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<th><strong>Title</strong></th>
<th>ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM (Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics)</th>
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</thead>
<tbody>
<tr>
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ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

DONGHO CHAE\dagger, MYEONGJU CHAE\dagger\dagger

ABSTRACT. In this paper we shall prove the global existence of solutions of the classical Maxwell-Chern-Simons-Higgs equations in $(2+1)$-dimensional Minkowski spacetime in the temporal gauge. We also prove that the topological solution of the Maxwell-Chern-Simons-Higgs system converges to that of Maxwell-Higgs system, as $\kappa$ goes to zero. Thus we reproduce the classical result by Moncrief [6] on the global existence of the Maxwell-Klein-Gordon system in $(2+1)$-dimension.

1. INTRODUCTION AND MAIN RESULTS

We are concerned on the global existence problem for the Maxwell-Chern-Simons-Higgs model in $(2+1)$-spacetime which was introduced to consider a self-dual system having both Maxwell and Chern-Simons terms [1]. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho - \langle D_\mu \phi, D^\mu \phi \rangle - \frac{1}{2} \partial_\mu N \partial^\mu N - \frac{1}{2} (e |\phi|^2 + \kappa N - e v^2)^2 - e^2 N^2 |\phi|^2,$$

(1.1)

where $g_{\mu\nu} = \text{diag}(1, -1, -1)$, $\phi$ is a complex scalar field, $N$ is a real scalar field, $A = (A_0, A_1, A_2)$ is a vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - ie A_\mu$, $e$ is the charge of the electron, and $\kappa$ is a coupling constant for the Chern-Simons term.

The Euler-Lagrange equations via variation of the action taken with respect to $(A, \phi, N)$ are

$$\partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + 2e Im(\phi \overline{D^\rho \phi}) = 0,$$

$$D_\mu D^\mu \phi + U_\phi(|\phi|^2, N) = 0,$$

$$\partial_\mu \partial^\mu N + U_N = 0.$$

(1.2)

Letting $\rho = 0$ in (1.2), we obtain the Gauss-Law constraint

$$\partial_2 F_{j0} - \kappa F_{12} - 2e Im(\phi \overline{D_0 \phi}) = 0.$$

(1.3)

\dagger partially supported by the BK 21 Project.
The static energy functional for the system is

$$E = \int_{\mathbb{R}^2} \left( \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N) \right),$$

where $U(|\phi|^2, N) = \frac{1}{2} (e|\phi|^2 + \kappa N - ev^2)^2 + e^2 N^2 |\phi|^2$, $(i = 1, 2, \mu = 0, 1, 2)$. We note that, if $(A, \phi, N)$ is a solution that makes $E$ finite, one of the following conditions should be required;

$$\phi \to 0 \quad \text{and} \quad N \to \frac{ev^2}{\kappa} \quad \text{(non-topological)} \quad (1.5)$$

$$|\phi|^2 \to v^2 \quad \text{and} \quad N \to 0 \quad \text{(topological)} \quad (1.6)$$

The terms of non-topological solution refers to the solution satisfying (1.5) and topological solution to the solution satisfying (1.6). [1], [2]

In the static case, above system are reduced to the system of elliptic equation. The static energy functional is

$$E = \int \left( |(D_1 \pm iD_2)\phi|^2 + |D_0 \phi \mp ie\phi N|^2 + \frac{1}{2} |F_{12} \pm (e|\phi|^2 + \kappa N - e)|^2 do \pm e \int F_{12} do. \right)$$

The solution saturating a lower bound for the energy is called self-dual solution, which studied extensively on both two conditions (1.5), (1.6) by D. Chae et al. ([2], [4]), and on a periodic boundary condition, by Tarantello [5]. They also studied the unifying feature of Maxwell-Chern-Simons-Higgs mathematically, which was formally described in [1].

For a time dependent solution to the Maxwell-Chern-Simons-Higgs, there is no result as we know, however, in [6], Mongrief proved the global existence for the classical Maxwell-Klein-Gordon equations using the Lorents gauge in $(2+1)$ spacetime. The Lagrangian of the Maxwell-Klein-Gordon is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \langle D_\mu \phi, D^\mu \phi \rangle.$$

He proved global existence by showing that a suitably defined higher order energy, though not strictly conserved, does not blow up in a finite time. In this article, we consider the global existence of the classical Maxwell-Chern-Simons-Higgs in the temporal gauge as well as a convergent result as the static case [3].

Before presenting main theorems, we state equations corresponding to the non-topological case in the temporal gauge.

Considering the non-topological solution of (1.2), (1.3), we put $\tilde{N}$ to
MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

$N - \frac{e v^2}{\kappa}$ in (1.2) to obtain the following system of semilinear wave equations with constraint ($\square = \partial_\mu - \Delta$).

$$\square A_1 = -\kappa \partial_0 A_2 + 2e Im(\phi \overline{D_1 \phi}),$$  \hspace{2cm} (1.7)
$$\square A_2 = \kappa \partial_0 A_1 + 2e Im(\phi \overline{D_2 \phi}),$$
$$\square \phi = -ie \phi \partial_j A_j - 2ie A_j \partial_j \phi - e^2 A_j^2 \phi - U_\phi$$
$$\square N = -U_N,$$
$$\partial_j F_{j0} - \kappa F_{12} - 2e Im(\phi \overline{\psi_0}) = 0.$$  \hspace{2cm} (1.8)

Above equations can be rewritten as Hamiltonian formalism;

$$\partial_0 A_j = F_{0j},$$
$$\partial_0 F_{0j} = -e^{jk} \partial_k F_{12} - \kappa e^{jk} F_{0k} - 2e \text{Im}(\phi \overline{\psi_j})$$
$$\partial_0 F_{12} = e^{\dot{j}l} \partial_{\dot{l}} F_{0j}$$
$$\partial_0 \phi = \psi_0$$
$$\partial_0 \psi_0 = D_j \psi_j - U_\phi$$
$$\partial_0 \psi_j = D_j \psi_0 - ie F_{0j} \phi$$
$$\partial_0 N = \Omega_0$$
$$\partial_0 \Omega_0 = \partial_j \Omega_j - U_N$$
$$\partial_0 \Omega_j = \partial_j \Omega_0$$  \hspace{2cm} (1.9)

supplemented by constrains,

$$F_{jk} = \partial_j A_k - \partial_k A_j$$
$$D_j \phi = \psi_j$$
$$\partial_j N = \Omega_j$$
$$\partial_j F_{j0} - \kappa F_{12} - 2e \text{Im}(\phi \overline{\psi_0}) = 0.$$  \hspace{2cm} (1.10)

For the topological solution we also have the equations corresponding to (1.7), (1.9) by introducing a new variable $\varphi$ such that $\varphi + \lambda = \phi$ to give a natural boundary conditions to (1.2). Let us remark on some notations. If no confuses are arisen, $u$ means a triple $(A, \phi, \tilde{N})$ or $(A, \varphi, N)$,

$$\|u(t, \cdot)\|_{H^s} = \|A(t, \cdot)\|_{H^s} + \|\phi(t, \cdot)\|_{H^s} + \|N(t, \cdot)\|_{H^s},$$
$$\|\partial_0 u(t, \cdot)\|_{H^s} = \|\partial A(t, \cdot)\|_{H^s} + \|\partial \phi(t, \cdot)\|_{H^s} + \|\partial N(t, \cdot)\|_{H^s},$$
$$\|u(t, \cdot)\|_{H^s \times H^{s-1}} = \|u(t, \cdot)\|_{H^s} + \|\partial_0 u(t, \cdot)\|_{H^{s-1}}.$$

Followings are our main theorems.
Theorem 1.1. (Global smooth solutions) Consider Maxwell-Chern-Simons-Higgs. Then any finite energy $H^s$ initial data set $(s \geq 2)$ admits a unique, global solution in the temporal gauge.

$$A, \phi, \tilde{N} \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))$$

in the non-topological case. Also in the same gauge, any finite energy $H^s$ initial data set $(s \geq 2)$ admits a unique, global solution

$$A, \varphi, N \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))$$

in the topological case.

Theorem 1.2. (Maxwell-Higgs Limit) Consider the topological case of Maxwell-Chern-Simons-Higgs. Let $u_\kappa$ be the global solution with coupling constant $\kappa$ of $H^s(s \geq 2)$ initial data $u_0$. Then $\|u_\kappa(t) - u(t)\|_{H^s} \to 0$ as $\kappa \to 0$. In the case of $\kappa = 0$, if we set $N$ initially zero then $N(t) = 0$ for all $t$.

Remark 1. In a succeeding section, we present the proof of the non-topological case only in Theorem 1.1 since the finite energy solution of the topological one can be found in the same way as non-topological case.

2. Outline of the proofs

i) local in time existence

Proposition 2.1. Given a data set $(A, \phi, \tilde{N}) \in H^s(s \geq 2)$ at $t = 0$, there exists $T^*$ depending only on $\|(A, \phi, \tilde{N})(0, \cdot)\|_{H^s}$ and a unique development $(A, \phi, \tilde{N})$ in the temporal gauge with

$$(A, \phi, \tilde{N}) \in C([0, T^*); H^s(R^2)) \cap C^1([0, T^*); H^{s-1}(R^2)).$$

This solution can be continued as long as $\|(A, \phi, \tilde{N})\|_{H^s(t)}$ remains bounded.

First we show that there exists $T^*$ such that (1.7) has a unique solution in $X_T$,

$$X_T = \{(u, \partial_0 u) \in C([0, T^*); H^2 \times H^1) : \|u\|_{X_T} < \infty\},$$

where $\|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^2 \times H^1}$. The solution is obtained by standard contraction argument using energy estimates, and has continuous dependence on the initial data. This solution can be continued as long as $\|u_n(t, \cdot)\|_{H^2 \times H^1}$ remains bounded. To complete the local existence of Maxwell-Chern-Simons-Higgs, we also show that the constraint (1.8) is preserved in time.

ii) global in time existence

The proofs follow Mongriff's method mentioned earlier and use usual
a priori estimates to show $||u(t,)||_{H^2 \times H^1}$ does not blow up in a finite time.
Let $u = (A, \phi, N) \in C([0, T); H^s(R^2)) \cap C^1([0, T); H^{s-1}(R^2))$ be a solution of Maxwell-Chern-Simons-Higgs obtained in the part $i)$. Define $E(t), E_1(t), F(t)$ as such

$$E(t) = \int_{R^2} \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |D_\mu N|^2 + U(|\phi|^2, N),$$

$$E_1(t) = \int_{R^2} \frac{1}{2} (\partial_i F_{0i})^2 + \frac{1}{2} (\partial_i F_{12})^2 + |D_i \psi_\mu|^2 + (\partial_i \Omega_\mu)^2,$$

where $\psi_\mu = D_\mu \phi$, $\Omega_\mu = \partial_\mu N$.

$$F(t) = ||\partial_\dot{t} A||_{L^2}(t) + ||\partial_\dot{t} \phi||_{L^2}(t) + ||\partial_\dot{t} N||_{L^2}(t).$$

The global result will be established after following Lemmas.

**Lemma 2.2.** Let $u = (A, \phi, \tilde{N}) \in C([0, T); H^2(R^2)) \cap C^1([0, T); H^1(R^2))$ be a solution of (1.7), (1.8). Then

1. $E(t) = E(0)$ for all $t \in [0, T)$
2. $(A, \phi, \tilde{N})$ are estimated in $L^2$ in terms of the initial data for all $t \in [0, T)$;

$$||u(t, \cdot)||_{L^2} \leq ||u(0, \cdot)||_{L^2} + tE.$$  \hspace{1cm} (2.4)

**Lemma 2.3.**
1. $E_1(t)$ is differentiable for all $t \in [0, T)$ and satisfies

$$\partial_0 E_1(t) = \int_{R^2} -\kappa e^{ik} \partial_0 F_{0i} \partial_0 F_{0k} - 2eIm(\psi_i \overline{\psi_i} + \phi D_i \overline{\psi_i})$$

$+ 2Re(D_i \overline{\psi_i} \cdot i \epsilon F_{ij} \psi_j - D_i U_\phi + i \epsilon F_{0i} \psi_0)$ \hspace{1cm} (2.5)

$+ 2Re(D_i \overline{\psi_j} \cdot i \epsilon F_{ij} \psi_0 - D_i (F_{0j} \phi + i \epsilon F_{0i} \psi_j) - 2\partial_1 U_N \partial_1 \Omega_0.$

2. $E_1(t), F_1(t)$ are estimated in terms of the initial data for all $t \in [0, T)$,

$$E_1(t) \leq C(E, E_1(0))(1 + t)^2,$$

$$F(t) \leq C(E, E_1(0), F(0))(1 + t)^{\frac{5}{2}}.$$  \hspace{1cm} (2.7)

It is not clear the energy norm, $||u(t, \cdot)||_{H^1}$ does not blow up in the temporal gauge, though the energy itself is preserved in Lemma 2.2. In Lemma 2.3, $E_1(t)$, the higher order energy, is shown to be initially bounded, from which $F_1(t) = ||\partial u(t, \cdot)||_{H^1}$ can be easily estimated in terms of the initial data. Combining (2.4), (2.7) we have $||u(t, \cdot)||_{H^1}$ is initially bounded.
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Proving two Lemmas we use the Hamiltonian formalism of this system (1.9) after taking time derivatives in $E(t), E_1(t), F(t)$. We depend on the covariant Sobolev inequalities in [7] (see Appendix of it), estimating each terms of $\partial_0 E_1(t)$ to show the right hand terms of (2.5) are at most linear with right to $E_1(t)$. Next we introduce Brezis-Gallouet inequality.

**Lemma 2.4.** [8] $s > 1$,

$$ ||u||_{L^\infty} \leq C||u||_{H^{1}}(1 + \sqrt{\log(1 + ||u||_{H^{s}})}). $$

Finally we carried out a priori estimate $||u(t, \cdot)||_{H^2 \times H^1}$ to get

$$ ||u||_{H^2}(t) + ||\partial_0u||_{H^1}(t) \leq ||u||_{H^2}(0) + C \int_0^t ||\partial_0u||_{H^1} $$

$$ + ((1 + ||u||_{L^\infty})^2 + ||u||_{H^1}^{\frac{1}{2}}(1 + ||u||_{L^2}))||u||_{H^2} $$

$$ + ||u||_{L^\infty} ||u||_{H^1}^{\frac{1}{2}} ||u||_{H^2} + (1 + ||u||_{L^\infty}) ||u||_{H^2}^2 $$

(2.8)

by energy estimates to (1.7), then we have

$$ ||u(0, \cdot)||_{H^2} + C(t) \int_0^t \log(1 + ||u(s, \cdot)||_{H^2}) ||u(s, \cdot)||_{H^2 \times H^1} ds. \quad (2.9) $$

applying above Brezis-Gallouet inequality. The desired result, thus, is given by the general Gronwall inequality.

For the case of an initial data $u \in H^s(\mathbb{R}^2)$, it is easy to obtain a local existence result as proposition 2.1. For a global result we state a next lemma omitting its simple proof.

**Lemma 2.5.** Let $(A, \phi, N) \in C([0, T); H^s(\mathbb{R}^2)) \cap C^1([0, T); H^{s-1}(\mathbb{R}^2))$ be a solution of (1.7), (1.8) for $s > 2$ then $||u||_{H^s}(t)$ is estimated in terms of the initial data for all $t \in [0, T)$.

iii) Maxwell-Higgs limit

Let $u^\kappa$ be a topological solution of Maxwell-Chern-Simons-Higgs obtained in Theorem 1.1 with coupling constant $\kappa$ of $H^2$ initial data $u_0$. It is easy to show $||(u^\kappa - u)(t, \cdot)||_{H^2 \times H^1}$ is estimated to be

$$ ||(u^\kappa - u)(t, \cdot)||_{H^2 \times H^1} \leq \int_0^t \kappa ||\partial_0 A_{\kappa}||_{H^1} + C(t)||u^\kappa - u||_{H^2}, \quad (2.10) $$

using $\sup_{0 \leq \kappa \leq 1} ||u^\kappa(t, \cdot)||_{H^2 \times H^1} \leq C(t)$ for a smooth function in the proof of Theorem 1.1. Then applying Gronwall inequality to (2.10), we have

$$ ||(u^\kappa - u)(t, \cdot)||_{H^2} \leq \kappa C(t) + ||(u^\kappa - u)(0, \cdot)||_{H^2}, $$

letting $\kappa \to 0$, we obtain the desired result.
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