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Spacelike stationary surfaces in semi-Riemannian space forms

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Let $N^n_p(c)$ denote the n-dimensional simply connected semi-Riemannian space form of constant curvature $c$ and index $p$, where we write $N^n(c)$ if $p = 0$. We say that a spacelike surface in $N^n_p(c)$ is stationary if its mean curvature vector vanishes identically. We are interested in comparing the geometries of spacelike stationary surfaces in $N^n_p(c)$ of various index $p$.

We discuss necessary and sufficient conditions for the existence of spacelike stationary surfaces in $N^n_1(c)$ and $N^n_2(c)$, together with isometric deformations preserving normal curvature.

THEOREM 1 ([S2]). (i) Let $M$ be a spacelike stationary surface in $N^n_1(c)$. We denote by $K$, $K_\nu$ and $\Delta$ the Gaussian curvature, the normal curvature and the Laplacian of $M$, respectively. Then

\begin{align*}
(1) \quad \Delta \log \{(c-K)^2 + K_\nu^2\} &= 8K \\
& \text{at points where } (c-K)^2 + K_\nu^2 > 0,
\end{align*}

and

\begin{align*}
(2) \quad \Delta \tan^{-1} \left( \frac{K_\nu}{c-K} \right) &= -2K_\nu \\
& \text{at points where } K \neq c.
\end{align*}

(ii) Conversely, let $M$ be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(\neq c)$ and Laplacian $\Delta$. If $K_\nu$ is a function on $M$ satisfying (1) and (2), then there exists an isometric stationary immersion of $M$ into $N^n_1(c)$ with normal curvature $K_\nu$.

THEOREM 2 ([S4]). Let $f : M \to N^n_1(c)$ be an isometric stationary immersion of a 2-dimensional simply connected Riemannian manifold $M$ into $N^n_1(c)$ with nowhere vanishing normal curvature $K_\nu$. Then there exists a $2\pi$-periodic family of isometric stationary immersions $f_\theta : M \to N^n_1(c)$ with the same normal curvature $K_\nu$. Moreover, if $\tilde{f} : M \to N^n_1(c)$ is another isometric stationary immersion with the same normal curvature $K_\nu$, then
there exists $\theta \in [0, \pi]$ such that $\tilde{f}$ and $f_\theta$ coincide up to congruence.

**THEOREM 3 ([S3]).** (i) Let $M$ be a spacelike stationary surface in $N^4_2(c)$. We denote by $K$, $K_\nu$ and $\Delta$ the Gaussian curvature, the normal curvature and the Laplacian of $M$, respectively. Then

\[(3) \quad \Delta \log(K - c + K_\nu) = 2(2K + K_\nu)\]

and

\[(4) \quad \Delta \log(K - c - K_\nu) = 2(2K - K_\nu)\]

at non-isotropic points where $(K - c)^2 - K_\nu^2 > 0$.

(ii) Conversely, let $M$ be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(>c)$ and Laplacian $\Delta$. If $K_\nu$ is a function on $M$ satisfying $(K - c)^2 - K_\nu^2 > 0$ and (3), (4), then there exists an isometric stationary immersion of $M$ into $N^4_2(c)$ with normal curvature $K_\nu$.

**THEOREM 4 ([S3]).** Let $f : M \rightarrow N^4_2(c)$ be a non-isotropic isometric stationary immersion of a 2-dimensional simply connected Riemannian manifold $M$ into $N^4_2(c)$ with normal curvature $K_\nu$. Then there exists a $2\pi$-periodic family of isometric stationary immersions $f_\theta : M \rightarrow N^4_2(c)$ with the same normal curvature $K_\nu$. Moreover, if $\tilde{f} : M \rightarrow N^4_2(c)$ is another isometric stationary immersion with the same normal curvature $K_\nu$, then there exists $\theta \in [0, \pi]$ such that $\tilde{f}$ and $f_\theta$ coincide up to congruence.

**THEOREM 5 ([S3]).** (i) Let $M$ be an isotropic spacelike stationary surface in $N^4_2(c)$ with Gaussian curvature $K$ and Laplacian $\Delta$. Then

\[(5) \quad \Delta \log(K - c) = 2(3K - c)\]

at points where $K > c$.

(ii) Conversely, let $M$ be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(>c)$ and Laplacian $\Delta$. If $M$ satisfies (5), then there exists an isotropic isometric stationary immersion $f$ of $M$ into $N^4_2(c)$. Moreover, if $\tilde{f} : M \rightarrow N^4_2(c)$ is another isotropic isometric stationary immersion, then $\tilde{f}$ and $f$ coincide up to congruence.

**REMARK.** For these theorems, see [GT] for the case of minimal surfaces in $N^4(c)$. 
We discuss spacelike stationary surfaces in $N_2^4(c)$ with constant Gaussian curvature, or constant normal curvature. We also give a rigidity type theorem.

**Theorem 6 ([S3]).** Let $M$ be a spacelike stationary surface with constant Gaussian curvature $K$ in $N_2^4(c)$. Then either (i) $K = c$ and $M$ is totally geodesic, (ii) $c < 0$, $K = c/3$ and $M$ is isotropic, or (iii) $c < 0$, $K = 0$ and $M$ is congruent to a certain surface in a totally geodesic $N_1^3(c)$.

**Remark.** Theorem 6 should be compared with [K] for minimal surfaces in $N^4(c)$.

**Theorem 7([S3]).** Let $M$ be a spacelike stationary surface with constant normal curvature $K_\nu$ in $N_2^4(c)$. Then either (i) $M$ lies in a totally geodesic $N_2^3(c)$, or (ii) $c < 0$ and $M$ has constant Gaussian curvature $c/3$.

**Theorem 8([S3]).** Let $M$ be a spacelike stationary surface in $N_2^4(c)$. If $M$ is locally isometric to a spacelike stationary surface in $N_1^3(c)$, then $M$ lies in a totally geodesic $N_1^3(c)$.

**Remark.** For Theorem 8, see [S1] for the case of minimal surfaces in $N^4(c)$.

We give two classes of 2-dimensional Riemannian manifolds which can be realized as spacelike stationary surfaces in $N_p^n(c)$.

Let $M$ be a 2-dimensional Riemannian manifold with Gaussian curvature $K$ and Laplacian $\Delta$. For each real number $c$, set

$$F_1^c = 2(K - c), \quad F_{p+1}^c = F_p^c + 2(p + 1)K - \sum_{q=1}^{p} \Delta \log(F_q^c) \quad \text{if } F_p^c > 0.$$ 

**Theorem 9([S5]).** Let $M$ be a 2-dimensional simply connected Riemannian manifold. Suppose that $F_p^c > 0$ for $p < m$, and $F_m^c = 0$ identically. Then there exists an isometric stationary immersion of $M$ into $N_{2[m/2]}^2(c)$, where $[\ ]$ denotes the Gauss symbol.

**Theorem 10([S5]).** Let $M$ be a 2-dimensional simply connected Riemannian manifold with metric $ds^2$. Suppose that $F_p^c > 0$ for $p \leq m$, and the
metric $d\tilde{s}^2 = \left(\prod_{p=1}^{m} F_p^c\right)^{1/(m+1)} ds^2$ is flat. Then there exists a $2\pi$-periodic family of isometric stationary immersions of $M$ into $N_{2m+1}^{2m}(c)$.

REMARK. The conditions of Theorems 9 and 10 may be seen as generalized Ricci conditions (cf. [L1], [J]). There are many 2-dimensional Riemannian manifolds which satisfy the conditions.

COROLLARY ([S5]). For every positive integer $m$, there exists an isometric stationary immersion of the hyperbolic plane of constant curvature $-2/m(m + 1)$ into $N_{2m}^{2m}(1)$.

REMARK. (i) For every positive integer $m$, there exists an isometric minimal immersion of the 2-sphere of constant curvature $2/m(m + 1)$ into the $2m$-dimensional unit sphere (cf. [C]).

(ii) The author does not know the explicit representations of the surfaces in the Corollary.

(iii) There exist many explicit flat spacelike stationary surfaces in pseudo-hyperbolic spaces (cf. [S5]).

REFERENCES


