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ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

DONGHO CHAE†, MYEONGJU CHAE††

ABSTRACT. In this paper we shall prove the global existence of solutions of the classical Maxwell-Chern-Simons-Higgs equations in (2 + 1)-dimensional Minkowski spacetime in the temporal gauge. We also prove that the topological solution of the Maxwell-Chern-Simons-Higgs system converges to that of Maxwell-Higgs system, as $\kappa$ goes to zero. Thus we reproduce the classical result by Moncrief [6] on the global existence of the Maxwell-Klein-Gordon system in (2 + 1) dimension.

1. INTRODUCTION AND MAIN RESULTS

We are concerned on the global existence problem for the Maxwell-Chern-Simons-Higgs model in (2 + 1)-spacetime which was introduced to consider a self-dual system having both Maxwell and Chern-Simons terms [1]. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^\mu{}_{\nu} F_{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_{\rho} - \langle D_\mu \phi, D^\mu \phi \rangle - \frac{1}{2} \partial_\mu N \partial^\mu N$$

$$- \frac{1}{2} (e|\phi|^2 + \kappa N - e v^2)^2 - e^2 N^2 |\phi|^2,$$

where $g_{\mu\nu} = \text{diag}(1, -1, -1)$, $\phi$ is a complex scalar field, $N$ is a real scalar field, $A = (A_0, A_1, A_2)$ is a vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, D_\mu = \partial_\mu - ie A_j, e$ is the charge of the electron, and $\kappa$ is a coupling constant for the Chern-Simons term.

The Euler-Lagrange equations via variation of the action taken with respect to $(A, \phi, N)$ are

$$\partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + 2e Im(\phi \overline{D^\rho \phi}) = 0,$$

$$D_\mu D^\mu \phi + U_\phi(|\phi|^2, N) = 0,$$

$$\partial_\mu \partial^\mu N + U_N = 0.$$

Letting $\rho = 0$ in (1.2), we obtain the Gauss-Law constraint

$$\partial_j F_{j0} - \kappa F_{12} - 2e Im(\phi \overline{D_o \phi}) = 0.$$

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The static energy functional for the system is

\[ E = \int_{\mathbb{R}^2} \frac{1}{2} F_{6i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N), \]  

(1.4)

where \( U(|\phi|^2, N) = \frac{1}{2} (e|\phi|^2 + \kappa N - ev^2)^2 + e^2 N^2 |\phi|^2, \) \( (i = 1, 2, \mu = 0, 1, 2). \) We note that, if \((A, \phi, N)\) is a solution that makes \( E \) finite, one of the following conditions should be required:

\[ \phi \rightarrow 0 \quad \text{and} \quad N \rightarrow \frac{ev^2}{\kappa} \quad \text{(non-topological)} \]  

(1.5)

\[ |\phi|^2 \rightarrow v^2 \quad \text{and} \quad N \rightarrow 0 \quad \text{(topological)} \]  

(1.6)

The terms of non-topological solution refers to the solution satisfying (1.5) and topological solution to the solution satisfying (1.6). [1], [2]

In the static case, above system are reduced to the system of elliptic equation. The static energy functional is

\[ E = \int |(D_1 \pm iD_2)\phi|^2 + |D_0 \phi \mp ie\phi N|^2 \]

\[ \frac{1}{2} |F_{12} \pm (e|\phi|^2 + \kappa N - e)|^2 do \pm e \int F_{12} do. \]

The solution saturating a lower bound for the energy is called self-dual solution, which studied extensively on both two conditions (1.5), (1.6) by D. Chae et al. ([2], [4]), and on a periodic boundary condition, by Tarantello [5]. They also studied the unifying feature of Maxwell-Chern-Simons-Higgs mathematically, which was formally discribed in [1].

For a time dependent solution to the Maxwell-Chern-Simons-Higgs, there is no result as we know, however, in [6], Mongrief proved the global existence for the classical Maxwell-Klein-Gordon equations using the Lorents gauge in \((2+1)\) spacetime. The Lagrangian of the Maxwell-Klein-Gordon is

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \langle D_\mu \phi, D^\mu \phi \rangle. \]

He proved global existence by showing that a suitably defined higher order energy, though not strictly conserved, does not blow up in a finite time. In this article, we consider the global existence of the classical Maxwell-Chern-Simons-Higgs in the temporal gauge as well as a convergent result as the static case [3].

Before presenting main theorems, we state equations corresponding to the non-topological case in the temporal gauge.

Considering the non-topological solution of (1.2), (1.3), we put \( \tilde{N} \) to
$N - \frac{e^{2}}{\kappa}$ in (1.2) to obtain the following system of semilinear wave equations with constraint ($\square = \partial_{tt} - \Delta$).

\[
\square A_1 = -\kappa \partial_0 A_2 + 2eIm(\phi \overline{D_1 \phi}),
\]
\[
\square A_2 = \kappa \partial_0 A_1 + 2eIm(\phi \overline{D_2 \phi}),
\]
\[
\square \phi = -ie \phi \partial_j A_j - 2ie A_j \partial_j \phi - e^2 A_j^2 \phi - U_{\overline{\phi}}
\]
\[
\square N = -U_{\overline{N}},
\]
\[
\partial_j F_j 0 - \kappa F_1 2 - 2eIm(\phi \overline{\psi_0}) = 0.
\]

Above equations can be rewritten as Hamiltonian formalism;

\[
\partial_0 A_j = F_0 j
\]
\[
\partial_0 F_0 j = -\epsilon^{jk} \partial_k F_1 2 - \kappa \epsilon^{jk} F_0 k - 2eIm(\phi \overline{\psi_j})
\]
\[
\partial_0 F_1 2 = \epsilon^{\dot{j}l} \partial_{\dot{l}} F_0 j
\]
\[
\partial_0 \phi = \psi_0
\]
\[
\partial_0 \psi_0 = D_j \psi_j - U_{\overline{\psi}}
\]
\[
\partial_0 \psi_j = D_j \psi_0 - ie F_0 j \phi
\]
\[
\partial_0 N = \Omega_0
\]
\[
\partial_0 \Omega_0 = \partial_j \Omega_j - U_{N}
\]
\[
\partial_0 \Omega_j = \partial_j \Omega_0
\]

supplemented by constrains,

\[
F_{jk} = \partial_j A_k - \partial_k A_j
\]
\[
D_j \phi = \psi_j
\]
\[
\partial_j N = \Omega_j
\]
\[
\partial_j F_j 0 - \kappa F_1 2 - 2eIm(\phi \overline{\psi_0}) = 0.
\]

For the topological solution we also have the equations corresponding to (1.7), (1.9) by introducing a new variable $\varphi$ such that $\varphi + \lambda = \phi$ to give a natural boundary conditions to (1.2). Let us remark on some notations. If no confuses are arisen, $u$ means a triple $(A, \phi, \tilde{N})$ or $(A, \varphi, N)$,

\[
\|u(t, \cdot)\|_{H^*} = \|A(t, \cdot)\|_{H^*} + \|\phi(t, \cdot)\|_{H^*} + \|N(t, \cdot)\|_{H^*},
\]
\[
\|\partial_0 u(t, \cdot)\|_{H^*} = \|\partial A(t, \cdot)\|_{H^*} + \|\partial \phi(t, \cdot)\|_{H^*} + \|\partial N(t, \cdot)\|_{H^*},
\]
\[
\|u(t, \cdot)\|_{H^{*} \times H^{*-1}} = \|u(t, \cdot)\|_{H^*} + \|\partial_0 u(t, \cdot)\|_{H^{*-1}}.
\]

Followings are our main theorems.
Theorem 1.1. (Global smooth solutions) Consider Maxwell-Chern-Simons-Higgs. Then any finite energy \( H^s \) initial data set \( (s \geq 2) \) admits a unique, global solution in the temporal gauge.

\[
A, \phi, \tilde{N} \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))
\]

in the non-topological case. Also in the same gauge, any finite energy \( H^s \) initial data set \( (s \geq 2) \) admits a unique, global solution

\[
A, \varphi, N \in C([0, \infty); H^s(R^2)) \cap C^{1}([0, \infty); H^{s-1}(R^2))
\]

in the topological case.

Theorem 1.2. (Maxwell-Higgs Limit) Consider the topological case of Maxwell-Chern-Simons-Higgs. Let \( u_\kappa \) be the global solution with coupling constant \( \kappa \) of \( H^s(s \geq 2) \) initial data \( u_0 \). Then \( \|u_\kappa(t) - u(t)\|_{H^s} \to 0 \) as \( \kappa \to 0 \). In the case of \( \kappa = 0 \), if we set \( N \) initially zero then \( N(t) = 0 \) for all \( t \).

Remark 1. In a succeeding section, we present the proof of the non-topological case only in Theorem 1.1 since the finite energy solution of the topological one can be found in the same way as non-topological case.

2. Outline of the proofs

i) local in time existence

Proposition 2.1. Given a data set \( (A, \phi, \tilde{N}) \in H^s(s \geq 2) \) at \( t = 0 \), there exists \( T^* \) depending only on \( \|(A, \phi, \tilde{N})(0, \cdot)\|_{H^s} \) and a unique development \( (A, \phi, \tilde{N}) \) in the temporal gauge with

\[
(A, \phi, \tilde{N}) \in C([0, T^*); H^s(R^2)) \cap C^1([0, T^*); H^{s-1}(R^2)).
\]

This solution can be continued as long as \( \|(A, \phi, \tilde{N})\|_{H^s(t)} \) remains bounded.

First we show that there exists \( T^* \) such that (1.7) has a unique solution in \( X_T \),

\[
X_T = \{(u, \partial_0 u) \in C([0, T^*); H^2 \times H^1) : \|u\|_{X_T} < \infty\},
\]

where \( \|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^2 \times H^1} \). The solution is obtained by standard contraction argument using energy estimates, and has continuous dependence on the initial data. This solution can be continued as long as \( \|u_\kappa(t, \cdot)\|_{H^2 \times H^1} \) remains bounded. To complete the local existence of Maxwell-Chern-Simons-Higgs, we also show that the constraint (1.8) is preserved in time.

ii) global in time existence

The proofs follow Mongrief's method mentioned earlier and use usual
a priori estimates to show $\|u(t, \cdot)\|_{H^2 \times H^1}$ does not blow up in a finite time.

Let $u = (A, \phi, N) \in C([0, T); H^2(R^2)) \cap C^1([0, T); H^{s-1}(R^2))$ be a solution of Maxwell-Chern-Simons-Higgs obtained in the part $i)$. Define $E(t), E_1(t), F(t)$ as such

$$E(t) = \int_{\mathbb{R}^2} \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N),$$

(2.1)

$$E_1(t) = \int_{\mathbb{R}^2} \frac{1}{2} (\partial_i F_{0i})^2 + \frac{1}{2} (\partial_i F_{12})^2 + |D_\mu \psi_\mu|^2 + (\partial_\mu \Omega_\mu)^2,$$

where $\psi_\mu = D_\mu \phi$, $\Omega_\mu = \partial_\mu N$,

(2.2)

$$F_1(t) = ||\partial_i A||_{L^2}(t) + ||\partial_i \phi||_{L^2}(t) + ||\partial_i N||_{L^2}(t).$$

(2.3)

The global result will be established after following Lemmas.

**Lemma 2.2.** Let $u = (A, \phi, \tilde{N}) \in C([0, T); H^2(R^2)) \cap C^1([0, T); H^1(R^2))$ be a solution of (1.7), (1.8). Then

1. $E(t) = E(0)$ for all $t \in [0, T)$
2. $(A, \phi, \tilde{N})$ are estimated in $L^2$ in terms of the initial data for all $t \in [0, T)$;

$$||u(t, \cdot)||_{L^2} \leq ||u(0, \cdot)||_{L^2} + tE.$$  

(2.4)

**Lemma 2.3.**

1. $E_1(t)$ is differentiable for all $t \in [0, T)$ and satisfies

$$\partial_0 E_1(t) = \int_{\mathbb{R}^2} -\kappa \epsilon^k \partial_i F_{0i} \partial_k F_{0k} - 2eIm(\psi_i \overline{\psi}_i + \phi \overline{D_i \psi}_i)$$

$$+ 2Re(D_i \overline{\psi}_j i eF_{ij} \psi_j - D_k U_\phi + i eF_{0i} \psi_0)$$

$$+ 2Re(D_i \overline{\psi}_j i eF_{ij} \psi_0 - D_k (F_{0j} \phi) + i eF_{0i} \psi_j) - 2\partial_0 U_N \partial_0 \Omega_0.$$  

(2.5)

2. $E_1(t), F_1(t)$ are estimated in terms of the initial data for all $t \in [0, T)$,

$$E_1(t) \leq C(E, E_1(0))(1 + t)^2,$$

(2.6)

$$F_1(t) \leq C(E, E_1(0), F(0))(1 + t)^{\frac{5}{2}}.$$

(2.7)

It is not clear the energy norm $\|u(t, \cdot)\|_{H^1}$ does not blow up in the temporal gauge, though the energy itself is preserved in Lemma 2.2. In Lemma 2.3, $E_1(t)$, the higher order energy, is shown to be initially bounded, from which $F_1(t) = ||\partial u(t, \cdot)||_{H^1}$ can be easily estimated in terms of the initial data. Combining (2.4), (2.7) we have $\|u(t, \cdot)\|_{H^1}$ is initially bounded.
Proving two Lemmas we use the Hamiltonian formalism of this system \((1.9)\) after taking time derivatives in \(E(t), E_1(t), F(t)\). We depend on the covariant Sobolev inequalities in \([7]\) (see Appendix of it), estimating each terms of \(\partial_0 E_1(t)\) to show the right hand terms of \((2.5)\) are at most linear with right to \(E_1(t)\). Next we introduce Brezis-Gallouet inequality.

**Lemma 2.4.** \([8]\) \(s > 1\),
\[
\|u\|_{L^\infty} \leq C\|u\|_H^1(1 + \sqrt{\log(1 + \|u\|_H^s)}).
\]

Finally we carried out a priori estimate \(\|u(t, \cdot)\|_{H^2 \times H^1}\) to get
\[
\|u\|_{H^2(t)} + \|\partial_0 u\|_{H^1(t)} \leq \|u\|_{H^2(0)} + C \int_0^t \|\partial_0 u\|_{H^1}
+ ((1 + \|u\|_{L^\infty})^2 + \|u\|_H^1(1 + \|u\|_{L^2}))\|u\|_H^1
+ \|u\|_{L^\infty}\|u\|_H^1\|u\|_H^1 + (1 + \|u\|_{L^\infty})\|u\|_H^2\]
\]
(2.8)

by energy estimates to \((1.7)\), then we have
\[
\|u(0, \cdot)\|_{H^2} + C(t) \int_0^t \log(1 + \|u(s, \cdot)\|_{H^2})\|u(s, \cdot)\|_{H^2 \times H^1} ds. \tag{2.9}
\]

applying above Brezis-Gallouet inequality. The desired result, thus, is given by the general Gronwall inequality.

For the case of an initial data \(u \in H^s(\mathbb{R}^2)\), it is easy to obtain a local existence result as proposition 2.1. For a global result we state a next lemma omitting its simple proof.

**Lemma 2.5.** Let \((A, \phi, N) \in C([0, T); H^s(\mathbb{R}^2)) \cap C^1([0, T); H^{s-1}(\mathbb{R}^2))\) be a solution of \((1.7), (1.8)\) for \(s > 2\) then \(\|u\|_{H^s(t)}\) is estimated in terms of the initial data for all \(t \in [0, T)\).

\(iii)\) Maxwell-Higgs limit

Let \(u^\kappa\) be a topological solution of Maxwell-Chern-Simons-Higgs obtained in Theorem1.1 with coupling constant \(\kappa\) of \(H^2\) initial data \(u_0\). It is easy to show \(\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1}\) is estimated to be
\[
\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1} \leq \int_0^t \kappa \|\partial_0 A_\kappa\|_{H^1} + C(t)\|u^\kappa - u\|_{H^2}, \tag{2.10}
\]

using \(\sup_{0 \leq \kappa \leq 1} \|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1} \leq C(t)\) for a smooth function in the proof of Theorem 1.1. Then applying Gronwall inequality to \((2.10)\), we have
\[
\|(u^\kappa - u)(t, \cdot)\|_{H^2} \leq \kappa C(t) + \|(u^\kappa - u)(0, \cdot)\|_{H^2},
\]

letting \(\kappa \to 0\), we obtain the desired result.
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