A generalization of the Lieb-Thirring inequality and its applications

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1 Introduction

In 1976 Lieb and Thirring proved the following theorem([9]).

Theorem 1.1 Let $n \in \mathbb{N}$ and $\gamma$ be a non-negative number such that

\[ \gamma > \frac{1}{2} \quad \text{if} \quad n = 1, \]
\[ \gamma > 0 \quad \text{if} \quad n = 2, \]
\[ \gamma \geq 0 \quad \text{if} \quad n \geq 3. \]

Suppose that $V \in L^{n/2+\gamma}(\mathbb{R}^n)$ and $V \geq 0$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the negative eigenvalues of the Schrödinger operator $-\Delta - V$. Then we have

\[ \sum_i |\lambda_i|^\gamma \leq c_{n,\gamma} \int_{\mathbb{R}^n} V^{n/2+\gamma} dx. \]

Remark

(i) The Lieb-Thirring inequality holds for $n = 1$ and $\gamma = 1/2$ (Weidl[17]).

(ii) The Lieb-Thirring inequality does not hold for $n = 1, \gamma < 1/2$ or $n = 2, \gamma = 0$ ([9]).

The Lieb-Thirring inequality has important applications in the study of the stability of matter or the estimate of the dimension of attractors of nonlinear equations.

In 1995 Egorov-Kondrat'ev provided a generalization of the Lieb-Thirring inequality([3]).

Theorem 1.2 Let $n \in \mathbb{N}$, $q \geq \frac{n}{2}$ and $\gamma$ be a non-negative number such that

\[ \gamma > q \quad \text{if} \quad n = 1, \]
\[ \gamma > 0 \quad \text{if} \quad n = 2 \]
\[ \gamma \geq 0 \quad \text{if} \quad n \geq 3. \]
Suppose $V \geq 0$ and \( \int_{\mathbb{R}^n} V^{q+\gamma} |x|^{2q-n} \, dx < \infty \). Let \( \lambda_1 \leq \lambda_2 \leq \cdots \) be the negative eigenvalues of the Schrödinger operator \(-\Delta - V\). Then we have
\[
\sum_i |\lambda_i|^{\gamma} \leq c_{n,\gamma,q} \int_{\mathbb{R}^n} V^{q+\gamma} |x|^{2q-n} \, dx.
\]

Theorem 1.2 is a special case of Egorov-Kondrat'ev's result in [3]. In fact Egorov and Kondrat'ev proved a generalization of Theorem 1.2 for an elliptic operator of order $2m$. In this paper we give a generalization of Egorov-Kondrat'ev’s result for certain degenerate elliptic partial differential operator, for which the rate of degeneracy is regulated by the weight $w \in A_2$.

First we recall the definition of $A_p$-weights. By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable function $w$ on $\mathbb{R}^n$ and $w > 0$ a.e. is an $A_p$-weight for some $p \in (1, \infty)$ if there exists a positive constant $C$ such that
\[
\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C
\]
for all cubes $Q \subset \mathbb{R}^n$. We say that $w$ is an $A_1$-weight if there exists a positive constant $C$ such that
\[
\frac{1}{|Q|} \int_Q w(y) \, dy \leq Cw(x) \quad \text{a.e.} \, x \in Q
\]
for all cubes $Q \subset \mathbb{R}^n$. We write $A_p$ for the class of $A_p$-weights.

Next we consider an elliptic partial differential operator of order $2m$. For $m \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$ let
\[
L_0 f(x) = \sum_{|\alpha|=|\beta|=m} (-1)^m D^\alpha (a_{\alpha\beta}(x) D^\beta f(x)),
\]
where
\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n,
\]
\[
a_{\alpha\beta} \in H^{m}_{\text{loc}}(\mathbb{R}^n), \quad \text{and} \quad a_{\alpha\beta} = \overline{a_{\beta\alpha}}.
\]

In the above definition the space $H^{m}_{\text{loc}}(\mathbb{R}^n)$ denotes the set of all $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that $D^\alpha f \in L^2_{\text{loc}}(\mathbb{R}^n)$ for all $|\alpha| \leq m$. 

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Let $n > 2m, q \geq n/(2m)$ and $\gamma \geq 0$. We assume that there exists a $w \in A_2$ such that

\[(1) \quad (L_0 f, f) \geq \int_{\mathbb{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2 \, dx\]

for all $f \in C_0^\infty(\mathbb{R}^n)$. Suppose that $u$ is a non-negative locally integrable function on $\mathbb{R}^n$ which satisfies $uw^{-q} \in A_q$ and

\[(2) \quad |Q|^{2m/n+1} \leq c_1 \int_Q w \left( \int_Q \frac{u}{w^q} \, dx \right)^{1/q}\]

for all cubes $Q \subset \mathbb{R}^n$, where $c_1$ is a positive constant not depending on $Q$. For a non-negative measurable function $V$ on $\mathbb{R}^n$ we assume that

\[(3) \quad \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx < \infty.\]

Let $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

\[\|f\|_{\mathcal{H}} = \{a(f, f) + \|f\|^2\}^{1/2}.\]

Then we have the following.

(i) There exists a unique self-adjoint operator $L$ in $L^2(\mathbb{R}^n)$ with domain $D \subset \mathcal{H}$ such that

\[(Lf, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx\]

for all $f \in D$ and $g \in \mathcal{H}$.

(ii) The negative spectrum of $L$ is discrete.

(iii) There exists a positive constant $c$ such that

\[(4) \quad \sum_i |\lambda_i|^{\gamma} \leq c \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx,\]

where $\{\lambda_i\}$ is the set of all negative eigenvalues of $L$ and $c$ does not depend on $V$. 
Remark

(i) Let \(L_0 = -\Delta, m = 1, w \equiv 1\), and \(u = |x|^{2q-n}\). Then we have the Egorov-Kondrat'ev theorem for \(n \geq 3\).

(ii) If \(u \equiv 1\) and \(q = n/(2m)\), then (2) is trivial by the Hölder inequality.

Next we consider the lower dimensional cases. First we recall the definition of dyadic cubes. For \(j \in \mathbb{Z}\) and \(k \in \mathbb{Z}^n\) the cube

\[ Q = \{(x_1, \ldots, x_n) : k_i \leq 2^j x_i < k_i + 1, \ i = 1, \ldots, n\} \]

is called a dyadic cube. Let \(Q\) be the set of all dyadic cubes in \(\mathbb{R}^n\). For each \(Q \in Q\) there is a unique \(Q' \in Q\) such that \(Q \subset Q'\) and the side-length of \(Q'\) is the double of that of \(Q\). We call \(Q'\) the parent of \(Q\) in this paper.

We have the following theorem.

Theorem 1.4 Let \(n \leq 2m, q \geq n/(2m), \gamma > 0\) and \(q + \gamma > 1\). We assume that there exists a \(w \in A_2\) such that

\[ (L_0 f, f) \geq \int_{\mathbb{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2 \, dx \]

for all \(f \in C_0^\infty(\mathbb{R}^n)\). We assume that

\[ \int_{Q'} w \, dx \leq 2^{2m} \int_Q w \, dx \]

for all dyadic cubes \(Q\) and its parent \(Q'\). Suppose that \(u\) is a non-negative locally integrable function on \(\mathbb{R}^n\) which satisfies \(uw^{-q} \in A_{q+\gamma}\) and

\[ |Q|^{2m/n+1} \leq c_1 \int_Q w \, dx \left( \int_Q \frac{u}{w^q} \, dx \right)^{1/q} \]

for all cubes \(Q \subset \mathbb{R}^n\), where \(c_1\) is a positive constant not depending on \(Q\). For a non-negative measurable function \(V\) on \(\mathbb{R}^n\) we assume that

\[ \int_{\mathbb{R}^n} V^{q+\gamma} u \frac{u}{w^q} \, dx < \infty. \]

Let \(H\) be the completion of \(C_0^\infty(\mathbb{R}^n)\) with respect to the norm

\[ \|f\|_H = \{a(f, f) + \|f\|^2\}^{1/2}. \]
Then we have the following.

(i) There exists a unique self-adjoint operator $L$ in $L^2(\mathbb{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf, g) = a(f, g) - \int_{\mathbb{R}^n} Vf \overline{g} \, dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$.

(ii) The negative spectrum of $L$ is discrete.

(iii) There exists a positive constant $c$ such that

$$\sum_i |\lambda_i|^\gamma \leq c \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx,$$

where $\{\lambda_i\}$ is the set of all negative eigenvalues of $L$ and $c$ does not depend on $V$.

**Remark**

(i) Let $L_0 = -\Delta$, $m = 1$, $w \equiv 1$, and $u = |x|^{2q-n}$. Then we have the Egorov-Kondrat'ev theorem for $n = 1$ or 2.

(ii) Since $w \in A_2$, there exists a positive constant $c$ such that

$$\int_{Q'} w \, dx \leq c \int_{Q} w \, dx$$

for all dyadic cubes $Q$ and its parent $Q'$ (c.f. Prop.3.1 (iv) in Section 3). Hence the condition (6) is satisfied if $m$ is sufficiently large.

In the proofs of Theorems 1.3 and 1.4 we use Meyer's wavelet basis.

## 2 Wavelets

First we recall the definition of Meyer's wavelet basis. Let $\theta$ be a function which satisfies the following condition.

- $\theta$ is an even function in $C_0^\infty(\mathbb{R})$.
- $0 \leq \theta(\xi) \leq 1$ and $\text{supp } \theta \subset [-4\pi/3, 4\pi/3]$.
- $\theta(\xi) = 1$ for all $\xi \in [-2\pi/3, 2\pi/3]$. 
\begin{itemize}
  \item $\theta(\xi)^2 + \theta(2\pi - \xi)^2 = 1$ for all $\xi \in [0, 2\pi]$.
\end{itemize}

We define a function $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) = \{\theta(\xi/2)^2 - \theta(\xi)^2\}^{1/2}e^{-i\xi/2}.$$  

For integers $j, k$ we set $\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k)$. Then it turns out that $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ which we call Meyer’s wavelet basis.

We define $n$-dimensional Meyer’s wavelet basis as follows. Let $\varphi$ be a function in $L^2(\mathbb{R})$ such that $\hat{\varphi}(x) = \theta(\xi)$. Set $E = \{0, 1\}^n \setminus \{0\}$ and

$$\psi^0(x) = \varphi(x), \quad \psi^1(x) = \varphi(x).$$

For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in E$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define

$$\psi^\epsilon(x) = \psi^{\epsilon_1}(x_1) \cdots \psi^{\epsilon_n}(x_n).$$

Let $\Lambda = \{(\epsilon, j, k) : \epsilon \in E, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^n\}$. For $\lambda = (\epsilon, j, k) \in \Lambda, \ x \in \mathbb{R}^n$, set

$$\psi^\lambda(x) = 2^{nj/2}\psi^\epsilon(2^jx - k).$$

Then $\{\psi^\lambda\}_{\lambda \in \Lambda}$ is Meyer’s wavelet basis of $L^2(\mathbb{R}^n)$.

\section{Weighted inequalities}

First we recall some properties of $A_p$-weights which will be used in the following sections. Let $M$ be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q$ which contain $x$.

\textbf{Proposition 3.1}

(i) Let $1 < p < \infty$ and $w$ be a non-negative locally integrable function on $\mathbb{R}^n$. Then $M$ is bounded on $L^p(w)$ if and only if $w \in A_p$.

(ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.  

(iii) Let $0 < \tau < 1$ and $f$ be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty \ a.e.. \ Then \ (M(f))^\tau \in A_1$.

(iv) Let $1 \leq p < \infty$ and $w \in A_p$. Then there exists a positive constant $c$ such that
\[
\int_{2Q} w \, dx \leq c \int_Q w \, dx
\]
for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of $Q$.

The proofs of these facts are in [6, Chapter IV] or [15, Chapter V]. Property (iv) is called the doubling property of $A_p$-weights.

Next we state some weighted inequalities. For $\alpha \geq 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ we define via inverse Fourier transform
\[
(-\Delta)^{\alpha/2} f(x) = \mathcal{F}^{-1}(|\xi|^{\alpha} \hat{f})(x).
\]
For $\lambda = (\epsilon, j, k) \in \Lambda$, set
\[
Q(\lambda) = \{(x_1, \ldots, x_n) : k_i \leq 2^j x_i < k_i + 1, \ i = 1, \ldots, n\}.
\]

Proposition 3.2 Let $\alpha \geq 0$ and $w \in A_2$. Then there exist positive constants $c_1$ and $c_2$ such that
\[
\frac{c_1}{\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f|^2 w \, dx} \leq \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-\frac{2\alpha}{n}} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx
\]
\[
\leq c_2 \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f|^2 w \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$.

This proposition is proved in [16, Prop. 2.2] for the $\varphi$-transform of Frazier-Jawerth. We can prove Proposition 3.2 by Proposition 2.2 in [16] by similar arguments in [5, p.72]. In our case we need the boundedness property of an almost orthogonal matrix on weighted spaces. This property is proved by the vector valued weighted inequality for maximal operators in [1] and similar arguments in [4, p.54].

4 Outline of the proof of Theorem 1.3

We shall prove Theorem 1.3 for the case $\gamma = 0$. The general case is proved by this special case. The detail of the proof is in [16]. By (ii) of Proposition 3.1 there exists a
constant $s$ such that $1 < s < q$ and $uw^{-q} \in A_{q/s}$. Let $v(x) = (M(V^s)(x))^{1/s}$. By the properties of the maximal operator we have $V(x) \leq v(x)$ a.e.. By (i) of Proposition 3.1 we get

$$
\int_{\mathbb{R}^n} \left( \frac{v}{w} \right)^q u \, dx = \int_{\mathbb{R}^n} \frac{M(V^s)^{q/s}}{w^q} u \, dx \leq c_1 \int_{\mathbb{R}^n} \left( \frac{V}{w} \right)^q u \, dx < \infty.
$$

Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 3.1.

Now we fix a $\delta > 0$ and set

$$
I = \{ \lambda \in \Lambda : \int_{Q(\lambda)} v(x) \, dx \geq \delta |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w(x) \, dx \}.
$$

**Lemma 4.1** $I$ is a finite set.

For $f \in C_0^{\infty}(\mathbb{R}^n)$ we have

$$
\int |f|^2 V \, dx \leq \int |f|^2 v \, dx \leq c_2 \sum_{\lambda \in I} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx,
$$

where we used Proposition 3.2 and the fact $v \in A_1 \subset A_2$. The last quantity is bounded by

$$
c_2 \sum_{\lambda \in I} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx + c_2 \sum_{\lambda \in \Lambda \setminus I} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\leq c_2 K \sum_{\lambda \in I} |(f, \psi_\lambda)|^2 + c_2 \delta \sum_{\lambda \in \Lambda \setminus I} |(f, \psi_\lambda)|^2 |Q(\lambda)|^{-2m/n} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx
\leq c_2 K \|f\|_2^2 + c_3 \delta \int |(-\Delta)^{m/2} f(x)|^2 w(x) \, dx,
$$

where

$$
K = \max_{\lambda \in I} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
$$

and we used Proposition 3.2.

Now we use the following lemma([16, Lemma 3.2]).

**Lemma 4.2** Let $m \in \mathbb{N}$ and $w \in A_2$. Then there exists a positive constant $c > 0$ such that

$$
\int_{\mathbb{R}^n} |(-\Delta)^{m/2} f(x)|^2 w(x) \, dx \leq c \int_{\mathbb{R}^n} \{ \sum_{|\alpha| = m} |D^\alpha f(x)|^2 \} w(x) \, dx
$$

for all $f \in C_0^{\infty}(\mathbb{R}^n)$. 

By Lemma 4.2 and the condition (1) we have
\[
\int_{\mathbb{R}^n} |f|^2 V \, dx \leq c_2 K \|f\|_2^2 + c_4 \delta \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha| = m} |D^\alpha f(x)|^2 \right\} w(x) \, dx \\
\leq c_2 K \|f\|_2^2 + c_4 \delta (L_0 f, f).
\]
We choose \( \delta \) such that \( c_4 \delta < 1 \). Then we have
\[
a(f, f) - \int_{\mathbb{R}^n} V |f|^2 \, dx \geq -c_2 K \|f\|_2^2
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). Hence
\[
b(f, g) = a(f, g) - \int_{\mathbb{R}^n} V f \bar{g} \, dx
\]
is a lower semi-bounded quadratic form on \( \mathcal{H} \).

We can show that \( b(f, g) \) is a closed form on \( \mathcal{H} \). Since \( b(f, g) \) is a closed and lower semi-bounded quadratic form on \( \mathcal{H} \), there exists a unique self-adjoint operator \( L \) in \( L^2(\mathbb{R}^n) \) with domain \( \mathcal{D} \subset \mathcal{H} \) such that
\[
(L f, g) = a(f, g) - \int_{\mathbb{R}^n} V f \bar{g} \, dx
\]
for all \( f \in \mathcal{D} \) and \( g \in \mathcal{H}([11, \text{Theorem VIII.15}]) \).

We shall estimate the number of negative eigenvalues of \( L \). Let
\[
F = \{ f \in \mathcal{D} : (f, \psi_\lambda) = 0 \text{ for all } \lambda \in \mathcal{I} \}.
\]
Then the similar arguments as before lead to the estimate
\[
\int |f|^2 V \, dx \leq c_4 \delta (L_0 f, f) \quad (f \in F).
\]
Hence we get
\[
(L f, f) \geq 0 \quad (f \in F).
\]
Therefore by Theorem 12 in [8, Chap.1] the negative spectrum of \( L \) is discrete. Furthermore we have
\[
N \leq \text{codim} \, F = \# \mathcal{I},
\]
where \( N \) is the number of negative eigenvalues of \( L \).
We shall estimate $\|I\|$. The following arguments are similar to those in [13, p.201]. Let
\[ B = \{Q \in \mathcal{Q} : \int_Q v(x) \, dx \geq \delta |Q|^{-2m/n} \int_Q w(x) \, dx \}. \]
Let $\tilde{B}$ be the set of all $Q \in B$ which satisfy the following condition: there exists a half size dyadic sub-cube $\tilde{Q} \subset Q$ such that $\tilde{Q}$ does not contain any dyadic cubes in $B$.

Then we have the following lemma.

**Lemma 4.3** $\|B\| \leq 2\|\tilde{B}\|$.

Lemma 4.3 is proved in Rochberg and Taibleson's paper([14, Lemma 1]). Let $Q \in \tilde{B}$ and $\tilde{Q}$ be a dyadic cube which satisfies the condition in the definition of $\tilde{B}$. Then we get
\[ 1 \leq c_5 \int_{\tilde{Q}} \left( \frac{v}{w} \right)^q u \, dx. \]
For each $Q \in \tilde{B}$ we choose a $\tilde{Q}$ as above. Then these $\{\tilde{Q}\}$ are disjoint. Therefore we get
\[ \|\tilde{B}\| = \|\{\tilde{Q}\}\| \leq \sum_{Q} c_5 \int_{\tilde{Q}} \left( \frac{v}{w} \right)^q u \, dx \]
\[ \leq c_5 \int_{\mathbb{R}^n} \left( \frac{v}{w} \right)^q u \, dx \leq c_6 \int_{\mathbb{R}^n} \left( \frac{V}{w} \right)^q u \, dx. \]
Hence we conclude
\[ N \leq \|I\| = (2^n - 1)\|B\| \leq c_7 \int_{\mathbb{R}^n} \left( \frac{V}{w} \right)^q u \, dx. \]
Therefore we proved Theorem 1.3 for the case $\gamma = 0$.

## 5 Outline of the proof of Theorem 1.4

By (ii) of Proposition 3.1 there exists a constant $s$ such that $1 < s < q + \gamma$ and $uw^{-q} \in A_{(q+\gamma)/s}$. Let $v(x) = (M(V^s)(x))^{1/s}$. Then we have $V(x) \leq v(x)$ a.e.. By (i) of Proposition 3.1 we get
\[ \int_{\mathbb{R}^n} v^{q+\gamma} \frac{u}{w^q} \, dx = \int_{\mathbb{R}^n} M(V^s)^{(q+\gamma)/s} u \, dx \leq c_1 \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx < \infty. \]
Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 3.1. By Proposition 3.2 and Lemma 4.2 we have the following lemmata.
Lemma 5.1 There exists a positive constant $\alpha$ such that
\[
\alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2m/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$.

Lemma 5.2 There exists a positive constant $\beta$ such that
\[
\int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$.

Now we set
\[
I = \{ \lambda \in \Lambda : \beta \int_{Q(\lambda)} v(x) \, dx > \alpha |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w(x) \, dx \}.
\]
Then the following lemma holds.

Lemma 5.3 There exists a $c > 0$ such that
\[
\sum_{\lambda \in I} \left( \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^{q+\gamma} \frac{u}{w^q} \, dx
\]
For $f \in C_0^\infty(\mathbb{R}^n)$ we have
\[
\int |f|^2 V \, dx \leq \int \left| \int |f|^2 v \, dx \right| \leq \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx,
\]
where we used Lemma 5.2. The last quantity is bounded by
\[
\beta \sum_{\lambda \in I} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx + \beta \sum_{\lambda \not\in I} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\]
\[
\leq \beta K \sum_{\lambda \in I} |(f, \psi_\lambda)|^2 + \alpha \sum_{\lambda \not\in I} |(f, \psi_\lambda)|^2 |Q(\lambda)|^{-2m/n} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx
\]
\[
\leq \beta K \|f\|_2^2 + \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w \, dx
\]
where
\[
K = \max_{\lambda \in I} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\]
and we used Lemma 5.1.
By the condition (5) we have
\[ \int_{\mathbb{R}^n} |f|^2 V \, dx \leq \beta K \|f\|^2 + (L_0 f, f). \]
Hence we have
\[ a(f, f) - \int_{\mathbb{R}^n} V |f|^2 \, dx \geq -\beta K \|f\|^2 \]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). Therefore
\[ b(f, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx \]
is a lower semi-bounded quadratic form on \( \mathcal{H} \). We can show that \( b(f, g) \) is a closed form on \( \mathcal{H} \). Since \( b(f, g) \) is a closed and lower semi-bounded quadratic form on \( \mathcal{H} \), there exists a unique self-adjoint operator \( L \) in \( L^2(\mathbb{R}^n) \) with domain \( \mathcal{D} \subset \mathcal{H} \) such that
\[ (Lf, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx \]
for all \( f \in \mathcal{D} \) and \( g \in \mathcal{H}([11, \text{Theorem VIII.15}]) \).

We set
\[ \lambda_1 = \inf_{f \in \mathcal{D}, \|f\|=1} (Lf, f) \]
and
\[ \lambda_k = \sup_{\phi_1, \ldots, \phi_{k-1} \in L^2, \|\phi_1\|=1, \ldots, \|\phi_{k-1}\|=1} \inf_{f \in \mathcal{D}, \|f\|=1, f \perp \phi_1, \ldots, \phi_{k-1}} (Lf, f) \]
for \( k \in \mathbb{N}, k \geq 2 \). There are two cases.

(i) \( \lambda_1 \leq \lambda_2 \leq \cdots \) are eigenvalues of \( L \).

(ii) \( \lambda_1 \leq \cdots \leq \lambda_{k_0} \) are eigenvalues of \( L \). Furthermore we have \( \lambda_{k_0+1} = \lambda_{k_0+2} = \cdots \) which value is the infimum of the essential spectrum of \( L \).

The following lemma holds.

**Lemma 5.4** For \( A > 0 \) we set
\[ I_A = \{ \lambda \in \Lambda : \alpha|Q(\lambda)|^{-1} \int_{Q(\lambda)} w \, dx - \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \leq -A \}. \]
Then \( I_A \) is a finite set.
Let \( \{\mu_k\}_{k=1}^{\infty} \) be the non-decreasing rearrangement of
\[
\left\{ \alpha|Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx - \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \right\}_{\lambda \in I}.
\]
Then
\[
\mu_1 \leq \mu_2 \leq \cdots
\]
and
\[
\lim_{k \to \infty} \mu_k = 0.
\]
When
\[
\mu_k = \alpha|Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx - \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx,
\]
we set \( \psi_k = \psi_\lambda \). Then we have
\[
\lambda_k \geq \inf_{f \in D, \|f\| = 1, f \perp \psi_1, \ldots, \psi_{k-1}} (Lf, f)
\]
\[
\geq \inf_{f \in D, \|f\| = 1, f \perp \psi_1, \ldots, \psi_{k-1}} \sum_{j=1}^{\infty} |(f, \psi_j)|^2 \mu_j
\]
\[
\geq \mu_k \sup_{f \in D, \|f\| = 1, f \perp \psi_1, \ldots, \psi_{k-1}} \sum_{j=k}^{\infty} |(f, \psi_j)|^2 \geq \mu_k,
\]
where we used the fact \( \mu_k < 0 \).

Since
\[
\lim_{k \to \infty} \mu_k = 0,
\]
the negative spectrum of \( L \) is discrete. By these inequalities we have
\[
\sum_{k, \lambda_k < 0} |\lambda_k|^\gamma \leq \sum_{k=1}^{\infty} |\mu_k|^\gamma
\]
\[
= \sum_{\lambda \in I} \left( \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx - \alpha|Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx \right)^\gamma
\]
\[
\leq \sum_{\lambda \in I} \left( \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \right)^\gamma
\]
\[
\leq c \int_{\mathbb{R}^n} v^{q+\gamma} \frac{u}{w^q} \, dx \leq c \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx,
\]
where we used Lemma 5.3.
6 The Sobolev-Lieb-Thirring inequality

As an application of Theorem 1.1 Lieb and Thirring proved the following inequality.

**Theorem 6.1** Suppose $n \in \mathbb{N}$, $\phi_i \in H^1(\mathbb{R}^n)$ ($i = 1, \ldots, N$), and that $\{\phi_i\}_{i=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^n)$. Then we have

$$\int_{\mathbb{R}^n} \rho^{1+2/n} dx \leq c_n \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \phi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

This inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations.

A generalization of the Sobolev-Lieb-Thirring inequality is known([7]).

**Theorem 6.2** Let $n, m \in \mathbb{N}$ and $\phi_i \in H^m(\mathbb{R}^n)$ ($i = 1, \ldots, N$). Suppose that $\{\phi_i\}_{i=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^n)$. Then we have

$$\int_{\mathbb{R}^n} \rho^{1+2m/n} dx \leq c \sum_{i=1}^N \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |D^\alpha \phi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

By Theorem 1.3 we have the following generalization of Theorem 6.2.

**Theorem 6.3** Let $m, n \in \mathbb{N}$, and $n > 2m$. Let $w$ be a weight in $A_2 \cap H_{loc}^m(\mathbb{R}^n)$ such that $w^{-n/(2m)} \in A_{n/(2m)}$. Suppose that $\{\phi_i\}_{i=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^n)$ such that

$$\sum_{i=1}^N \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) dx < \infty.$$

Then we have

$$\int_{\mathbb{R}^n} \rho(x)^{1+2m/n} w(x) dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) dx,$$
\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]

and \( c \) is a positive constant which does not depend on \( \{\phi_i\}_{i=1}^{N} \).

**Example of weights** Let \( a \) be a number satisfying \( m - n/2 < a < 2m \). Then

\[ w(x) = |x|^a \]

is an example of weights which satisfy the conditions of Theorem 6.3.

We have a similar theorem in low dimensional cases.

**Theorem 6.4** Let \( m, n \in \mathbb{N} \), and \( n \leq 2m \). Let \( w \) be a weight in \( A_2 \cap H_{loc}^m(\mathbb{R}^n) \) such that \( w^{-n/(2m)} \in A_{1+n/(2m)} \) and

\[ \int_{Q'} w \, dx \leq 2^{2m} \int_{Q} w \, dx \]

for all dyadic cubes \( Q, Q' \) such that \( Q' \) is the parent of \( Q \). Suppose that \( \{\phi_i\}_{i=1}^{N} \) is an orthonormal family in \( L^2(\mathbb{R}^n) \) such that

\[ \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) \, dx < \infty. \]

Then we have

\[ \int_{\mathbb{R}^n} \rho(x)^{1+2m/n} w(x) \, dx \leq c \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) \, dx, \]

where

\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]

and \( c \) is a positive constant which does not depend on \( \{\phi_i\}_{i=1}^{N} \).

The proofs of these theorems will appear elsewhere.
参考文献


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