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ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

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ABSTRACT. In this paper we shall prove the global existence of solutions of the classical Maxwell-Chern-Simons-Higgs equations in $(2 + 1)$-dimensional Minkowski spacetime in the temporal gauge. We also prove that the topological solution of the Maxwell-Chern-Simons-Higgs system converges to that of Maxwell-Higgs system, as $\kappa$ goes to zero. Thus we reproduce the classical result by Mongrief [6] on the global existence of the Maxwell-Klein-Gordon system in $(2 + 1)$-dimension.

1. INTRODUCTION AND MAIN RESULTS

We are concerned on the global existence problem for the Maxwell-Chern-Simons-Higgs model in $(2 + 1)$-spacetime which was introduced to consider a self-dual system having both Maxwell and Chern-Simons terms [1]. The Lagrangian is

$$
\mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_{\rho} - \langle D_\mu \phi, D^\mu \phi \rangle - \frac{1}{2} \partial_\mu N \partial^\mu N
$$

$$
- \frac{1}{2} (e|\phi|^2 + \kappa N - ev^2)^2 - e^2 N^2 |\phi|^2,
$$

(1.1)

where $g_{\mu\nu} = \text{diag}(1, -1, -1)$, $\phi$ is a complex scalar field, $N$ is a real scalar field, $A = (A_0, A_1, A_2)$ is a vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - ie A_\mu$, $e$ is the charge of the electron, and $\kappa$ is a coupling constant for the Chern-Simons term.

The Euler-Lagrange equations via variation of the action taken with respect to $(A, \phi, N)$ are

$$
\partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + 2eIm(\phi \overline{D^\rho \phi}) = 0,
$$

$$
D_\mu D^\mu \phi + U_{\overline{\phi}}(|\phi|^2, N) = 0,
$$

$$
\partial_\mu \partial^\mu N + U_N = 0.
$$

(1.2)

Letting $\rho = 0$ in (1.2), we obtain the Gauss-Law constraint

$$
\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi \overline{D_0 \phi}) = 0.
$$

(1.3)

\dagger partially supported by the BK 21 Project.
The static energy functional for the system is
\[ E = \int_{\mathbb{R}^2} \frac{1}{2} F_{6i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N), \]  
(1.4)
where \( U(|\phi|^2, N) = \frac{1}{2} (e|\phi|^2 + \kappa N - ev^2)^2 + e^2 N^2 |\phi|^2 \), \( (i = 1, 2, \mu = 0, 1, 2) \). We note that, if \((A, \phi, N)\) is a solution that makes \( E \) finite, one of the following conditions should be required;
\[ \phi \to 0 \quad \text{and} \quad N \to \frac{ev^2}{\kappa} \quad \text{(non-topological)} \quad (1.5) \]
\[ |\phi|^2 \to v^2 \quad \text{and} \quad N \to 0 \quad \text{(topological)} \quad (1.6) \]
The terms of non-topological solution refers to the solution satisfying (1.5) and topological solution to the solution satisfying (1.6). [1], [2]

In the static case, above system are reduced to the system of elliptic equation. The static energy functional is
\[ E = \int |(D_1 \pm iD_2)\phi|^2 + |D_0 \phi \pm i e \phi N|^2 \]
\[ \frac{1}{2} |F_{12} \pm (e|\phi|^2 + \kappa N - e)|^2 do \pm e \int F_{12} do. \]

The solution saturating a lower bound for the energy is called self-dual solution, which studied extensively on both two conditions (1.5), (1.6) by D. Chae et al. ([2], [4]), and on a periodic boundary condition, by Tarantello [5]. They also studied the unifying feature of Maxwell-Chern-Simons-Higgs mathematically, which was formally discribed in [1].

For a time dependent solution to the Maxwell-Chern-Simons-Higgs, there is no result as we know, however, in [6], Mongrief proved the global existence for the classical Maxwell-Klein-Gordon equations using the Lorentz gauge in \( (2+1) \) spacetime. The Lagrangian of the Maxwell-Klein-Gordon is
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \langle D_\mu \phi, D^\mu \phi \rangle. \]

He proved global existence by showing that a suitably defined higher order energy, though not strictly conserved, does not blow up in a finite time. In this article, we consider the global existence of the classical Maxwell-Chern-Simons-Higgs in the temporal gauge as well as a convergent result as the static case [3].

Before presenting main theorems, we state equations corresponding to the non-topological case in the temporal gauge.

Considering the non-topological solution of (1.2), (1.3), we put \( \tilde{N} \) to
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$N - \frac{e\nu^2}{\kappa}$ in (1.2) to obtain the following system of semilinear wave equations with constraint ($\Box = \partial_{tt} - \Delta$).

\[
\begin{align*}
\Box A_1 &= -\kappa \partial_0 A_2 + 2eIm(\phi \overline{D_1 \phi}), \\
\Box A_2 &= \kappa \partial_0 A_1 + 2eIm(\phi \overline{D_0 \phi}), \\
\Box \phi &= -ie\phi \partial_j A_j - 2ieA_j \partial_j \phi - e^2 A_j^2 \phi - U_{\overline{\phi}}, \\
\Box N &= -U_{\overline{N}}, \\
\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi \overline{\psi_0}) &= 0.
\end{align*}
\]

Above equations can be rewritten as Hamiltonian formalism;

\[
\begin{align*}
\partial_0 A_j &= F_{0j} \\
\partial_0 F_{0j} &= -\epsilon^{jk} \partial_k F_{12} - \kappa \epsilon^{jk} F_{0k} - 2eIm(\phi \overline{\psi_j}) \\
\partial_0 F_{12} &= \epsilon^{\dot{j}l} \partial_{\dot{l}} F_{0j} \\
\partial_0 \phi &= \psi_0 \\
\partial_0 \psi_0 &= D_j \psi_j - U_{\overline{\psi}} \\
\partial_0 \psi_j &= D_j \psi_0 - ieF_{0j} \phi \\
\partial_0 N &= \Omega_0 \\
\partial_0 \Omega_0 &= \partial_j \Omega_j - U_N \\
\partial_0 \Omega_j &= \partial_j \Omega_0 \\
\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi \overline{\psi_0}) &= 0.
\end{align*}
\]

supplemented by constrains,

\[
\begin{align*}
F_{jk} &= \partial_j A_k - \partial_k A_j \\
D_j \phi &= \psi_j \\
\partial_j N &= \Omega_j \\
\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi \overline{\psi_0}) &= 0.
\end{align*}
\]

For the topological solution we also have the equations corresponding to (1.7), (1.9) by introducing a new variable $\varphi$ such that $\varphi + \lambda = \phi$ to give a natural boundary conditions to (1.2). Let us remark on some notations. If no confuses are arisen, $u$ means a triple $(A, \phi, \tilde{N})$ or $(A, \varphi, N)$,

\[
\begin{align*}
||u(t, \cdot)||_{H^*} &= ||A(t, \cdot)||_{H^*} + ||\phi(t, \cdot)||_{H^*} + ||N(t, \cdot)||_{H^*}, \\
||\partial_0 u(t, \cdot)||_{H^+} &= ||\partial A(t, \cdot)||_{H^*} + ||\partial \phi(t, \cdot)||_{H^*} + ||\partial N(t, \cdot)||_{H^*}, \\
||u(t, \cdot)||_{H^* \times H^* + 1} &= ||u(t, \cdot)||_{H^*} + ||\partial_0 u(t, \cdot)||_{H^* + 1}.
\end{align*}
\]

Followings are our main theorems.
**Theorem 1.1.** (Global smooth solutions) Consider Maxwell-Chern-Simons-Higgs. Then any finite energy \( H^s \) initial data set \( (s \geq 2) \) admits a unique, global solution in the temporal gauge.

\[
A, \phi, \overline{N} \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))
\]

in the non-topological case. Also in the same gauge, any finite energy \( H^s \) initial data set \( (s \geq 2) \) admits a unique, global solution

\[
A, \varphi, N \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))
\]

in the topological case.

**Theorem 1.2.** (Maxwell-Higgs Limit) Consider the topological case of Maxwell-Chern-Simons-Higgs. Let \( u_\kappa \) be the global solution with coupling constant \( \kappa \) of \( H^s(s \geq 2) \) initial data \( u_0 \). Then \( \|u_\kappa(t) - u(t)\|_{H^s} \to 0 \) as \( \kappa \to 0 \). In the case of \( \kappa = 0 \), if we set \( N \) initially zero then \( N(t) = 0 \) for all \( t \).

**Remark 1.** In a succeeding section, we present the proof of the non-topological case only in Theorem 1.1 since the finite energy solution of the topological one can be found in the same way as non-topological case.

2. Outline of the proofs

i) local in time existence

**Proposition 2.1.** Given a data set \( (A, \phi, \overline{N}) \in H^s(s \geq 2) \) at \( t = 0 \), there exists \( T^* \) depending only on \( \|(A, \phi, \overline{N})(0, \cdot)\|_{H^s} \) and a unique development \( (A, \phi, \overline{N}) \) in the temporal gauge with

\[
(A, \phi, \overline{N}) \in C([0, T^*); H^s(R^2)) \cap C^1([0, T^*); H^{s-1}(R^2)).
\]

This solution can be continued as long as \( \|(A, \phi, \overline{N})\|_{H^s(t)} \) remains bounded.

First we show that there exists \( T^* \) such that (1.7) has a unique solution in \( X_T \),

\[
X_T = \{(u, \partial_0 u) \in C([0, T^*); H^2 \times H^1) : \|u\|_{X_T} < \infty\},
\]

where \( \|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^2 \times H^1} \). The solution is obtained by standard contraction argument using energy estimates, and has continuous dependence on the initial data. This solution can be continued as long as \( \|u_n(t, \cdot)\|_{H^2 \times H^1} \) remains bounded. To complete the local existence of Maxwell-Chern-Simons-Higgs, we also show that the constraint (1.8) is preserved in time.

ii) global in time existence

The proofs follow Mengrief's method mentioned earlier and use usual
a priori estimates to show \( \|u(t, \cdot)\|_{H^2 \times H^1} \) does not blow up in a finite time.

Let \( u = (A, \phi, N) \in C([0, T); H^s(R^2)) \cap C^1([0, T); H^{s-1}(R^2)) \) be a solution of Maxwell-Chern-Simons-Higgs obtained in the part \( i \). Define \( E(t), E_1(t), F(t) \) as such

\[
E(t) = \int_{R^2} \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N),
\]

\[
E_1(t) = \int_{R^2} \frac{1}{2} (\partial_t F_{0i})^2 + \frac{1}{2} (\partial_t F_{12})^2 + |D_i \psi_\mu|^2 + (\partial_t \Omega_\mu)^2,
\]

where \( \psi_\mu = D_\mu \phi \), \( \Omega_\mu = \partial_\mu N \),

\[
F(t) = ||\partial u(t, \cdot)||_{L^2}(t) + ||\partial_\mu \phi||_{L^2}(t) + ||\partial_\mu N||_{L^2}(t).
\]

The global result will be established after following Lemmas.

\textbf{Lemma 2.2.} Let \( u = (A, \phi, \tilde{N}) \in C([0, T); H^2(R^2)) \cap C^1([0, T); H^1(R^2)) \) be a solution of (1.7), (1.8). Then

1) \( E(t) = E(0) \) for all \( t \in [0, T) \)

2) \( (A, \phi, \tilde{N}) \) are estimated in \( L^2 \) in terms of the initial data for all \( t \in [0, T) \);

\[
\|u(t, \cdot)\|_{L^2} \leq \|u(0, \cdot)\|_{L^2} + t E.
\]

\textbf{Lemma 2.3.} \( E_1(t) \) is differentiable for all \( t \in [0, T) \) and satisfies

\[
\partial_0 E_1(t) = \int_{R^2} \kappa e^{ik} \partial_t F_{0i} \partial_t F_{0k} - 2e \text{Im}(\psi_i \overline{\psi_i} + \phi \overline{D_l \psi_i})
\]

\[
+ 2 \text{Re}(D_l \overline{\psi_j} \cdot i e F_{ij} \psi_j - D_l \overline{U_\phi} + i e F_{0i} \psi_0)
\]

\[
+ 2 \text{Re}(D_l \overline{\psi_j} \cdot i e F_{ij} \psi_0 - D_l (F_{0j} \phi) + i e F_{0i} \psi_j) - 2 \partial_t U_\Omega \partial_0 \Omega_0.
\]

2) \( E_1(t), F_1(t) \) are estimated in terms of the initial data for all \( t \in [0, T) \),

\[
E_1(t) \leq C(E, E_1(0))(1 + t)^2,
\]

\[
F(t) \leq C(E, E_1(0), F(0))(1 + t)^{\frac{3}{2}}.
\]

It is not clear the energy norm, \( \|u(t, \cdot)\|_{H^1} \) does not blow up in the temporal gauge, though the energy itself is preserved in Lemma 2.2. In Lemma 2.3, \( E_1(t) \), the higher order energy, is shown to be initially bounded, from which \( F_1(t) = ||\partial u(t, \cdot)||_{H^1} \) can be easily estimated in terms of the initial data. Combining (2.4), (2.7) we have \( \|u(t, \cdot)\|_{H^1} \) is initially bounded.
Proving two Lemmas we use the Hamiltonian formalism of this system (1.9) after taking time derivatives in $E(t), E_1(t), F(t)$. We depend on the covariant Sobolev inequalities [7] (see Appendix of it), estimating each terms of $\partial_0 E_1(t)$ to show the right hand terms of (2.5) are at most linear with right to $E_1(t)$. Next we introduce Brezis-Gallouet inequality.

Lemma 2.4. [8] $s > 1$, 

$$
\|u\|_{L^\infty} \leq C \|u\|_{H^1} (1 + \sqrt{\log(1 + \|u\|_{H^s})}).
$$

Finally we carried out a priori estimate $\|u(t, \cdot)\|_{H^2 \times H^1}$ to get

$$
\|u\|_{H^2(t) + \|\partial_0 u\|_{H^1(t)} \leq \|u\|_{H^2(0)} + C \int_0^t \|\partial_0 u\|_{H^1} + ((1 + \|u\|_{L^\infty})^2 + \|u\|_{H^1}^2) \|u\|_{H^1} + \|u\|_{L^\infty} \|u\|_{H^1} \|u\|_{H^1}^2 + (1 + \|u\|_{L^\infty}) \|u\|_{H^1}^2,
$$

(2.8)

by energy estimates to (1.7), then we have

$$
\|u(0, \cdot)\|_{H^2} + C(t) \int_0^t \log(1 + \|u(s, \cdot)\|_{H^2}) \|u(s, \cdot)\|_{H^2 \times H^1} ds.
$$

(2.9)

applying above Brezis-Gallouet inequality. The desired result, thus, is given by the general Gronwall inequality.

For the case of an initial data $u \in L^s(\mathbb{R}^2)$, it is easy to obtain a local existence result as proposition 2.1. For a global result we state a next lemma omitting its simple proof.

Lemma 2.5. Let $(A, \phi, N) \in C([0, T); H^s(\mathbb{R}^2)) \cap C^1([0, T); H^{s-1}(\mathbb{R}^2))$ be a solution of (1.7), (1.8) for $s > 2$ then $\|u\|_{H^2(t)}$ is estimated in terms of the initial data for all $t \in [0, T)$.

iii) Maxwell-Higgs limit

Let $u^\kappa$ be a topological solution of Maxwell-Chern-Simons-Higgs obtained in Theorem 1.1 with coupling constant $\kappa$ of $H^2$ initial data $u_0$. It is easy to show $\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1}$ is estimated to be

$$
\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1} \leq \int_0^t \kappa \|\partial_0 A\|_{H^1} + C(t) \|u^\kappa - u\|_{H^2},
$$

(2.10)

using $\sup_{0 \leq \kappa \leq 1} \|u^\kappa(t, \cdot)\|_{H^2 \times H^1} \leq C(t)$ for a smooth function in the proof of Theorem 1.1. Then applying Gronwall inequality to (2.10), we have

$$
\|(u^\kappa - u)(t, \cdot)\|_{H^2} \leq \kappa C(t) + \|(u^\kappa - u)(0, \cdot)\|_{H^2},
$$

letting $\kappa \to 0$, we obtain the desired result.
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