ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM (Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics)

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ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

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ABSTRACT. In this paper we shall prove the global existence of solutions of the classical Maxwell-Chern-Simons-Higgs equations in $(2+1)$-dimensional Minkowski spacetime in the temporal gauge. We also prove that the topological solution of the Maxwell-Chern-Simons-Higgs system converges to that of Maxwell-Higgs system, as $\kappa$ goes to zero. Thus we reproduce the classical result by Moncrief [6] on the global existence of the Maxwell-Klein-Gordon system in $(2+1)$-dimension.

1. INTRODUCTION AND MAIN RESULTS

We are concerned on the global existence problem for the Maxwell-Chern-Simons-Higgs model in $(2+1)$-spacetime which was introduced to consider a self-dual system having both Maxwell and Chern-Simons terms [1]. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_{\rho} - \langle D_\mu \phi, D^\mu \phi \rangle - \frac{1}{2} \partial_\mu N \partial^\mu N$$

$$- \frac{1}{2} (e|\phi|^2 + \kappa N - ev^2)^2 - e^2 N^2 |\phi|^2,$$

where $g_{\mu\nu} = \text{diag}(1, -1, -1)$, $\phi$ is a complex scalar field, $N$ is a real scalar field, $A = (A_0, A_1, A_2)$ is a vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - ie A_\mu$, $e$ is the charge of the electron, and $\kappa$ is a coupling constant for the Chern-Simons term.

The Euler-Lagrange equations via variation of the action taken with respect to $(A, \phi, N)$ are

$$\partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + 2e Im(\phi \overline{D^\rho \phi}) = 0,$$

$$D_\mu D^\mu \phi + U_{\overline{\phi}}(|\phi|^2, N) = 0,$$

$$\partial_\mu \partial^\mu N + U_N = 0.$$

Letting $\rho = 0$ in (1.2), we obtain the Gauss-Law constraint

$$\partial_j F_{j0} - \kappa F_{12} - 2e Im(\phi \overline{D_0 \phi}) = 0.$$

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The static energy functional for the system is
\[
E = \int_{\mathbb{R}^2} \left( \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_{\mu}\phi|^2 + |\partial_{\mu}N|^2 + U(|\phi|^2, N) \right),
\]  
(1.4)
where \(U(|\phi|^2, N) = \frac{1}{2}(e|\phi|^2 + \kappa N - ev^2)^2 + e^2 N^2 |\phi|^2\), \((i = 1, 2, \mu = 0, 1, 2)\). We note that, if \((A, \phi, N)\) is a solution that makes \(E\) finite, one of the following conditions should be required;
\[
\phi \to 0 \quad \text{and} \quad N \to \frac{ev^2}{\kappa} \quad (\text{non-topological}) \quad (1.5)
\]
\[
|\phi|^2 \to v^2 \quad \text{and} \quad N \to 0 \quad (\text{topological}) \quad (1.6)
\]
The terms of non-topological solution refers to the solution satisfying (1.5) and topological solution to the solution satisfying (1.6). [1], [2]

In the static case, above system are reduced to the system of elliptic equation. The static energy functional is
\[
E = \int \left| (D_1 \pm iD_2)\phi \right|^2 + |D_0\phi \mp ie\phi N|^2
\]
\[
\frac{1}{2} \left| F_{12} \pm (e|\phi|^2 + \kappa N - e) \right|^2 d\sigma \pm e \int F_{12} d\sigma.
\]

The solution saturating a lower bound for the energy is called self-dual solution, which studied extensively on both two conditions (1.5), (1.6) by D. Chae et al. ([2], [4]), and on a periodic boundary condition, by Tarantello [5]. They also studied the unifying feature of Maxwell-Chern-Simons-Higgs mathematically, which was formally discribed in [1].

For a time dependent solution to the Maxwell-Chern-Simons-Higgs, there is no result as we know, however, in [6], Mongrief proved the global existence for the classical Maxwell-Klein-Gordon equations using the Lorentz gauge in \((2+1)\) spacetime. The Lagrangian of the Maxwell-Klein-Gordon is
\[
\mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} - \langle D_\mu\phi, D^\mu\phi \rangle.
\]
He proved global existence by showing that a suitably defined higher order energy, though not strictly conserved, does not blow up in a finite time. In this article, we consider the global existence of the classical Maxwell-Chern-Simons-Higgs in the temporal gauge as well as a convergent result as the static case [3].

Before presenting main theorems, we state equations corresponding to the non-topological case in the temporal gauge.
Considering the non-topological solution of (1.2), (1.3), we put \(\bar{N}\) to
$N - \frac{e\nu^2}{\kappa}$ in (1.2) to obtain the following system of semilinear wave equations with constraint ($\Box = \partial_{tt} - \Delta$).

\begin{align*}
\Box A_1 &= -\kappa \partial_0 A_2 + 2e \text{Im}(\phi D_1 \phi), \\
\Box A_2 &= \kappa \partial_0 A_1 + 2e \text{Im}(\phi D_2 \phi), \\
\Box \phi &= -ie\phi \partial_j A_j - 2ie A_j \partial_j \phi - e^2 A_j^2 \phi - U_{\overline{\phi}} \\
\Box N &= -U_{\overline{N}}, \\
\partial_j F_{j0} - \kappa F_{12} - 2e \text{Im}(\phi \overline{\psi_0}) &= 0.
\end{align*}

Above equations can be rewritten as Hamiltonian formalism;

\begin{align*}
\partial_0 A_j &= F_{0j} \\
\partial_0 F_{0j} &= -\epsilon^{jk} \partial_k F_{12} - \kappa \epsilon^{jk} F_{0k} - 2e \text{Im}(\phi \overline{\psi_{j}}) \\
\partial_0 F_{12} &= \epsilon^{ij} \partial_i F_{0j} \\
\partial_0 \phi &= \psi_0 \\
\partial_0 \psi_0 &= D_j \psi_j - U_{\overline{\psi}} \\
\partial_0 \psi_j &= D_j \psi_0 - ie F_{0j} \phi \\
\partial_0 N &= \Omega_0 \\
\partial_0 \Omega_0 &= \partial_j \Omega_j - U_{N} \\
\partial_0 \Omega_j &= \partial_j \Omega_0
\end{align*}

supplemented by constrains,

\begin{align*}
F_{jk} &= \partial_j A_k - \partial_k A_j \\
D_j \phi &= \psi_j \\
\partial_j N &= \Omega_j
\end{align*}

For the topological solution we also have the equations corresponding to (1.7), (1.9) by introducing a new variable $\varphi$ such that $\varphi + \lambda = \phi$ to give a natural boundary conditions to (1.2). Let us remark on some notations. If no confuses are arisen, $u$ means a triple $(A, \phi, \tilde{N})$ or $(A, \varphi, N),$

\begin{align*}
\|u(t, \cdot)\|_{H^s} &= \|A(t, \cdot)\|_{H^s} + \|\phi(t, \cdot)\|_{H^s} + \|N(t, \cdot)\|_{H^s}, \\
\|\partial_0 u(t, \cdot)\|_{H^s} &= \|\partial A(t, \cdot)\|_{H^s} + \|\partial \phi(t, \cdot)\|_{H^s} + \|\partial N(t, \cdot)\|_{H^s}, \\
\|u(t, \cdot)\|_{H^s \times \dot{H}^{s-1}} &= \|u(t, \cdot)\|_{H^s} + \|\partial_0 u(t, \cdot)\|_{\dot{H}^{s-1}}.
\end{align*}

Followings are our main theorems.
**Theorem 1.1. (Global smooth solutions)** Consider Maxwell-Chern-Simons-Higgs. Then any finite energy $H^s$ initial data set $(s \geq 2)$ admits a unique, global solution in the temporal gauge.

$$A, \phi, \tilde{N} \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))$$

in the non-topological case. Also in the same gauge, any finite energy $H^s$ initial data set $(s \geq 2)$ admits a unique, global solution

$$A, \varphi, N \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))$$

in the topological case.

**Theorem 1.2. (Maxwell-Higgs Limit)** Consider the topological case of Maxwell-Chern-Simons-Higgs. Let $u_\kappa$ be the global solution with coupling constant $\kappa$ of $H^s(s \geq 2)$ initial data $u_0$. Then $\|u_\kappa(t) - u(t)\|_{H^s} \to 0$ as $\kappa \to 0$. In the case of $\kappa = 0$, if we set $N$ initially zero then $N(t) = 0$ for all $t$.

**Remark 1.** In a succeeding section, we present the proof of the non-topological case only in Theorem 1.1 since the finite energy solution of the topological one can be found in the same way as non-topological case.

## 2. Outline of the proofs

**i) local in time existence**

**Proposition 2.1.** Given a data set $(A, \phi, \tilde{N}) \in H^s(s \geq 2)$ at $t = 0$, there exists $T^*$ depending only on $\|(A, \phi, \tilde{N})(0, \cdot)\|_{H^s}$ and a unique development $(A, \phi, \tilde{N})$ in the temporal gauge with

$$(A, \phi, \tilde{N}) \in C([0, T^*); H^s(R^2)) \cap C^1([0, T^*); H^{s-1}(R^2)).$$

This solution can be continued as long as $\|(A, \phi, \tilde{N})\|_{H^s(t)}$ remains bounded.

First we show that there exists $T^*$ such that (1.7) has a unique solution in $X_T$,

$$X_T = \{(u, \partial_0 u) \in C([0, T^*); H^2 \times H^1) : \|u\|_{X_T} < \infty\},$$

where $\|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^2 \times H^1}$. The solution is obtained by standard contraction argument using energy estimates, and has continuous dependence on the initial data. This solution can be continued as long as $\|u_\kappa(t, \cdot)\|_{H^2 \times H^1}$ remains bounded. To complete the local existence of Maxwell-Chern-Simons-Higgs, we also show that the constraint (1.8) is preserved in time.

**ii) global in time existence**

The proofs follow Mongrief's method mentioned earlier and use usual
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a priori estimates to show \( \| u(t, \cdot) \|_{H^2 \times H^1} \) does not blow up in a finite time.

Let \( u = (A, \phi, N) \in C([0, T); H^s(R^2)) \cap C^1([0, T); H^{s-1}(R^2)) \) be a solution of Maxwell-Chern-Simons-Higgs obtained in the part \( i \). Define \( E(t), E_1(t), F(t) \) as such

\[
E(t) = \int_{\mathbb{R}^2} \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N),
\]

\[
E_1(t) = \int_{\mathbb{R}^2} \frac{1}{2} (\partial_t F_{0i})^2 + \frac{1}{2} (\partial_t F_{12})^2 + |D_\mu \psi_\mu|^2 + (\partial_t \Omega_\mu)^2,
\]

where \( \psi_\mu = D_\mu \phi \), \( \Omega_\mu = \partial_\mu N \),

\[
F_1(t) = ||\partial_t A||_{L^2}(t) + ||\partial_t \phi||_{L^2}(t) + ||\partial_t N||_{L^2}(t).
\]

The global result will be established after following Lemmas.

Lemma 2.2. Let \( u = (A, \phi, \bar{N}) \in C([0, T); H^2(R^2)) \cap C^1([0, T); H^1(R^2)) \) be a solution of (1.7), (1.8). Then

1. \( E(t) = E(0) \) for all \( t \in [0, T) \)
2. \( (A, \phi, \bar{N}) \) are estimated in \( L^2 \) in terms of the initial data for all \( t \in [0, T) \);
   \[
   ||u(t, \cdot)||_{L^2} \leq ||u(0, \cdot)||_{L^2} + tE.
   \]

Lemma 2.3. (1) \( E_1(t) \) is differentiable for all \( t \in [0, T) \) and satisfies

\[
\partial_0 E_1(t) = \int_{\mathbb{R}^2} -\kappa e^{ik} \partial_t F_{0i} \partial_t F_{0k} - 2e Im(\psi_i \bar{\psi}_i + \phi \bar{D_i \psi_i}) \\
+ 2Re(D_i \bar{\psi}_j \cdot ie F_{ij} \psi_j - D_i \bar{F}_{ij} \psi_j - F_{0j} \psi_j + ie F_{0j} \psi_j) - 2\partial_t U_N \partial_t \Omega_0.
\]

2. \( E_1(t), F_1(t) \) are estimated in terms of the initial data for all \( t \in [0, T) \),
   \[
   E_1(t) \leq C(E, E_1(0))(1 + t)^2,
   \]
   \[
   F(t) \leq C(E, E_1(0), F(0))(1 + t)^{\frac{5}{2}}.
   \]

It is not clear the energy norm, \( ||u(t, \cdot)||_{H^1} \) does not blow up in the temporal gauge, though the energy itself is preserved in Lemma 2.2. In Lemma 2.3, \( E_1(t) \), the higher order energy, is shown to be initially bounded, from which \( F_1(t) = ||\partial u(t, \cdot)||_{H^1} \) can be easily estimated in terms of the initial data. Combining (2.4), (2.7) we have \( ||u(t, \cdot)||_{H^1} \) is initially bounded.
Proving two Lemmas we use the Hamiltonian formalism of this system (1.9) after taking time derivatives in \(E(t), E_1(t), F(t)\). We depend on the covariant Sobolev inequalities in [7] (see Appendix of it), estimating each terms of \(\partial_0 E_1(t)\) to show the right hand terms of (2.5) are at most linear with right to \(E_1(t)\). Next we introduce Brezis-Gallouet inequality.

**Lemma 2.4.** [8] \(s > 1\),

\[
\|u\|_{L^\infty} \leq C\|u\|_{H^1}(1 + \sqrt{\log(1 + \|u\|_{H^s})}.
\]

Finally we carried out a priori estimate \(\|u(t, \cdot)\|_{H^2 \times H^1}\) to get

\[
\|u\|_{H^2(t)} + \|\partial_0 u\|_{H^1(t)} \leq \|u\|_{H^2(0)} + C\int_0^t \|\partial_0 u\|_{H^1}
\]

\[
+ ((1 + \|u\|_{L^\infty})^2 + \|u\|_{H^1}^\frac{1}{2}(1 + \|u\|_{L^2}))\|u\|_{H^1}
\]

\[
+ \|u\|_{L^\infty}\|u\|_{H^1}^\frac{1}{2}\|u\|_{H^2}^\frac{3}{2} + (1 + \|u\|_{L^\infty})\|u\|_{H^1}^2
\]

(2.8)

by energy estimates to (1.7), then we have

\[
\|u(0, \cdot)\|_{H^2} + C(t)\int_0^t \log(1 + \|u(s, \cdot)\|_{H^2})\|u(s, \cdot)\|_{H^2 \times H^1} ds
\]

(2.9)

applying above Brezis-Gallouet inequality. The desired result, thus, is given by the general Gronwall inequality.

For the case of an initial data \(u \in H^s(\mathbb{R}^2)\), it is easy to obtain a local existence result as proposition 2.1. For a global result we state a next lemma omitting its simple proof.

**Lemma 2.5.** Let \((A, \phi, N) \in C([0, T); H^s(\mathbb{R}^2)) \cap C^1([0, T); H^{s-1}(\mathbb{R}^2))\) be a solution of (1.7), (1.8) for \(s > 2\) then \(\|u\|_{H^2(t)}\) is estimated in terms of the initial data for all \(t \in [0, T)\).

iii) Maxwell-Higgs limit

Let \(u^\kappa\) be a topological solution of Maxwell-Chern-Simons-Higgs obtained in Theorem 1.1 with coupling constant \(\kappa\) of \(H^2\) initial data \(u_0\). It is easy to show \(\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1}\) is estimated to be

\[
\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1} \leq \int_0^t \kappa\|\partial_0 A_\kappa\|_{H^1} + C(t)\|u^\kappa - u\|_{H^2},
\]

(2.10)

using \(\sup_{0 \leq \kappa \leq 1} ||u_\kappa(t, \cdot)||_{H^2 \times H^1} \leq C(t)\) for a smooth function in the proof of Theorem 1.1. Then applying Gronwall inequality to (2.10), we have

\[
||u_\kappa(t, \cdot)||_{H^2} \leq \kappa C(t) + ||(u^\kappa - u)(0, \cdot)||_{H^2},
\]

letting \(\kappa \to 0\), we obtain the desired result.
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