ON $L^p$ BOUNDEDNESS OF A CLASS OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Let $p(x,\xi)$ be a symbol in Hörmander class $S_{1,\delta}^0$. Then it is known that the pseudodifferential operator $p(X,D_x)$ is $L^p(\mathbb{R}^n)$ bounded. In the present paper we give a class of pseudodifferential operators and study the $L^p(\mathbb{R}^n)$ boundedness of the operators. The class of operators is closely related to the Schrödinger operators with magnetic potentials.

Keywords: pseudodifferential operators, BMO, interpolation

1. INTRODUCTION

Let $S_{\rho,\delta}^m$ be the set of Hörmander class symbols, that is,

$$S_{\rho,\delta}^m = \{ p(x,\xi) : |p_{(\beta)}^{(\alpha)}(x,\xi)| \leq C \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \text{ for any } \alpha \text{ and } \beta \}$$

Here we use that for any multiintegers $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$

$$p_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta p(x,\xi)$$

and

$$\partial_\xi^\alpha = \left( \frac{\partial}{\partial \xi} \right)^\alpha = \left( \frac{\partial}{\partial \xi_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial \xi_n} \right)^{\alpha_n}$$

$$D_x^\beta = \left( \frac{\partial}{i\partial x} \right)^\beta = \left( \frac{\partial}{i\partial x_1} \right)^{\beta_1} \cdots \left( \frac{\partial}{i\partial x_n} \right)^{\beta_n}$$

We define the pseudodifferential operator of symbol $p(x,\xi)$ by

$$p(X,D_x)u(x) = \frac{1}{(2\pi)^n} \int e^{ix \xi} p(x,\xi) \hat{u}(\xi) d\xi$$

where the integration is taken in $\mathbb{R}^n$ and $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is,

$$\hat{u}(\xi) = \int e^{-ix \xi} u(x) dx$$

We denote that the set of pseudodifferential operators with symbol of class $S_{\rho,\delta}^m$ by the same notation as the symbol class.

We say that a linear operator $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is $L^p$ bounded if there is a constant $C$ such that

$$||Tu||_{L^p(\mathbb{R}^n)} \leq C ||u||_{L^p(\mathbb{R}^n)} \text{ for any } u \in S$$

We denote the set of all $L^p$ bounded operators by $\mathcal{L}(L^p(\mathbb{R}^n))$. The following theorem is known as Calderón-Vaillancourt theorem([1]).
Theorem 1. Let $0 \leq \delta \leq \rho \leq 1, \delta < 1$. Then we have

$$S^0_{\rho,\delta} \subset \mathcal{L}(L^2(\mathbb{R}^n))$$

For the general $L^p$ boundedness, we can see the following theorem([2],[4]).

Theorem 2. Let $0 \leq \delta \leq \rho \leq 1, (\delta < 1)$ and $1 < p < \infty$. Then we have

$$S^m_{\rho,\delta} \subset \mathcal{L}(L^p)$$

if and only if $m \leq -n(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|$.

We want to generalize these results to a class of pseudodifferential operators which is useful to the study of Schrödinger operators with magnetic potentials.

2. PRELIMINARY RESULTS

Let $a(x) = (a_1(x), \cdots, a_n(x))$ be an $\mathbb{R}^n$ valued function such that $\partial_x^\alpha a_j(x)$ are bounded for any multiinteger $\alpha \neq 0$. Then we define a smooth function $\lambda(x, \xi)$ by

$$\lambda(x, \xi) = \sqrt{|\xi - a(x)|^2 + 1}$$

Then it is not difficult that the function $\lambda(x, \xi)$ satisfies

1. $\lambda(x, \xi) \geq 1$
2. $|\partial_\xi^\alpha \partial_x^\beta \lambda(x, \xi)| \leq C_{\alpha,\beta} \lambda(x, \xi)^{1-|\alpha|}$

By using the function $\lambda(x, \xi)$ we define a class $S^m_{\rho,\delta,\lambda}$ of symbols by

$$S^m_{\rho,\delta,\lambda} = \{p(x, \xi) : |p^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_{\alpha,\beta} \lambda(x, \xi)^{m-\rho|\alpha|+\delta|\beta|} \text{ for any } \alpha; \text{ and } \beta\}$$

and denote

$$S^\infty_{\rho,\delta,\lambda} = \bigcup_{m \in \mathbb{R}} S^m_{\rho,\delta,\lambda}.$$ 

This class of symbols is useful for the study of Schrödinger operators with magnetic potentials (see for example [7]). Then it is known that if $0 \leq \delta < \rho \leq 1$, $\delta < 1$ then the class of pseudodifferential operators with symbols $S^\infty_{\rho,\delta,\lambda}$ makes an algebra. Moreover we can show the following $L^2$ boundedness theorem by using the method in [5].

Theorem 3. We assume that $0 \leq \delta < \rho \leq 1$. If a symbol $p(x, \xi)$ is in $S^0_{\rho,\delta,\lambda}$, then the pseudodifferential operator $P = p(X, D_x)$ is $L^2$ bounded. That is, there is a constant $C$ such that

$$\|p(X, D_x)u\| \leq C\|u\|$$

where $\|\cdot\|$ means the usual $L^2(\mathbb{R}^n)$ norm.

3. $L^p$ BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS

Let $a(x) = (a_1(x), \cdots, a_n(x))$ be an $\mathbb{R}^n$ valued function, and let

$$\lambda(x, \xi) = \sqrt{|\xi - a(x)|^2 + 1}.$$

In the following we don’t assume that the vector function $a(x)$ is not smooth, we need only the fact that $a(x)$ is $\mathbb{R}^n$ valued and measurable.
In the following we use always $C$ as constant independent of variables. Hence the value of $C$ in inequalities are not the same at each occurrence. First we give simple boundedness lemmas of the pseudodifferential operators.

**Lemma 1.** If the support of symbol $p(x, \xi)$ is contained in $\{(x, \xi) : |\xi - a(x)| \leq R\}$ for some positive constant $R$ and $p(x, \xi)$ satisfies

\[
|p^{(\alpha)}(x, \xi)| \leq C_\alpha
\]

for any $\alpha$ with $|\alpha| \leq n + 1$. Then the operator $p(X, D_x)$ is written as

\[
p(X, D_x)u(x) = \int K(x, x - y)u(y)dy
\]

where the kernel $K(x, z)$ satisfies

\[
|k(x, z)| \leq \frac{C}{\langle z \rangle^{n+1}}
\]

**Proof.** We can write

\[
p(X, D_x)u(x) = \int K(x, x - y)u(y)dy
\]

where

\[
K(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi}p(x, \xi)d\xi
\]

Then for $|\alpha| \leq n + 1$ we have

\[
z^\alpha K(x, z) = (i)^{|\alpha|} \frac{1}{(2\pi)^n} \int e^{iz\xi}p^{(\alpha)}(x, \xi)d\xi
\]

Hence we have

\[
|z^\alpha K(x, z)| \leq \frac{1}{(2\pi)^n} \int |p^{(\alpha)}(x, \xi)|d\xi
\]

\[
\leq \frac{1}{(2\pi)^n} \int_{\xi : |\xi - a(x)|} C_\alpha d\xi
\]

\[
\leq C
\]

where the last constant $C$ is independent of the variable $x$. Thus we have the kernel estimate (3). \qed

Because of the estimate (3), we have

\[
\int |K(x, z)|dz \leq M
\]

Therefore we have

**Proposition 1.** Let $p(x, \xi)$ satisfy the same assumption as in Lemma 1, then the pseudodifferential operator $p(X, D_x)$ is $L^p$ bounded for $1 \leq p \leq \infty$ and the bound norm is estimated by $M$ in (4).

For $2 \leq p \leq \infty$ we have
Lemma 2. If the support of symbol $p(x, \xi)$ contained in $\{(x, \xi) : | \xi - a(x)| \leq R\}$ for some positive constant $R$ and $p(x, \xi)$ satisfies the inequality (2) for $|\alpha| \leq \kappa = \left[\frac{n}{2}\right] + 1$, then the pseudodifferential operator $p(X, D_x)$ is $L^p$ bounded for $2 \leq p \leq \infty$.

Proof. We can write
\[ p(X, D_x)u(x) = \int K(x, x-y)u(y)dy \]
where
\[ K(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi}p(x, \xi)d\xi \]
Then by the Schwarz inequality and the Plancherel formula we have
\[ \int |K(x, z)|dz \leq C \left\{ \int (z)^{-2\kappa}dz \right\}^{1/2} \left\{ \int (z)^{2\kappa}|K(x, z)|^2dz \right\}^{1/2} \leq C \sum_{|\alpha| \leq \kappa} \left\{ \int |p^{(\alpha)}(x, \xi)|^2d\xi \right\}^{1/2} \]
Thus we have
\[ \|p(X, D_x)u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{L^\infty(\mathbb{R}^n)} \]
In a similar way we have
\[ \|p(X, D_x)u\|^2 = \int \left| \int K(x, x-y)u(y)dy \right|^2 dx \leq \int \left\{ \int (x-y)^{2\kappa}|K(x, x-y)|^2dx \right\} \left\{ \int (x-y)^{-2\kappa}|u(x)|^2dx \right\} \leq C \int \int (x-y)^{-2\kappa}|u(x)|^2dydx = C\|u\|^2 \]
Hence by the Riesz-Thorin interpolation we get the Lemma. \square

One of the main results in the present note is the following.

Theorem 4. Let $a(x)$ be the same as in Lemma 1, and $\lambda(x, \xi)$ be defined by (1). Let $\omega(t)$ be a nonnegative and nondecreasing function on $[0, \infty)$ such that
\[ \int_0^\infty \frac{\omega(t)}{t}dt < \infty \]
We assume that a symbol $p(x, \xi)$ satisfies
\[ |p^{(\alpha)}(x, \xi)| \leq C_\alpha \lambda(x, \xi)^{-|\alpha|}\omega(\lambda(x, \xi)^{-1}) \]
for any $\alpha$ with $|\alpha| \leq n + 1$. Then the pseudodifferential operator $p(X, D_x)$ is $L^p$ bounded for $1 \leq p \leq \infty$.

Proof. By Lemma 1, we may assume that the support of the symbol $p(x, \xi)$ is contained in $\{(x, \xi) : |\xi - a(x)| \geq 2\}$. Now we take a smooth nonnegative function $f(t)$ such that the support of $f(t)$ is contained in the interval $[\frac{1}{2}, 1]$ and

$$\int_{0}^{\infty} \frac{f(t)}{t} dt = 1$$

Then since the support of the symbol $p(x, \xi)$ is contained in $\{(x, \xi) : |\xi - a(x)| \geq 2\}$, we have

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_{0}^{1} \frac{1}{t} dt \int e^{i(x-y)\xi} p(t|x|) u(y) dy d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{0}^{1} \frac{1}{t^{n+1}} dt \int e^{i(x-y)\xi} e^{i(x-y)\alpha(x)} p(x, \xi + a(x)) f(|\xi|) u(y) d\xi dy$$

$$= \frac{1}{(2\pi)^n} \int_{0}^{1} \frac{1}{t} dt \int e^{t\alpha(x)} K_t(x, z) u(x - tz) dz$$

where

$$K_t(x, z) = \frac{1}{(2\pi)^n} \int e^{i\xi p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi}$$

If we put $\tilde{p}(x, \xi) = p(x, \frac{\xi}{t} + a(x))$, then it is easy to see that

$$|\tilde{p}^{(\alpha)}(x, \xi)| \leq C_\alpha \langle \xi \rangle^{-|\alpha|} \omega(\langle \xi \rangle^{-1})$$

for $|\alpha| \leq n + 1$. Since the equality

$$z^\alpha K_t(x, z) = \frac{1}{(2\pi)^n} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \left(\frac{\alpha}{\alpha'}\right) \int e^{i\xi p^{(\alpha)} (x, \frac{\xi}{t})} \delta^{\alpha - \alpha'} f(|\xi|) d\xi$$

holds for $|\alpha| \leq n + 1$, we have

$$|z^\alpha K_t(x, z)| \leq \frac{1}{(2\pi)^n} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \left(\frac{\alpha}{\alpha'}\right) \int |\tilde{p}^{(\alpha)}(x, \frac{\xi}{t})| \delta^{\alpha - \alpha'} f(|\xi|) d\xi$$

$$\leq C \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \left(\frac{\alpha}{\alpha'}\right) \int_{1/2 \leq |\xi| \leq 1} \frac{1}{t} |\xi| |\xi|^{-1} \omega(\frac{|\xi|}{t}) d\xi$$

$$\leq C \omega(t)$$

for $|\alpha| \leq n + 1$. Therefore we have

$$|K_t(x, z)| \leq C(\xi)^{-n-1} \omega(t)$$

By the inequality (5) and the equality

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_{0}^{1} \frac{1}{t} dt \int e^{t\alpha(x)} K_t(x, z) u(x - tz) dz$$

$$= \frac{1}{(2\pi)^n} \int_{0}^{1} \frac{1}{t} dt \int e^{t\alpha(x)} \sum_{\alpha} \frac{1}{t^{n+1}} \left(\frac{\alpha}{\alpha'}\right) \int e^{i\xi p^{(\alpha)} (x, \frac{\xi}{t})} \delta^{\alpha - \alpha'} f(|\xi|) d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{0}^{1} \frac{1}{t} dt \int e^{t\alpha(x)} K_t(x, z) u(x - tz) dz$$
we can see that the operator $p(X,D_x)$ is $L^1$ bounded and $L^\infty$ bounded. That is inequalities

$$\|p(X,D_x)u\|_{L^1(\mathbb{R}^n)} \leq C\|u\|_{L^1(\mathbb{R}^n)}$$
$$\|p(X,D_x)u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}$$

holds. So by the Riesz-Thorin interpolation theorem we have the $L^p$ boundedness for $1 \leq p \leq \infty$.

When $2 \leq p$, we can show a little more general result than Theorem 4, by using the Plancherel Theorem.

**Theorem 5.** Let $a(x)$ and $\lambda(x,\xi)$ be the same as in Theorem 4. Let $\omega(t)$ be a nonnegative and nondecreasing function on $[0,\infty)$ such that

$$\int_0^\infty \frac{\omega(t)}{t} dt < \infty$$

We assume that a symbol $p(x,\xi)$ satisfies

$$|p^{(\alpha)}(x,\xi)| \leq C_\alpha \lambda(x,\xi)^{-|\alpha|} \omega(\lambda(x,\xi)^{-1})$$

for any $\alpha$ with $|\alpha| \leq \kappa = \left[\frac{n}{2}\right] + 1$. Then the pseudodifferential operator $p(X,D_x)$ is $L^p$ bounded for $2 \leq p \leq \infty$.

**Proof.** We first show the $L^\infty$ boundedness. We write the operator $p(X,D_x)$, as in the proof of Theorem 4, by

$$p(X,D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \int e^{itza(x)} K_t(x,z) u(x-tz) dz$$

where

$$K_t(x,z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p\left(x, \frac{\xi}{t} + a(x)\right) f(|\xi|) d\xi$$

Then writing $\kappa = \left[\frac{n}{2}\right] + 1$, we have

$$\int |K_t(x,z)| dz = \int (z)^{-\kappa} |K_t(x,z)| dz$$

$$\leq \left\{ \int (z)^{-2\kappa} dz \right\}^{1/2} \left\{ \int (z)^{2\kappa} |K_t(x,z)|^2 dz \right\}^{1/2}$$

$$\leq C \sum_{|\alpha| \leq \kappa} \left\{ \int |z^\alpha K_t(x,z)|^2 dz \right\}^{1/2}$$

Using the Plancherel equality, we have

$$\int |z^\alpha K_t(x,z)|^2 dz = \int |\partial_\xi^\alpha \{ p\left(x, \frac{\xi}{t}\right) f(|\xi|) \}|^2 d\xi$$

$$\leq C_\alpha \omega(t)$$

Hence we have

$$|p(X,D_x)u(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}$$
In order to show $L^2$ boundedness of the operator $p(X, D_x)$, we use the same representation

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \int e^{itza(x)} K_t(x, z) u(x - tz) dz$$

where

$$K_t(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi$$

From this representation we have

$$\|p(X, D_x)\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) d\cdot \right\|_{L^2(\mathbb{R}^n)}$$

Then we have

$$\left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) d\cdot \right\|^2 = \int \left\| \int e^{itza(x)} K_t(x, z) u(x - tz) dz \right\|^2 dx$$

$$\leq \int \left\| |K_t(x, z) u(x - tz)| dz \right\|^2 dx$$

By using the Schwarz inequality we have

$$\left\| \int |K_t(x, z) u(x - tz)| dz \right\|^2 \leq \int (z)^{2\kappa} |K_t(x, z)|^2 dz \int (z)^{-2\kappa} |u(x - tz)|^2 dz$$

As above we can see

$$\int (z)^{2\kappa} |K_t(x, z)|^2 dz \leq \sum_{|\alpha| \leq \kappa} \int |z^\alpha K_t(x, z)|^2 dz$$

$$\leq \sum_{|\alpha| \leq \kappa} \int |\partial_\xi \{ p(x, \frac{\xi}{t} + a(x)) f(|\xi|) \}|^2 d\xi$$

Therefore we get

$$\left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) d\cdot \right\|^2 \leq C \int \int (z)^{-2\kappa} |u(x - tz)|^2 dz dx$$

$$\leq C \omega(t)^2 \|u\|^2$$

Thus from the assumption of $\omega(t)$ we have the $L^2$ estimate

$$\|p(X, D_x)u\| \leq C \|u\|$$

Again by the Riesz-Thorin interpolation theorem we have the $L^p$ boundedness for $2 \leq p \leq \infty$. \qed

**Remark 1.** Theorems in this section we don't always assume that the vector function $a(x) = (a_1(x), \cdots, a_n(x))$ satisfies the estimate

$$|\partial_\xi a_j(x)| \leq C$$

In several theorems we can prove the theorem under only the measurability of $a(x)$.  

Remark 2. If the vector function \( a(x) \) is bounded, then the symbol class \( S^{m}_{p,d,\lambda} \) coincide with the usual Hörmander class \( S^{m}_{p,d} \). Hence using the similar method of usual class, the \( L^{p} \) boundedness in Theorem 4 can be shown (see for example, [6]). Even if \( a(x) \) is not bounded, we have

Proposition 2. Let \( a(x) \) and \( \lambda(x,\xi) \) be the same as in Theorem 4. For any smooth function \( \varphi \) with compact support, we have

\[
||\varphi(x)p(X,D_{x})u||_{L^{p}(\mathbb{R}^{n})} \leq C||u||_{L^{p}(\mathbb{R}^{n})}
\]

4. CONJECTURE

As we see in the previous sections we can expect that the following \( L^{p} \) boundedness theorem.

Conjecture 1. If the vector function \( a(x) = (a_{1}(x), \cdots, a_{n}(x)) \) satisfies

\[
|\partial^{\alpha}a_{j}(x)| \leq C_{\alpha}
\]

for any \( \alpha \neq 0 \). Then for \( 1 < p < \infty \), the operator \( p(X,D) \) in \( S^{0}_{1,\delta,\lambda} \) is \( L^{p} \) bounded. That is,

\[
S^{0}_{1,\delta,\lambda} \subset \mathcal{L}(L^{p}^{\infty}(\mathbb{R}^{n}))
\]

holds.

As we stated in section 2 it is known that if the vector function \( a(x) \) satisfies the estimates in the above conjecture, the operators in \( S^{0}_{1,\delta,\lambda} \) with \( (\delta < 1) \) are \( L^{2} \) bounded. So if we can show the weak type \((1,1)\) estimates or boundedness from \( L^{\infty}(\mathbb{R}^{n}) \) to \( BMO \), then we can get the above conjecture, that is, \( L^{p} \) boundedness for \( 1 < p < \infty \) by using the interpolation theorems (see for example [8], [3]). The fundamental conjecture is

Conjecture 2. If the vector function \( a(x) = (a_{1}(x), \cdots, a_{n}(x)) \) satisfies

\[
|\partial^{\alpha}a_{j}(x)| \leq C_{\alpha}
\]

for any \( \alpha \neq 0 \). Then the operator \( p(X,D) \) in \( S^{0}_{1,\delta,\lambda} \) is bounded from \( L^{\infty}(\mathbb{R}^{n}) \) to \( BMO \), that is, there is a constant \( C \) such that

\[
||p(X,D_{x})u||_{BMO} \leq C||u||_{L^{\infty}(\mathbb{R}^{n})}
\]

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