{f} = 0$.

By Key lemma, the solution $(\mathbf{v}_c, \pi_c)$ of (4.5) can be written by the formula :

$$\mathbf{v}_c = P(\mathbf{u}_\infty)(I + S(\mathbf{u}_\infty))^{-1} r(\mathbf{u}_\infty)[\mathbf{f}], \quad \pi_c = Q(I + S(\mathbf{u}_\infty))^{-1} r(\mathbf{u}_\infty)[\mathbf{f}],$$

which completes our proof of Linear Theorem.

5. Uniform Stability Theorem of Stationary Solutions With Respect to $\mathbf{u}_\infty$ near 0

In this section, we consider the stability of stationary solution obtained in Theorem 4.1, focusing on the uniform estimate for solutions to (1.5) with respect to $\mathbf{u}_\infty$ near 0. In order to state our main result precisely, first we formulate the problem. Since $\nabla \cdot \mathbf{w} = 0$ and $\nabla \cdot \mathbf{z} = 0$, we have

$$(\mathbf{w} \cdot \nabla)\mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{z} = \nabla \cdot (\mathbf{w} \otimes \mathbf{z} + \mathbf{z} \otimes \mathbf{w} + \mathbf{z} \otimes \mathbf{z}).$$

Noting this and applying the Helmholtz projection P to (1.5), we have the Cauchy problem of the semilinear evolution equation :

$$\mathbf{z}_t + \mathbb{O}(\mathbf{u}_\infty)\mathbf{z} + P[\nabla \cdot (\mathbf{w} \otimes \mathbf{z} + \mathbf{z} \otimes \mathbf{w} + \mathbf{z} \otimes \mathbf{z})] = 0 \ \text{ for } t > 0, \ \ \mathbf{z}|_{t=0} = \mathbf{b}.$$

Applying the Duhamel's principle, we have

$$\mathbf{z}(t) = T_{\mathbf{u}_\infty}(t)\mathbf{b} - \int_0^t T_{\mathbf{u}_\infty}(t-s)P[\nabla \cdot (\mathbf{w} \otimes \mathbf{z}(s) + \mathbf{z}(s) \otimes \mathbf{w} + \mathbf{z}(s) \otimes \mathbf{z}(s))]\, ds.$$

Testing the equation by $\varphi \in C_{0,\sigma}^\infty(\Omega)^3 = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\Omega)^3 \mid \nabla \cdot \varphi = 0 \text{ in } \Omega\ \}$, we have

$$\begin{aligned}
(\mathbf{z}(t), \varphi) &= (T_{\mathbf{u}_\infty}(t)\mathbf{b}, \varphi) \\
&\quad - \int_0^t (T_{\mathbf{u}_\infty}(t-s)P[\nabla \cdot (\mathbf{w} \otimes \mathbf{z}(s) + \mathbf{z}(s) \otimes \mathbf{w} + \mathbf{z}(s) \otimes \mathbf{z}(s))], \varphi)\, ds \\
&= (T_{\mathbf{u}_\infty}(t)\mathbf{b}, \varphi) \\
&\quad + \int_0^t (\mathbf{w} \otimes \mathbf{z}(s) + \mathbf{z}(s) \otimes \mathbf{w} + \mathbf{z}(s) \otimes \mathbf{z}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\, ds.
\end{aligned}$$

We introduce the following definition.

Definition. Let $3 < p < \infty$. We call $\mathbf{z}$ a mild solution of (1.5) in the class S_p if $\mathbf{z}$ satisfies the following conditions :

(i) $\quad \mathbf{z} \in BC((0,\infty); L_{3,\infty}(\Omega)), \ \ \nabla \cdot \mathbf{z} = 0, \ \ t^{(1/2-3/2p)}\mathbf{z}(t,\cdot) \in BC((0,\infty); L_{p,\infty}(\Omega));$

(ii)
$$\begin{aligned}
(\mathbf{z}(t), \varphi) &= (T_{\mathbf{u}_\infty}(t)\mathbf{b}, \varphi) \\
&\quad + \int_0^t (\mathbf{w} \otimes \mathbf{z}(s) + \mathbf{z}(s) \otimes \mathbf{w} + \mathbf{z}(s) \otimes \mathbf{z}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\, ds;
\end{aligned}$$

(iii)
$$\lim_{t \to 0+} (\mathbf{z}(t), \varphi) = (\mathbf{b}, \varphi) \qquad \forall\, \varphi \in C_{0,\sigma}^\infty(\Omega).$$

If a mild solution is regular in the usual sense, then it satisfies (1.5). To prove the regularity is now rather standard (cf. Kozono and Yamazaki [32], also Yamazaki [48]), and therefore we only give a sketch of our proof about the following existence theorem of mild solutions below.

Theorem 5.1. *Let $3 < p < \infty$. Then, there exists a $\sigma > 0$ such that if $\|\mathbf{b}\|_{L_{3,\infty}(\Omega)} + |\mathbf{u}_\infty| \leqq \sigma$ and $\nabla \cdot \mathbf{b} = 0$, then (1.5) admits a mild solution $\mathbf{z}$ in class S_p. Moreover, $\mathbf{z}$ satisfies the following estimate :*

$$[\mathbf{z}]_{3,\infty,t} + [\mathbf{z}]_{p,\infty,t} \leqq C\sigma \qquad \forall t \in (0,\infty), \tag{5.1}$$

where $C > 0$ is a constant independent of $\mathbf{u}_\infty$ and $\mathbf{b}$,

$$\begin{aligned} [\mathbf{z}]_{3,\infty,t} &= \sup_{0<s<t} \|\mathbf{z}(s,\cdot)\|_{L_{3.\infty}(\Omega)}, \\ [\mathbf{z}]_{p,\infty,t} &= \sup_{0<s<t} s^{(1/2-3/2p)} \|\mathbf{z}(s,\cdot)\|_{L_{p,\infty}(\Omega)}. \end{aligned} \tag{5.2}$$

Remark. By Marcinkiewitz interpolation theorem, for any $r \in (3,p)$ we have

$$\|\mathbf{z}(t,\cdot)\|_{L_r(\Omega)} \leqq C_r t^{-(1/2-3/2r)}\sigma \qquad \forall t \in (0,\infty).$$

Open Problem. Show the following decay property of our mild solution $\mathbf{z}$:

$$\begin{aligned} &\sup_{0<s<t} s^{1/2} \|\mathbf{z}(s,\cdot)\|_{L_\infty(\Omega)} \leqq C\sigma, \\ &\sup_{0<s<t} s^{1/2} \|\nabla\mathbf{z}(s,\cdot)\|_{L_{3,\infty}(\Omega)} \leqq C\sigma. \end{aligned}$$

Our proof of Theorem 5.1 is based on the following L_p - L_q estimate of the Oseen semigroup $\{T_{\mathbf{u}_\infty}\}_{t\geqq 0}$.

Estimate of Oseen semigroup II. (i) *When $t \geqq 1$ and $|\mathbf{u}_\infty| \leqq M$, we have the following estimates :*

$$\begin{aligned} &|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-\nu}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \\ &\qquad 1 < p < \infty, \ \nu = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \ 1 < p \leqq q < \infty, \ 1 \leqq r \leqq \infty. \\ &\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_\infty(\Omega)} \leqq C t^{-3/2p}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \qquad 1 < p < \infty, \ 1 \leqq r \leqq \infty. \\ &\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-(\nu+1/2)}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \ 1 < p \leqq q \leqq 3, \ 1 \leqq r \leqq \infty. \\ &\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-3/2p}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \\ &\qquad 1 < p \leqq q, \ 3 < q < \infty, \ 1 \leqq r \leqq \infty. \\ &\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_\infty(\Omega)} \leqq Ct^{-3/2p}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \qquad 1 < p < \infty, \ 1 \leqq r \leqq \infty. \end{aligned}$$

Here, C is independent of $\mathbf{u}_\infty$ while C depends on p, q, r and M.
(ii) *When $0 < t \leqq 1$ and $|\mathbf{u}_\infty| \leqq M$, we have the following estimates :*

$$\begin{aligned} &\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-\nu}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \qquad 1 < p \leqq q < \infty, \ 1 \leqq r \leqq \infty. \\ &\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-(\nu+1/2)}\|\mathbf{a}\|_{L_{q,r}(\Omega)}, \ 1 < p \leqq q < \infty, \ 1 \leqq r \leqq \infty. \end{aligned}$$

Here, C is also independent of $\mathbf{u}_\infty$ while C depends on p, q, r and M.

A Sketch of A Proof of Estimate of Oseen semigroup II

In order to show Estimate of Oseen semigroup II, we use the following estimate due to Kobayashi and Shibata [26] :

$$\sum_{j=0}^{1} \|\partial_t^j T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{W_\infty^m(\Omega_R)} \leqq C_{p,m,R}(1+t)^{-3/2p}\|\mathbf{a}\|_{L_p(\Omega)} \tag{5.3}$$

for any $1 < p < \infty$, $m \geqq 0$ and $R >> 1$ with a suitable constant $C_{p,d,R}$ independent of $\mathbf{u}_\infty$. Interpolating this inequality, we have

$$\sum_{j=0}^{1} \|\partial_t^j T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{W_\infty^m(\Omega_R)} \leqq C_{p,m,R}(1+t)^{-3/2p}\|\mathbf{a}\|_{L_{p,q}(\Omega)} \tag{5.4}$$

for any $1 < p < \infty$ and $1 \leqq q \leqq \infty$. Let $S_{\mathbf{u}_\infty}(t)\mathbf{a}$ denote a solution of the evolutional Oseen equation in the whole space. By the usual L_p - L_q estimate and the interpolation theorem, we have

$$\|\partial_t^j \partial_x^\alpha S_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\mathbf{R}^3)} \leqq C_{p,q,r,j,\alpha} t^{-(\nu+j+|\alpha|/2)}\|\mathbf{a}\|_{L_{p,r}(\mathbf{R}^3)}, \tag{5.5}$$

with $\nu = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$ for $1 < p \leqq q < \infty$, $1 \leqq r \leqq \infty$, and

$$\|\partial_t^j \partial_x^\alpha S_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_\infty(\mathbf{R}^3)} \leqq C_{p,q,r,j,\alpha} t^{-(3/2p+j+|\alpha|/2)}\|\mathbf{a}\|_{L_{p,r}(\mathbf{R}^3)} \tag{5.6}$$

for $1 < p < \infty$ and $1 \leqq r \leqq \infty$, when $t > 0$. By using the cut-off function and combining (5.4) - (5.6) and employing the same argument due to Kobayashi and Shibata [26] and also Iwashita [24], we have Estimate of Oseen semigroup II.

A Sketch of A Proof of Theorem 5.1.

Now, we will give a sketch of our proof of Theorem 5.1. We shall prove Theorem 5.1 by the contraction mapping principle. As the underlying space, we put

$$\begin{aligned}\mathcal{I}_\sigma = \{\mathbf{u}(t,\cdot) \in BC((0,\infty); L_{3,\infty}(\Omega)^3) \mid \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\\ [\mathbf{u}]_{3,\infty,t} + [\mathbf{u}]_{p,\infty,t} \leqq \sigma \text{ for } \forall t > 0\}.\end{aligned}$$

Given $\mathbf{u}(t) = \mathbf{u}(t,\cdot) \in \mathcal{I}_\sigma$, let us define $\mathbf{v}(t) = \mathbf{v}(t,\cdot)$ for each $t > 0$ by the formula :

$$\begin{aligned}(\mathbf{v}(t),\varphi) &= (T_{\mathbf{u}_\infty}(t)\mathbf{b},\varphi)\\ &- \int_0^t (\mathbf{w}\otimes\mathbf{u}(s) + \mathbf{u}(s,\cdot)\otimes\mathbf{w} + \mathbf{u}(s)\otimes\mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\,ds\end{aligned}$$

for all $\varphi \in C^{\infty}_{0,\sigma}(\Omega)$. What we have to show is that

$$(5.7)\qquad |(\mathbf{v}(t),\varphi)| \leqq C\{\|\mathbf{b}\|_{L_{3,\infty}(\Omega)} + \|\mathbf{w}\|_{L_{3,\infty}(\Omega)}[\mathbf{u}]_{3,\infty,t} + [\mathbf{u}]^2_{3,\infty,t}\}\|\varphi\|_{L_{3/2,1}(\Omega)},$$

$$(5.8)\qquad |(\mathbf{v}(t),\varphi)| \leqq Ct^{-(1/2-3/2p)}\{\|\mathbf{b}\|_{L_{3,\infty}(\Omega)} + \|\mathbf{w}\|_{L_{3,\infty}(\Omega)}[\mathbf{u}]_{p,\infty,t} + [\mathbf{u}]_{3,\infty,t}[\mathbf{u}[_{p,\infty,t}\}\|\varphi\|_{L_{p,1}(\Omega)},$$

Since we can get the continuity with respect to $t > 0$ of $\mathbf{v}(t,\cdot)$ by considering the difference : $(\mathbf{v}(t_1) - \mathbf{v}(t_2), \varphi)$, we see that $\mathbf{v} \in \mathcal{I}_\sigma$. Taking σ smaller if necessary, we can also see easily that the map : $\mathbf{u} \mapsto \mathbf{v}$ is a contraction one from $\mathcal{I}_\sigma$ into iteself, which completes the proof of Theorem 5.1.

Therefore, we shall explain how to get (5.7) and (5.8) from now on. The key is the following lemma.

LEMMA. *If $1 < q < r \leqq 3$ and $1/q - 1/r = 1/3$, then we have*

$$\int_0^\infty \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}\,dt \leqq C_{r,q}\|\varphi\|_{L_{q,1}(\Omega)}.$$

Remark. From the usual L_p - L_q estimate, we have

$$\|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_r(\Omega)} \leqq C_{r,q}t^{-1}\|\varphi\|_{L_q(\Omega)}$$

when $1/q - 1/r = 1/3$, which does not imply the integrability. In order to get the integrability, we used a little bit smaller spaces $L_{r,1}$ and $L_{q,1}$ than L_r and L_q, which is a crusial part of our argument.

Proof of LEMMA. Observe that

$$\int_0^\infty \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}\,dt \leqq \sum_{j=-\infty}^{\infty}\int_{2^{j-1}}^{2^j} \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}\,dt \leqq \frac{1}{2}\sum_{j=-\infty}^{\infty} 2^j m_j$$

where

$$m_j = \sup_{2^{j-1}\leqq t\leqq 2^j} \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi\|_{L_{r,1}(\Omega)}.$$

By $L_{p,1}$ - $L_{q,1}$ estimate,

$$\|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)} \leqq d_{p_k}t^{-\frac{3}{2}\left(\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)}\|\varphi\|_{L_{p_k,1}(\Omega)}$$

with suitable constant d_{p_k} independent of $\mathbf{u}_\infty$ for $k = 0, 1$, where $1 < p_0 < q < p_1 < r \leqq 3$. Since $2^{j-1} \leqq t \leqq 2^j$, we see that

$$m_j \leqq d_{p_k}2^{\left(\frac{3}{2}\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)}(2^j)^{-\left(\frac{3}{2}\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)}\|\varphi\|_{L_{p_k,1}(\Omega)}.$$

$$C_{p_k} = d_{p_k} 2^{\left(\frac{3}{2}\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)} \quad \text{and} \quad s_k = \frac{3}{2}\left(\frac{1}{p_k} - \frac{1}{r}\right) + \frac{1}{2},$$

and then

$$\sup_{j\in\mathbb{Z}} (2^j)^{s_k} m_j \leqq C_{p_k} \|\varphi\|_{L_{p_k,1}(\Omega)}, \quad k = 0, 1.$$

By the real interpolation, we see that

$$(\ell_\infty^{s_0}, \ell_\infty^{s_1})_{\theta,1} = \ell_1^q, \quad s = (1-\theta)s_0 + \theta s_1, \quad 0 < \theta < 1$$

(cf. Jöran Bergh and Jörgen Löfström [1, Theorem 5.6.1]). Therefore, we have

$$\sum_{j=-\infty}^{\infty} 2^{js} m_j \leqq C_q \|\varphi\|_{L_{q,1}(\Omega)}, \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

In particular,

$$s = (1-\theta)s_0 + \theta s_1 = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1}{2} = 1$$

because $1/q - 1/r = 1/3$, and therefore we have

$$\sum_{j=-\infty}^{\infty} 2^j m_j \leqq C_q \|\varphi\|_{L_{q,1}(\Omega)},$$

which completes the proof of the lemma.

To show (5.7), observe that

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{b}\|_{L_{3,\infty}(\Omega)} \leqq C\|\mathbf{b}\|_{L_{3,\infty}(\Omega)};$$

$$\begin{aligned}
&\left|\int_0^t (\mathbf{w}\otimes\mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\, ds\right| \\
&\quad \leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} \int_0^t \|\mathbf{u}(s)\|_{L_{3,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds \\
&\quad \leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{3,\infty,t} \int_0^\infty \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds
\end{aligned}$$

using LEMMA and noting that $2/3 - 1/3 = 1/3$,

$$\begin{aligned}
&\quad \leqq C\|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{3,\infty,t} \|\varphi\|_{L_{3/2,1}(\Omega)}; \\
&\left|\int_0^t (\mathbf{u}(s)\otimes\mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\, ds\right| \\
&\quad \leqq \int_0^t \|\mathbf{u}(s)\|^2_{L_{3,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds
\end{aligned}$$

$$\leqq C[\mathbf{u}]^2_{3,\infty,t} \int_0^\infty \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds$$
$$\leqq C[\mathbf{u}]^2_{3,\infty,t} \|\varphi\|_{L_{3/2,1}(\Omega)}.$$

To show (5.8), observe that

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{b}\|_{L_{p,\infty}(\Omega)} \leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})}\|\mathbf{b}\|_{L_{3,\infty}(\Omega)}.$$

Choose r so that $1/3 + 1/p + 1/r = 1$, and then $1/q - 1/r = 1/3$. Therefore,

$$\left|\int_0^t (\mathbf{w}\otimes\mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\, ds\right|$$
$$\leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} \int_0^t \|\mathbf{u}(s)\|_{L_{p,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{p,\infty,t} \int_0^t s^{-(\frac{1}{2}-\frac{3}{2p})} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})}\|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{p,\infty,t} \|\varphi\|_{L_{q,1}(\Omega)}.$$

In fact, since

$$\|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi\|_{L_{r,1}(\Omega)} \leqq C(t-s)^{-1}\|\varphi\|_{L_{q,1}(\Omega)}$$

as follows from that $(3/2)(1/q - 1/r) + 1/2 = 1$, we have

$$\int_0^{t/2} s^{-(\frac{1}{2}-\frac{3}{2p})} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq C \int_0^{t/2} s^{-(\frac{1}{2}-\frac{3}{2p})} (t-s)^{-1}\, ds \|\varphi\|_{L_{q,1}(\Omega)}$$
$$\leqq C(t/2)^{-1} \int_0^{t/2} s^{-(\frac{1}{2}-\frac{3}{2p})}\, ds \|\varphi\|_{L_{q,1}(\Omega)}$$
$$\leqq C(t/2)^{-1}(t/2)^{(\frac{1}{2}+\frac{3}{2p})}\|\varphi\|_{L_{q,1}(\Omega)}$$
$$\leqq C^{-(\frac{1}{2}-\frac{3}{2p})}\|\varphi\|_{L_{q,1}(\Omega)}.$$

On the other hand,

$$\int_{t/2}^t s^{-(\frac{1}{2}-\frac{3}{2p})} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq (t/2)^{-(\frac{1}{2}-\frac{3}{2p})} \int_{t/2}^t \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})} \int_0^\infty \|\nabla[T_{-\mathbf{u}_\infty}(s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})}\|\varphi\|_{L_{q,1}(\Omega)},$$

and therefore we have

$$\int_0^t s^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)} \leqq Ct^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\varphi\|_{L_{q,1}(\Omega)}.$$

In the same manner, we have

$$\begin{aligned}
&\left| \int_0^t (\mathbf{u}(s) \otimes \mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi])\, ds \right| \\
&\quad \leqq \int_0^t \|\mathbf{u}(s)\|_{L_{3,\infty}(\Omega)} \|\mathbf{u}(s)\|_{L_{p,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds \\
&\quad \leqq C[\mathbf{u}]_{3,\infty,t}[\mathbf{u}]_{p,\infty,t} \int_0^t s^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds \\
&\quad \leqq Ct^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} [\mathbf{u}]_{3,\infty,t}[\mathbf{u}]_{p,\infty,t} \|\varphi\|_{L_{q,1}(\Omega)}.
\end{aligned}$$

Combining these estimations implies (5.8), which completes the proof of Theorem 5.1.

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