Title
Asymptotic Shape of a Solution for the Plasma Problem in Higher Dimension (Variational Problems and Related Topics)

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Asymptotic Shape of a Solution
for the Plasma Problem in Higher Dimension

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1 Introduction and Main Theorem

In this paper, we consider a simple model of a confined plasma which is described by

\[
\begin{align*}
\Delta u - \lambda u_+ &= 0 \quad \text{in } \Omega, \\
u &= u(\Gamma) \quad \text{on } \Gamma, \\
\int_{\Gamma} \frac{\partial u}{\partial \nu} \, dS(x) &= I
\end{align*}
\]

(1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n (n \geq 3)\) with \(C^2\) boundary \(\Gamma\), \(u_+ = \max\{u, 0\}\), \(u = u_+ - u_-\), \(u(\Gamma)\) is an unknown constant, \(\lambda\) and \(I\) are given positive parameters. In this paper, we denote by \(\lambda_i\) the \(i\)th eigenvalue of \(-\Delta\) with Dirichlet zero boundary condition on \(\Omega\). For physical background of this problem, see [10], [11].

Many authors treat this problem (cf. [2] [3], [7], [8], [11], [12]). In the case \(n \geq 2\), Temam [11, 12] showed that there exists a solution \(u\) of (1) if and only if \(\lambda > 0\) and it holds that

\[ u(\Gamma) > 0 \text{ if } \lambda > \lambda_1, \quad u(\Gamma) = 0 \text{ if } \lambda = \lambda_1, \quad u(\Gamma) < 0 \text{ if } \lambda < \lambda_1, \]

furthermore, if \(0 < \lambda < \lambda_2\) then (1) has unique solution.

If \(\lambda > \lambda_1\), we can easily to obtain that \(\{x \in \Omega; u(x) < 0\}\) is nonempty by using (1) and the maximum principle. In this case, the set

\[ \Omega_p = \{x \in \Omega; u(x) < 0\} \]

is called the plasma set, and \(\Gamma_p = \partial \Omega_p\) is called the free boundary. In [7, 8], they proved \(\Gamma_p\) is a simple closed analytic curve.

We consider this problem by using variational method. Put

\[ W := \{u \in H^1(\Omega); u \equiv \text{constant on } \Gamma\}, \quad X := \{u \in W; \int_{\Omega} u_- = \frac{I}{\lambda}\} \]

and we define energy functional \(E_\lambda\) on \(W\) by

\[ E_\lambda[u] := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} u_-^2 \, dx - Iu(\Gamma). \]
Temam [12] showed that there is a global minimizer \( u_\lambda \) and \( u_\lambda \) is a weak solution of (1) i.e.

\[
E_\lambda[u_\lambda] = \min_{u \in X} E_\lambda[u],
\]

\[
\int_\Omega \nabla u_\lambda \nabla v + (u_\lambda)_- v \, dx = Iv(\Gamma)
\]

for all \( v \in W \). Hereafter, we denote by \( u_\lambda \) obtained in [12]. In [3], Caffarelli and Friedman consider the shape, size and location of \( \Omega_p \) where \( \lambda \) increases to infinity in the case \( n = 2 \). They proved that

\[
\text{diameter}(\Omega_p) < C\lambda^{-\frac{1}{2}}, \quad |\Omega_p| \geq C\lambda^{-1}
\]

for some \( C > 0 \). Furthermore,

\[
\max_{x \in \Gamma_p} |\lambda^{\frac{1}{2}}|x - x_\lambda| - R| \to 0 \quad \text{if } \lambda \to \infty.
\]

for suitable point \( x_\lambda \) and some \( R \). It means the shape of \( \Gamma_p \) is approximated by a circle with center \( x_\lambda \) and radius \( R\lambda^{-1/2} \). About the location of \( \Gamma_p \), they showed that \( x_\lambda \) converges to a point which is called the harmonic center determined by the geometry of \( \Omega \). Moreover, they concerned the case \( n = 3 \) but they proved only

\[
|\Omega_p| < C\lambda^{-\frac{3}{2}}.
\]

In this paper, we consider the case \( n \geq 3 \) and prove Caffarelli and Friedman's result is valid if \( n \geq 3 \). To prove our result, we need to approximate \( u_\lambda \) as \( \lambda \to \infty \). For it, the following limiting problem is very important.

\[
\begin{cases}
\Delta w_0 + (w_0 - 1)_+ = 0, & w_0 > 0 \quad \text{in } \mathbb{R}^n, \\
\nabla w_0(0) = 0, & \lim_{|y| \to \infty} w_0(y) = 0.
\end{cases}
\]

This equation has a unique solution \( w_0 \) (see Lemma 2.1). Now we state Theorem A.

**Theorem A.** Suppose \( u_\lambda \) a solution of (1) obtained in Temam [12] then

(i) There exists a constant \( \lambda_0 > 0 \) such that \( u_\lambda \) has only one local maximal point \( x_\lambda \) in \( \Omega \) if \( \lambda \) is sufficiently large.

(ii) \( u_\lambda \) is approximated by \( w_0 \) in the following sense:

\[
w_\lambda(y) = \frac{u_\lambda(\Gamma) - u_\lambda(x)}{u_\lambda(\Gamma)} \to w_0(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \text{ as } \lambda \to \infty
\]

where \( y = \lambda^{\frac{1}{2}}(x - x_\lambda) \).
(iii) $\max_{z \in \Omega} |1^{\mathrm{A}}| x - x_\lambda - x_\lambda^{1/2} | \rightarrow 0$ as $\lambda \rightarrow \infty$. Furthermore the free-boundary $\partial \Omega_\lambda$ is of class $C^2$ and the plasma $\lambda_\lambda$ is strictly convex.

In Theorem A, one find the plasma set $\Gamma_\lambda$ is approximately a ball with center $x_\lambda$ and radius $\lambda_1^{1/2} \lambda^{-1/2}$. Next, we state Theorem B about the location of $x_\lambda$. To state Theorem B, the geometry of $\Omega_\lambda$, namely the Robin function for $\Omega$, plays an important role. The Robin function is defined by

$$t(x) := H_x(x),$$

where $H_x(y)$ is a solution of

$$\begin{cases}
\Delta_y H_x(y) = 0 & \text{in } \Omega, \\
H_x(y) = (n - 2)^{-1} |\partial B_1|^{-1} |x - y|^{2-n} & \text{on } \partial \Omega.
\end{cases}$$

Here $B_1$ is a ball with radius 1. It is well-known that the Robin function $t(x)$ is a positive continuous function with $t(x) \rightarrow \infty$ as $x \rightarrow \partial \Omega$. A minimal point of $t(x)$ is called a harmonic center. So there exists at least one harmonic center for any bounded domain $\Omega$. For the details of the harmonic center, see e.g. [1]. We denote by $\Omega_h$ the set of all harmonic center i.e.

$$\Omega_h = \{ x \in \Omega; x \text{ is a harmonic center} \}.$$

Now we state Theorem B.

**Theorem B.** In addition to Theorem A, the following properties holds:

(i) $\lim_{\lambda \rightarrow \infty} \text{dist}(x_\lambda, \Omega_h) = 0$.

(ii) The energy $E_{\lambda}[u_\lambda]$ has the following asymptotic formula:

$$E_{\lambda}[u_\lambda] = \frac{I^2 \lambda^{\frac{n-2}{2}}}{k_0} \left\{ -1 + k_0 \lambda^{-\frac{n-2}{2}} \min_{x \in \Omega} t(x) + o(\lambda^{-\frac{n-2}{2}}) \right\}$$

where $k_0$ is a positive constant defined by $k_0 = (n - 2)|\partial B_1| \lambda^{\frac{n-2}{12}}$.

In Section 2, we define $w_{\lambda,z}$ for approximate the solution and we note the properties of $w_0$ and $w_{\lambda,z}$. In Section 3 and Section 4, we give the proof of Theorem A and B. In Section 5, we give the proof of Lemma 4.2 which is used in Section 4 for the proof of Theorem B.
2 Preliminaries

In this section, we define $w_{\lambda,z}$ and note the properties of $w_{0}, w_{\lambda,z}$. $w_{0}, w_{\lambda,z}$ will be used in Section 3 and Section 4 for approximation of the solution.

Lemma 2.1. There is a unique solution in $C^{2}(\mathbb{R}^{n})$ for

\begin{equation}
\begin{cases}
\Delta w_{0}(y) + (w_{0}(y) - 1)^{+} = 0, & u(y) > 0 \quad \text{in } \mathbb{R}^{n}, \\
\nabla w_{0}(0) = 0, & \lim_{|y| \to \infty} w_{0}(y) = 0.
\end{cases}
\end{equation}

Moreover, $w_{0}$ has the following formula.

\begin{equation}
w_{0}(y) = \begin{cases}
\lambda^{\frac{n-2}{12}} |y|^{2-n} & \text{if } |x| > \lambda^{\frac{1}{12}}, \\
\phi_{1}(\lambda_{1}^{- \frac{1}{2}} y) + 1 & \text{if } |x| \leq \lambda^{\frac{1}{12}}.
\end{cases}
\end{equation}

Here $\phi_{1}$ is a first eigenfunction of $-\Delta$ on $B_{1}$ which satisfies $|\nabla \phi_{1}| = n - 2$ on $\partial B_{1}$.

Proof. First, we show uniqueness of the solution. If $w_{0} \in C^{2}(\mathbb{R}^{n})$ is a solution, by [9, Theorem 2], we obtain $u(y) = u(r)$ for $r = |y|$ and $u'(r) < 0$ if $r > 0$. So there is a unique positive constant $R$ with $u(R) = 1$. Since $u(r) < 1$ if $r > R$, we have $-\Delta u = 0$ in $\mathbb{R}^{n} \setminus \overline{B_{R}}$. It follows from (4) that $u(x) = c|x|^{2-n}$ on $\mathbb{R}^{n} \setminus \overline{B_{R}}$ for some positive constant $c$. Since $u(R) = 1$, we have $c = R^{n-2}$. We define $v$ by $v(x) = w_{0}(y) - 1$ for $y = Rx$. Then we have

$$\Delta v(x) = \Delta_{x} w_{0}(Rx) = R^{2} \Delta w_{0}(Rx) = -R^{2}(w_{0} - 1) = -R^{2} v$$

if $x \in B_{1}$ and $v = 0$ if $x \in \partial B_{1}$. It means $v$ is first eigenfunction of $-\Delta$ on $B_{1}$ with Dirichlet zero boundary condition and $R^{2}$ is its first eigenvalue. Hence, $R = \lambda_{1}^{\frac{1}{2}}$. Since $w_{0}'$ is continuous, we have

$$\frac{2-n}{R} = w_{0}'(R) = \frac{v'(1)}{R}.$$ 

Such $v$ is unique and we get $v \equiv \phi_{1}$. Consequently, $w_{0}$ is a unique solution.

On the other hand, $w_{0}$ defined by (5) is a $C^{2}$ solution of (4). It completes the proof of this lemma. \hfill \square

It follows from 2.1 that the following corollary.

Corollary 2.2.

$$\int_{\mathbb{R}^{n}} (w_{0} - 1)^{+} \, dy = k_{0} = (n - 2) |\partial B_{1}| \lambda_{1}^{\frac{n-2}{2}}.$$
For \( \lambda > 0 \), \( z \in \Omega \), we denote by \( w_{\lambda,z} \) the unique solution of

\[
\begin{align*}
\Delta w_{\lambda,z} + (w_0 - 1)_+ &= 0 \quad \text{in } \Omega_{\lambda,z}, \\
w_{\lambda,z} &= 0 \quad \text{on } \Omega_{\lambda,z}
\end{align*}
\]

where \( \Omega_{\lambda,z} = \lambda^{\frac{1}{2}}(\Omega - z) \), and we define \( h_z \) by \( h_z(y) = H_z(\lambda^{-\frac{1}{2}}y + z) \).

**Lemma 2.3.** For \( w_{\lambda,z} \), the following properties hold:

(i) \( w_0 > w_{\lambda,z} \).

(ii) \( w_0(y) = w_{\lambda,z}(y) + k_0 \lambda^{-\frac{n-2}{2}} h_z(y) \) if \( B_{\lambda^{1/2}} \subset \Omega_{\lambda,z} \).

(iii) \( h_z(y) \to t(z) \) in \( L_{\text{loc}}^{\infty}(\mathbb{R}^n) \) as \( \lambda \to \infty \).

(iv) \( w_{\lambda,z}(y) = w_0(y) - k_0 \lambda^{-\frac{n-2}{2}} (t(z) + o(1)) \) as \( \lambda \to \infty \) in \( L_{\text{loc}}^{\infty}(\mathbb{R}^n) \).

Remark that Lemma 2.3 (iii) may be not valid if \( z \in \Omega \) is depend on \( \lambda \) since \( t(x) \not\in C(\overline{\Omega}) \).

**Proof of Lemma 2.3.** By the equation and Lemma 2.1, \( w(y) := w_0(y) - w_{\lambda,z}(y) \) satisfies

\[
\begin{align*}
\Delta w(y) &= 0 \quad \text{in } \Omega_{\lambda,z}, \\
w(y) &= w_0(y) = \lambda^{\frac{n-2}{12}} |y|^{2-n} \quad \text{on } \partial \Omega_{\lambda,z}
\end{align*}
\]

if \( |y| \geq \lambda^{\frac{1}{12}} \). By the definition of \( h_z \), we find

\[
\begin{align*}
\Delta h_z(y) &= 0 \quad \text{in } \Omega_{\lambda,z}, \\
h_z(y) &= (n-2)^{-1}|B_1|^{-1} |y|^{2-n} \lambda^{\frac{n-2}{2}} \quad \text{on } \partial \Omega_{\lambda,z}.
\end{align*}
\]

Consequently, (ii) holds. It follows from (ii) and \( h_z > 0 \) that (i) holds. (iii) is clear because of \( H_z \) is continuous. (ii),(iii) mean (iv).

\( \square \)

3 Proof of Theorem A

**Proposition 3.1.** Let \( u_\lambda \) be a global minimizer, then the following asymptotic formula holds as \( \lambda \to \infty \).

\[
E_\lambda[u_\lambda] \leq \frac{I^2 \lambda^{\frac{n-2}{2}}}{2k_0} \left\{ -1 + k_0 \lambda^{-\frac{n-2}{2}} \min_{x \in \Omega} t(x) + o(\lambda^{-\frac{n-2}{2}}) \right\}
\]

Here \( k_0 \) is a positive constant defined by \( k_0 = (n-2)|\partial B_1| \lambda^{\frac{n-2}{6}} \).
Remark. To prove Theorem A, the second order term is not necessary.

Proof. Take \( z \in \Omega \) with \( t(z) = \min_{x \in \Omega} t(x) \). Then there is a large constant \( \beta \) such that \( B_{\lambda_1^{1/2}} \subset \Omega_{\lambda, z} \) if \( \lambda > \beta \). We define \( v \) by \( v(x) = c(1 - w_{\lambda, z}(y)) \) where \( y = \lambda^{1/2}(x - z) \). Here, we choose \( c \) which satisfies \( \int_{\Omega} v_- dx = \frac{I}{\lambda} \). Then we have

\[
I \lambda^{\frac{n-2}{2}} = c \int_{\Omega_{\lambda, z}} (w_0 - 1 - k_0 \lambda^{-\frac{n-2}{2}} h_z)_+ dy
\]

\[
= c \int_{\{w_0(y) > 1\}} (w_0 - 1 - k_0 \lambda^{-\frac{n-2}{2}} t(z)) dy + o(\lambda^{-\frac{n-2}{2}})
\]

\[
= c(k_0 - k_0 \lambda^{-\frac{n-2}{2}} |B_{\lambda_1^{1/2}}| t(z) + o(\lambda^{-\frac{n-2}{2}}))
\]

because of Corollary 2.2. So we obtain

\[
c = \frac{I \lambda^{\frac{n-2}{2}}}{k_0} \left( 1 + \lambda^{-\frac{n-2}{2}} |B_{\lambda_1^{1/2}}| t(z) + o(\lambda^{-\frac{n-2}{2}}) \right).
\] (6)

Using \( \Delta v(x) = -\lambda c \Delta w_{\lambda, z}(y) = \lambda c(w_0(y) - 1)_+ \), we obtain

\[
\int_{\Omega} |\nabla v|^2 dx = \int_{\Omega} \nabla v \nabla v dx - \int_{\Omega} v \Delta v dx
\]

\[
= \int_{\Omega} \nabla v \nabla S(x) - \int_{\Omega} v \Delta v dx = \int_{\Omega} \Delta v dx - \int_{\Omega} v \Delta v dx
\]

\[
= \int_{\Omega} (c - v) \Delta v dx = c^2 \lambda^{-\frac{n-2}{2}} \int_{\Omega} (w_0 - 1)_+ w_{\lambda, z} dy,
\]

\[
\lambda \int_{\Omega} v^2 dx = c^2 \lambda^{-\frac{n-2}{2}} \int_{\Omega} (w_{\lambda, z} - 1)^2 dy
\]

\[
= \lambda \int_{\Omega} \nabla w_{\lambda, z}^2 dx - \int_{\Omega} v \Delta v dx - cI.
\]

So we have

\[
E_\lambda[v] = \frac{c^2 \lambda^{-\frac{n-2}{2}}}{2} \left\{ \int_{\Omega} (w_0 - 1)_+ w_{\lambda, z} - (w_{\lambda, z} - 1)_+ w_{\lambda, z} \right\} dy - \frac{Ic}{2}.
\]

Noting \( w_{\lambda, z} < w_0 \) and

\[
|\int_{\{w_{\lambda, z} < 1 < w_0\}} w_{\lambda, z} - 1| \leq \int_{\{w_{\lambda, z} < 1 < w_0\}} |w_{\lambda, z} - w_0| dy = o(\lambda^{-\frac{n-2}{2}}),
\]

we obtain

\[
E_\lambda[v] = \frac{c^2 \lambda^{-\frac{n-2}{2}}}{2} \left\{ \int_{\{w_0 > 1\}} (w_0 - 1 - w_{\lambda, z} + 1) w_{\lambda, z} dy + o(\lambda^{-\frac{n-2}{2}}) \right\} - \frac{Ic}{2}
\]

\[
= \frac{c^2 \lambda^{-\frac{n-2}{2}}}{2} \left\{ \int_{\{w_0 > 1\}} \lambda^{-\frac{n-2}{2}} k_0 t(z) w_0 dy + o(\lambda^{-\frac{n-2}{2}}) \right\} - \frac{Ic}{2}.
\]
Using (6) and \( c^2 = I^2\lambda^{n-2}k_0^{-2}(1 + o(1)) \),

\[
E_{\lambda}[v] = \frac{I^2}{2k_0} \left\{ \int_{\{w_0 > 1\}} t(z) w_0 \, dy + o(1) \right\} - \frac{Ic}{2}
\]

\[
= \frac{I^2}{2k_0} \left\{ \int_{\{w_0 > 1\}} t(z)(w_0 - 1 + 1) \, dy + o(1) \right\} - \frac{I}{2k_0} \left( \lambda^{\frac{n-2}{2}} + |B_{\lambda_{1}^{1/2}}|t(z) + o(1) \right)
\]

\[
= \frac{I^2}{2k_0} \left\{ k_0 t(z) + t(z)|B_{\lambda_{1}^{1/2}}| - \lambda^{\frac{n-2}{2}} - |B_{\lambda_{1}^{1/2}}|t(z) + o(1) \right\}
\]

\[
= \frac{I^2\lambda^{\frac{n-2}{2}}}{2k_0} \left\{ -1 + k_0\lambda^{-\frac{n-2}{2}}t(z) + o(\lambda^{-\frac{n-2}{2}}) \right\}
\]

Hereafter, we denote by \( x_{\lambda} \) a local minimal point of \( u_{\lambda} \) in \( \Omega \) for each \( \lambda > 0 \) and define \( w_{\lambda} \) and \( \Omega_{\lambda} \) by \( \Omega_{\lambda} = \lambda^{\frac{1}{2}}(\Omega - x_{\lambda}) \), \( w_{\lambda}(y) = (u_{\lambda}(\Gamma) - u_{\lambda}(x))/u_{\lambda}(\Gamma) \) where \( y = \lambda^{\frac{1}{2}}(x - x_{\lambda}) \). Then \( w_{\lambda} \) is a solution of

\[
\begin{cases}
\Delta w_{\lambda} + (w_{\lambda} - 1)^+ = 0, & \text{in } \Omega_{\lambda}, \\
w_{\lambda} = 0 & \text{on } \Omega_{\lambda}.
\end{cases}
\]  

(7)

Using the maximum principle, we find \( w_{\lambda}(y) > 1 \) if \( y \) is a local maximal point of \( w_{\lambda} \).

**Lemma 3.2.** Suppose \( \lambda > \lambda_1 \). Then \( ||w_{\lambda}||_{C^{1, \alpha}(\Omega_{\lambda})} \) and \( ||w_{\lambda}||_{W^{2,p}(\Omega_{\lambda})} \) is uniformly bounded with respect to \( \lambda \) where \( \alpha > 0 \) and \( 2 < p < n \) is some constant. Moreover, \( w_{\lambda} \) is a classical solution.

**Proof.** By (3) we have

\[
-\frac{Iu_{\lambda}(\Gamma)}{2} = E_{\lambda}[u_{\lambda}].
\]  

(8)

Using Proposition 3.1, we obtain

\[
u_{\lambda}(\Gamma) \geq \frac{I}{2k_0}\lambda^{-\frac{n-2}{2}}(1 + o(1)).
\]  

(9)

as \( \lambda \to \infty \). First, we show the following claim.

**Claim.**

\[
\int_{\Omega_{\lambda}} (w_{\lambda} - 1)^2 \, dy \leq C,
\]  

(10)

\[
\int_{\Omega_{\lambda}} |\nabla w_{\lambda}|^2 \, dy \leq C
\]  

(11)

where \( C \) is a positive constant independent of \( \lambda \).
Suppose $\lambda > \lambda_1$ and define $v_\lambda$ by $v_\lambda(x) = (u_\lambda(\Gamma) - u_\lambda(x))/u_\lambda(\Gamma)$. Noting that $u_\lambda(\Gamma) > 0$ and $v_\lambda \in W^{1,2}_0(\Omega)$, it follows from interpolation inequality, Sobolev's inequality and $u_\lambda \in X$ that

$$
\|(v_\lambda - 1)_+\|_{L^2(\Omega)} \leq \|(v_\lambda - 1)_+\|_{L^1(\Omega)}\|(v_\lambda - 1)_+\|_{L^{2^*}(\Omega)}
$$

$$
\leq C \left( \frac{I}{\lambda u_\lambda(\Gamma)} \right)^{\theta} \left( \int_{\{u_\lambda > 1\}} |\nabla v_\lambda|^2 \, dx \right)^{\frac{1-\theta}{2}}
$$

where $\theta = 2/(n+2)$, $2^* = 2n/(n-2)$ and $C$ is a positive constant depend on $n$. By $E'[u_\lambda][u_\lambda] = 0$, we have

$$
\int_{\{u_\lambda < 0\}} |\nabla u_\lambda|^2 \, dx = \int_{\Omega} (u_\lambda)_-^2 \, dx, \quad \int_{\{v_\lambda < 1\}} |\nabla v_\lambda|^2 \, dx = \int_{\Omega} (v_\lambda - 1)_+^2 \, dx.
$$

So we obtain

$$
\|(v_\lambda - 1)_+\|_{L^2(\Omega)} \leq \left( \frac{I}{\lambda u_\lambda(\Gamma)} \right)^{\theta} \|\lambda(v_\lambda - 1)_+\|_{L^2(\Omega)}^{1-\theta}.
$$

It follows from this inequality and (9) that

$$
\left( \int_{\Omega} (v_\lambda - 1)_+^2 \right)^{\frac{\theta}{2}} \leq C \lambda^{-\frac{n\theta}{2}} \lambda^{\frac{1-\theta}{2}} = C \lambda^{-\frac{n}{2(n+2)}}.
$$

where $C$ is a positive constant depend on $I, n$. Consequently,

$$
\int_{\Omega} (v_\lambda - 1)_+^2 \, dx \leq C \lambda^{-\frac{n}{2}}
$$

holds and it means (10). By (8), (9) and (10), we have

$$
\int_{\Omega} |\nabla v_\lambda|^2 \, dx = \frac{I}{u_\lambda(\Gamma)} + \int_{\Omega} \lambda(v_\lambda - 1)_+^2 \, dx \leq C \lambda^{-\frac{n-2}{2}}.
$$

It means (11) and this claim is valid.

Secondly, we show the following claim.

**Claim.** For $1 < p < n$, $p^* = np/(n-p)$, there is a positive constant $C$ independent of $\lambda$ such that

$$
\|\nabla w_\lambda\|_{L^{p^*}(\Omega_\lambda)} \leq C \|\Delta w_\lambda\|_{L^p(\Omega_\lambda)}, \quad (12)
$$

$$
\|(w_\lambda - 1)_+\|_{L^{p^*}(\Omega_\lambda)} \leq C \|\nabla w_\lambda\|_{L^p(\Omega_\lambda)}, \quad (13)
$$

$$
\|D^2 w_\lambda\|_{L^p(\Omega_\lambda)} \leq C \|\Delta w_\lambda\|_{L^p(\Omega_\lambda)}. \quad (14)
$$
By the $L^p$ regularity theorem, we have

$$||D^2 v_\lambda||_{L^p(\Omega)} \leq ||v_\lambda||_{W^{2,p}(\Omega)} \leq C||\Delta v_\lambda||_{L^p(\Omega)}$$

where $C$ is a positive constant independent of $\lambda$. It asserts (14) immediately. Let $B$ be a ball with $\Omega \subset\subset B$. By the extension theorem (cf. [6, Theorem 7.25]), there is a bounded linear operator $E$ from $W^{2,p^*}(\Omega)$ to $W^{2,p^*}_0(B)$ such that $Eu = u$ on $\Omega$. This and Sobolev's inequality assert

$$||(v_\lambda - 1)_+||_{L^{p^*}(\Omega)} \leq ||v_\lambda||_{L^{p^*}(\Omega)} \leq C||\nabla v_\lambda||_{L^{p^*}(\Omega)}$$

where $C$ is a positive constant independent of $\lambda$. So we have

$$||\nabla v_\lambda||_{L^{p^*}(\Omega)} \leq C||\Delta v_\lambda||_{L^p(\Omega)}.$$
for $q > n$. By using (14), we have

$$\|w_\lambda\|_{W^{2,q}(\Omega_\lambda)} \leq C$$

for some $q > n$. The definition of $\Omega_\lambda$ and the assumption of $\partial\Omega$ assert that there exists a constant $r > 0$ such that for any $x \in \Omega_\lambda$, there is a ball $B$ with radius $r$ satisfying $x \in B \subseteq \Omega$. By Morrey's inequality, the extension theorem, we have $w_\lambda \in C^{2,\alpha}(\Omega_\lambda)$ and

$$\|w_\lambda\|_{C^{1,\alpha}(B)} \leq C\|w_\lambda\|_{W^{2,q}(B)} \leq C\|w_\lambda\|_{W^{2,q}(\Omega_\lambda)}.$$

where $\alpha$ is a constant in $(0, 1)$ and $C$ is a constant independent of $\lambda$, $x$. Consequently, $\|w_\lambda\|_{C^{1,\alpha}(\Omega_\lambda)}$ is a uniformly bounded. Moreover, Schauder’s regularity theorem asserts $w_\lambda \in C^{2,\alpha}(\Omega_\lambda)$ and $w_\lambda$ is a classical solution. $\square$

**Lemma 3.3.**

$$\text{dist}\lambda^{\frac{1}{2}}(x_\lambda, \partial\Omega) = \infty$$

holds. Especially, it holds that $\lim_{\lambda \to \infty} \Omega_\lambda = \mathbb{R}^n$ as $\lambda \to \infty$.

**Proof.** If not, there exists a subsequence $\{\lambda_j\}_{j=1}^\infty$ and a positive constant $C$ such that $\text{dist}(x_{\lambda_j}, \partial\Omega)\lambda_j^{1/2} \leq C$. By passing to a subsequence if necessary, we may assume there exists $\delta \in [0, \infty)$ such that

$$\lim_{j \to \infty} \lambda_j^{1/2}\text{dist}(x_{\lambda_j}, \partial\Omega) = \delta.$$

If $\delta = 0$, take $\hat{x}_\lambda \in \partial\Omega$ with $\text{dist}(x_{\lambda_j}, \partial\Omega) = \text{dist}(x_{\lambda_j}, \hat{x}_\lambda)$. Put $\hat{y}_\lambda := (\hat{x}_\lambda - x_{\lambda_j})\lambda_j^{1/2}$. By $O\hat{y}_\lambda \subset \overline{\Omega}_{\lambda_j}$ and the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$\hat{y}_\lambda \cdot \nabla w_\lambda(\theta \hat{y}_\lambda) = w_\lambda(\hat{y}_\lambda) - w_\lambda(O) = -w_\lambda(O)$$

We can apply Lemma 3.2 to obtain

$$1 \leq |w_\lambda(y_\lambda)| \leq |\hat{y}_\lambda| |\nabla w_\lambda(\theta \hat{y}_\lambda)| \leq \lambda_j^{1/2}\text{dist}(x_{\lambda_j}, \partial\Omega) |\nabla w_\lambda(\theta \hat{y}_\lambda)| \leq C\lambda_j^{1/2}\text{dist}(x_{\lambda_j}, \partial\Omega),$$

for some constant $C$. This is a contradiction.

If $\delta \neq 0$, by using a rotation and a translation of coordinates, we can assume $x_{\lambda_j} = O$ and $\lim_{j \to \infty} \Omega_{\lambda_j} = \mathbb{R}^n_{\delta+} := \{x \in \mathbb{R}^n; x_n > -\delta\}$ because of smoothness of $\partial\Omega$. By Lemma 3.2 and $C^{1,\alpha'}(B)$ is compactly imbeded to $C^{1,\alpha}(B)$ if $0 < \alpha' < \alpha$ for any ball $B$, by passing to a subsequence if necessary, there is a $w \in C^{1,\alpha'}(\mathbb{R}^n_{\delta+})$ such that

$$w_{\lambda_j} \to w \quad \text{in} \quad C^{1,\alpha'}_{\text{loc}}(\mathbb{R}^n_{\delta+}).$$
Moreover, we can apply the interior Schauder estimate to obtain
\[ w_{\lambda_{j}} \to w \quad \text{in} \quad C^{2,\alpha}_{\text{loc}}(\mathbb{R}^{n}_{\delta+}) \]
and \( w \in C^{2,\lambda}(\mathbb{R}^{n}) \) by passing to a subsequence if necessary. By equation, we have \( \Delta w + (w - 1)_+ = 0 \) in \( \mathbb{R}^{n}_{\delta+} \), \( w(0) \geq 1 \) and \( \nabla w(0) = 0 \). Denote by \( \tilde{w}_{\lambda_{j}} \) the extension of \( w_{\lambda_{j}} \) then we can easily to see \( \|\tilde{w}_{\lambda_{j}}\|_{C^{0,1}(\mathbb{R}^{n})} = \|w_{\lambda_{j}}\|_{C^{0,1}(\Omega_{\lambda_{j}})} \) and \( \tilde{w}_{\lambda_{j}} \to \tilde{w} \) in \( L^{\infty}_{\text{loc}}(\mathbb{R}^{n}) \). It mean \( w = 0 \) on \( \partial \mathbb{R}^{n}_{\delta+} \). Consequently, \( w \) satisfies
\[
\begin{cases}
\Delta w + (w - 1)_+ = 0 & \text{in} \ \mathbb{R}^{n}_{\delta+}, \\
 w = 0 & \text{on} \ \mathbb{R}^{n}_{\delta+}.
\end{cases}
\]

By using global Schauder estimate, we can find \( w \in C^{2}(\mathbb{R}^{n}_{\delta+}) \). The definition of \( w \) and the uniform estimate for \( w_{\lambda_{j}} \) assert
\[
\int_{\mathbb{R}^{n}_{\delta+}} |\nabla w_{0}|^{2} \, dy < \infty, \quad \int_{\mathbb{R}^{n}_{\delta+}} (w_{0} - 1)_+ \, dy < \infty.
\]

By Esteban-Lions's result [4], \( w \) must be the trival solution i.e. \( w_{0} = 0 \) in \( \mathbb{R}^{n}_{\delta+} \).
It contradicts to \( w(O) \geq 1 \).

Based on Lemma 3.3, we can approximate the solution \( u_{\lambda} \) by using the ground state when \( \lambda \) is sufficiently large.

**Lemma 3.4.** Let \( x_{\lambda} \) be a local minimal point of \( u_{\lambda} \). Then
\[ w_{\lambda} \to w_{0} \quad \text{in} \quad C^{2}_{\text{loc}}(\mathbb{R}^{n}) \]
holds as \( \lambda \to \infty \).

**Proof.** By Lemma 3.3, \( \lim_{j \to \infty} \Omega_{\lambda_{j}} = \mathbb{R}^{n} \). Using similar argument in Lemma 3.3, by passing to a subsequence if necessary, there exists \( w \in C^{2}(\Omega) \) such that
\[ \lim_{j \to \infty} w_{\lambda_{j}} = w \quad \text{in} \quad C^{2}_{\text{loc}}(\mathbb{R}^{n}). \] (15)

Here, \( w \) is a solution of
\[
\begin{cases}
\Delta w + (w - 1)_+ = 0 & \text{in} \ \mathbb{R}^{n}, \\
 \nabla w(0) = 0
\end{cases}
\]
and \( \|w\|_{C^{0,1}(\mathbb{R}^{n})} < \infty, \|w\|_{W^{1,p}(\mathbb{R}^{n})} < \infty \). Obviously, it mean \( \lim_{|y| \to \infty} w(y) = 0 \).
By Lemma 2.1, such \( w \) is unique. Hence \( w \equiv w_{0} \). So we obtain
\[ w_{\lambda_{j}} \to w_{0} \quad \text{in} \quad C^{2}_{\text{loc}}(\mathbb{R}^{n}). \] (16)
Finally, we show (3.4). If not, there exists a subsequence \( \{\lambda_j\}_{j=1}^{\infty} \) of \( \lambda \to \infty \), \( \epsilon > 0 \) and \( R > 0 \) such that
\[
\|w_{\lambda_j} - w_0\|_{C^2(B_R)} > \epsilon.
\]
By the above argument asserts (16) by passing to a subsequence if necessary. It contradicts to the assumption. Hence (3.4) was proved. \( \square \)

Now, we can prove the following proposition.

**Proposition 3.5.** \( u_\lambda \) has only one local minimal point if \( \lambda \) is sufficiently large.

**Proof.** If not, then there exists a subsequence \( \{\lambda_j\}_{j=1}^{\infty} \) of \( \lambda \to \infty \) such that \( u_{\lambda_j} \) have two maximal points \( x_{\lambda_j} \) and \( \tilde{x}_{\lambda_j} \). Define \( \delta_{\lambda_j} := |x_{\lambda_j} - \tilde{x}_{\lambda_j}| \lambda_j^{1/2} \). Then, by passing to a subsequence if necessary, there exists \( \delta \in [0, \infty) \) such that \( \lim_{j \to \infty} \delta_{\lambda_j} = \delta \).

First, consider the case \( \delta \in (0, \infty) \). Define \( \tilde{y}_{\lambda_j} = (\tilde{x}_{\lambda_j} - x_{\lambda_j}) \lambda_j^{1/2} \). Then \( \nabla w_{\lambda_j}(O) = 0 \), \( \nabla w_{\lambda_j}(\tilde{y}_{\lambda_j}) = 0 \). Since \( \lim_{j \to \infty} |\tilde{y}_{\lambda_j}| = \delta \), by passing to a subsequence if necessary, we may assume \( \lim_{j \to \infty} \tilde{y}_{\lambda_j} = \tilde{y}_0 \) and \( \tilde{y}_0 = \delta \). By Lemma 3.4, we may assume \( u_{\lambda_j} \to w_0 \) in \( C^2_{\text{loc}}(\mathbb{R}^n) \). So \( \Delta w_0(\tilde{y}_0) = 0 \) and it contradicts to Lemma 2.1.

Next, consider the case \( \delta = 0 \). Let \( R_{\lambda_j} \) be the rotation of coordinates so that \( \tilde{y}_{\lambda_j} = (\tilde{y}_{\lambda_j,1}, 0, \ldots, 0) \) and we define \( w_{\lambda_j}(y) = (u_{\lambda_j}(\Gamma) - u_{\lambda_j}(x))/u_{\lambda_j}(\Gamma) \) where \( y = R_{\lambda_j}(x - x_{\lambda_j})/\lambda_j, \) and \( \Omega_{\lambda_j} = R_{\lambda_j}(\Omega - x_{\lambda_j})/\lambda_j \). In a similar way to the proof of Lemma 3.4, we have
\[
0 = \frac{\partial_1^{1} w_{\lambda_j}(O) - \partial_1^{1} w_{\lambda_j}(\tilde{y}_{\lambda_j})}{\tilde{y}_{\lambda_j,1}}.
\]
Since \( \delta = 0 \), we have \( \lim_{j \to \infty} \tilde{y}_{\lambda_j} = 0 \), and hence \( \partial_1^{2} w_0(O) = 0 \). Since \( w_0 \) is radially symmetric about the origin, it follows \( \partial_i^{2} w_0(O) = 0 \) \( (i = 1, 2, \ldots, n) \), and hence \( \Delta w_0(O) = 0 \). Since \( \Delta w_0(O) + (w_0(O) - 1)_+ = 0 \), it follows \( w_0(O) \leq 1 \) and which contradicts to Lemma 2.1.

Finally, we consider the case \( \delta = \infty \). Fix \( R > 0 \), then \( B(x_{\lambda_j}, \lambda_j^{-\frac{1}{2}}R) \cap B(\tilde{x}_{\lambda_j}, \lambda_j^{-\frac{1}{2}}R) = \emptyset \) holds for sufficiently large \( j \). We define
\[
w_{\lambda_j}(y) = (u_{\lambda_j}(\Gamma) - u_{\lambda_j}(\lambda^{-1/2}y + x_{\lambda_j}))/u_{\lambda_j}(\Gamma),
\]
\[
\tilde{w}_{\lambda_j}(y) = (u_{\lambda_j}(\Gamma) - u_{\lambda_j}(\lambda^{-1/2}y + \tilde{x}_{\lambda_j}))/u_{\lambda_j}(\Gamma).
\]
From Lemma 3.4, we have
\[
w_{\lambda_j} \to w_0 \quad \text{in} \ C^2_{\text{loc}}(\mathbb{R}^n),
\]
\[
\tilde{w}_{\lambda_j} \to w_0 \quad \text{in} \ C^2_{\text{loc}}(\mathbb{R}^n)
\]
as \( \lambda \to \infty \) where \( w_0 \) is the unique solution to \( \Delta w_0 + (w_0 - 1)_+ = 0 \) in \( \mathbb{R}^n \). On the other hand, using (8) and the definition of \( w_\lambda \), we have

\[
Iu_\lambda^{-1} = \int_{\Omega_{\lambda_j}} |\nabla w_{\lambda_j}|^2 - (w_{\lambda_j} - 1)_+^2 \, dy \lambda^{-\frac{n-2}{2}}.
\]

It follows from (7) that

\[
\int_{\Omega_{\lambda_j}} |\nabla w_{\lambda_j}|^2 \, dy = \int_{\Omega_{\lambda_j}} (w_{\lambda_j} - 1)_+ w_{\lambda_j} \, dy = \int_{\Omega_{\lambda_j}} (w_{\lambda_j} - 1)_+^2 + (w_{\lambda_j} - 1)_+ \, dy.
\]

So we have

\[
Iu_{\lambda_j}^{-1} \lambda^{-\frac{n-2}{2}} = \int_{\Omega_{\lambda_j}} (w_{\lambda_j} - 1)_+ \, dy.
\]

Noting the definition of \( \bar{w}_{\lambda_j} \), we have

\[
\int_{B_R} (w_{\lambda_j} - 1)_+ \, dy + \int_{B_R} (\bar{w}_{\lambda_j} - 1)_+ \, dy \leq Iu_{\lambda_j}^{-1} \lambda^{-\frac{n-2}{2}}.
\]

Taking \( \lambda \to \infty \) and using Proposition 3.1, we obtain

\[
2 \int_{B_R} (w_0 - 1)_+ \, dy \leq k_0.
\]

If \( R > \lambda_1^{1/2} \), Corollary 2.2 asserts that the left hand side equals to \( 2k_0 \) and it is contradiction. \( \square \)

The following proposition completes the proof of Theorem A.

**Proposition 3.6.** \( \max_{x \in \Gamma_p} |\lambda^\frac{1}{2} |x - x_{\lambda}| - \lambda^\frac{1}{2} | \to 0 \) as \( \lambda \to \infty \). Furthermore the free-boundary \( \partial \Omega_p \) is of class \( C^2 \) and the plasma \( \Omega_p \) is strictly convex if \( \lambda \) is sufficiently large.

**Proof.** \( \Omega_p \) has only one component if \( \lambda \) is sufficiently large, because each component has a maximal point and \( u_\lambda \) has only one maximal point if \( \lambda \) is large. By Lemma 2.1, \( w_0(y) \) is radially symmetric and strictly decreasing, and hence there are unique \( s \) and \( t \) such that \( s > 1 > t \) and

\[
B_r = \{ y \in \mathbb{R}^n | w_0(y) > s \} \subset B_{\lambda_1^{1/2}} \subset \{ y \in \mathbb{R}^n | w_0(y) > t \} = B_R.
\]

By Lemma 3.4, we obtain

\[
w_\lambda \to w_0 \quad \text{in} \ C^2_{\text{loc}}(\mathbb{R}^n)
\]
as $\lambda \to \infty$. Since $B_R \subset \Omega_\lambda$ if $\lambda$ is large,

$$w_\lambda \to w_0 \text{ in } C^2(B_R)$$

(17)
as $\lambda \to \infty$. So, if $\lambda$ is large, then $|w_\lambda - w_0| \leq \min \{s - 1, 1 - t\}/2$ and

$$w_\lambda > \frac{s + 1}{2} > 1 \text{ in } B_r, \quad w_\lambda < \frac{t + 1}{2} < 1 \text{ in } B^c_R.$$Since $\Omega_p$ has only one component,

$$B_r \subset \{y \in \Omega_\lambda|w_\lambda(y) > 1\} \subset B_R.$$Hence $B(x_\lambda, \lambda^{-1/2}r) \subset \Omega_p \subset B(x_\lambda, \lambda^{-1/2}R)$ holds if $\lambda$ is sufficiently large.

It means

$$\max_{x \in \Gamma_p} |\lambda^{1/2}|x - x_\lambda| - \lambda^{1/12} \to 0 \text{ as } \lambda \to \infty.$$Next, we show that $\partial \Omega_p$ is of class $C^2$ if $\lambda$ is large. Since $w_0'(s) < 0$ on $(0, \infty)$, there exists $a > 0$ such that

$$|\nabla w_0(y)| = |w_0'(|y|)| > a \text{ in } B \setminus B_r.$$As (17), $|\nabla w_0| - |\nabla w_\lambda| < a/2$ in $\overline{B_R}$ if $\lambda$ is large. So we have $|\nabla w_\lambda| > a/2$ in $\overline{B_R} \setminus B_r$. Especially $\nabla w_\lambda \neq 0$ on $\partial \Omega_p$. Since $w_\lambda$ is of class $C^2$, the implicit function theorem asserts that $\partial \Omega_p$ is of class $C^2$ if $\lambda$ is sufficiently large.

Finally, we show that $\Omega_p$ is strictly convex if $\lambda$ is sufficiently large. As above, $\Omega_p \subset B_R$ for all small $\lambda$ and

$$w_\lambda \to w_0 \text{ in } C^2(B_R)$$

(18)as $\lambda \to \infty$. On the other hand, the principal curvature of $\partial \Omega_p$ is determined by $D^2w_\lambda$. Consequently, $\Omega_p$ is strictly convex for sufficiently small $\lambda$ because of the strict positivity of $D^2w_0$. \qed

4 Proof of Theorem B

To prove Theorem B, we need precisely lower estimate for $E_\lambda[u_\lambda]$. The argument of the proof of Theorem B is dependent on Flucher and Wei [5]. To estimate $E_\lambda[u_\lambda]$, we need the following two lemmas.

Lemma 4.1.

$$\lim_{\mu \to 0} \frac{h_{z_\lambda}}{t(x_\lambda)} = 1 \text{ in } C^0(\overline{B_{2\lambda^{1/2}}}).$$In particular, $\lambda^{-n/2}h_{z_\lambda} = \lambda^{-n/2}t(x_\lambda)(1 + o(1))$ as $\lambda \to \infty$ and $h_{z_\lambda}/t(x_\lambda)$ is uniformly bounded on $B_{2\lambda^{1/2}}$ for sufficiently large $\lambda$.\[\square\]
We can obtain this Lemma by using similar argument as [1, p196]

**Lemma 4.2.** Suppose $q > n/(n - 2)$ and $R > 0$. We define the operator $L$ by

$$Lv := \Delta v + \chi_{B_{R}}v$$

for $v \in W^{2,q}(\mathbb{R}^{n}) \cap W_{0}^{1,2}(\mathbb{R}^{n})$.

Then $\ker L = \text{span}\{\partial_{1}w_{0}, \ldots, \partial_{n}w_{0}\}$ holds.

For the proof of this lemma, see Appendix.

**Lemma 4.3.** We have the following formula for $w_{\lambda}$ as $\lambda \to \infty$:

$$w_{\lambda} - w_{\lambda,x_{\lambda}} - t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} + o(1)) = 0 \quad \text{in } \mathbb{R}^{n}.$$  \hspace{1cm} (19)

**Proof.** Define $\phi_{\lambda}$ by

$$t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}\phi_{\lambda} = w_{\lambda} - w_{\lambda,x_{\lambda}} - t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}w_{0}.$$

Then

$$t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}\Delta\phi_{\lambda} = (w_{0} - 1)_{+} - (w_{\lambda} - 1)_{+} + t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} - 1)_{+}.$$

So we have

$$t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}|\Delta\phi_{\lambda}|$$

$$\leq |(w_{0} - 1)_{+} - (w_{\lambda} - 1)_{+}| + t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} - 1)_{+}$$

$$\leq |w_{0} - w_{\lambda}| + t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} - 1)_{+}$$

$$= |k_{0}\lambda^{-\frac{n-2}{2}}h_{x_{\lambda}} - t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}w_{0} - t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}\phi_{\lambda}|$$

$$+ t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} - 1)_{+}.$$

Hence

$$|\phi_{\lambda}| \leq \left| \frac{h_{x_{\lambda}}}{t(x_{\lambda})} - w_{0} - \phi_{\lambda} \right| + (w_{0} - 1)_{+} \quad \text{in } \mathbb{R}^{n}.$$  

Since $w_{\lambda} \to w_{0}$ in $C^{2}_{\text{loc}}(\mathbb{R}^{n})$, for any $\epsilon > 0$, if $\lambda$ is sufficiently large, we have $w_{\lambda} > 1, w_{0} > 1$ on $R_{R-\epsilon}$ and

$$t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}\Delta\phi_{\lambda} = w_{0} - w_{\lambda} + t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} - 1)_{+}.$$  

Hence, we obtain

$$\begin{cases} \Delta\phi_{\lambda} = \left( \frac{h_{x_{\lambda}}}{t(x_{\lambda})} - 1 \right) - \phi_{\lambda} & \text{in } B_{R-\epsilon}, \\ \Delta\phi_{\lambda} = 0 & \text{in } \mathbb{R}^{n} \setminus B_{R+\epsilon}. \end{cases}$$
To show $||\phi_\lambda||_{L^\infty(R^n)}$ is bounded as $\lambda \to \infty$, we suppose $||\phi_\lambda||_{L^\infty(R^n)} \to \infty$ for some subsequence. Define $\psi_\lambda$ by $\psi_\lambda = \phi_\lambda / ||\phi_\lambda||_{L^\infty(R^n)}$. Then $\psi_\lambda$ satisfies the following properties:

\[
\begin{align*}
|\Delta \psi_\lambda| &\leq C & \text{in } R^n, \\
\Delta \psi_\lambda &\left(\frac{h_{x_\lambda}}{t(x_\lambda)} - 1\right)/||\phi_\lambda||_{L^\infty(R^n)} - \psi_\lambda & \text{in } B_{R-\epsilon}, \\
\Delta \psi_\lambda &= 0 & \text{in } R^n \setminus \overline{B_{R+\epsilon}}.
\end{align*}
\]

Furthermore, the support of $\psi_\lambda$ is bounded for each $\lambda$. By the maximum principle, we obtain

$$|\psi_\lambda| \leq c|y|^{2-n}$$

for some positive constant $c$ which is independent of $\lambda$. And the maximal point of $\psi_\lambda$ is contained in $B_{R+\epsilon}$ because of $\psi_\lambda$ is harmonic in $R^n \setminus \overline{B_{R+\epsilon}}$. The standard elliptic estimate and Ascoli-Arzela's Theorem assert

$$\psi_\lambda \to \psi_0 \quad \text{in } C^{2,\alpha}_{loc}(R^n) \quad \text{as } \lambda \to \infty$$

by passing to a subsequence if necessary. Here, $\psi_0$ is a solution of

\[
\begin{align*}
\Delta \psi_0 &= -\psi_0 & \text{in } B_R, \\
\Delta \psi_0 &= 0 & \text{in } R^n \setminus \overline{B_R}, \\
|\psi(y)| &\leq c|y|^{2-n} & \text{in } R^n.
\end{align*}
\]

So we obtain $\psi_0 \in W^{2,q}(R^n)$ for some $q > n/(n-2)$ and $\psi_0 \in \ker L$. It follows from Lemma 4.2 that

$$\psi_0 = \sum_{j=1}^n a_j \partial_j w_0$$

for some $a = (a_1, \ldots, a_n) \in R^n$. It follows from $\partial_{ij} w_0(0) = \delta_{ij} w_0''(0)$ that $\nabla \phi_0(0) = w_0''(0)a$. On the other hand,

$$\frac{w_0 - w_{\lambda,x_\lambda} - k_0 \lambda^{-\frac{n-2}{2}} t(x_\lambda)}{k_0 \lambda^{-\frac{n-2}{2}} t(x_\lambda)} = \frac{h_{x_\lambda} - t(x_\lambda)}{t(x_\lambda)}$$

is uniformly bounded on $B_R$ and

$$\Delta (w_0 - w_{\lambda,x_\lambda} - k_0 \lambda^{-\frac{n-2}{2}} t(x_\lambda))) = 0 \quad \text{in } R^n.$$

By the interior Schauder estimates, we have

$$\left\| \frac{w_0 - w_{\lambda,x_\lambda} - k_0 \lambda^{-\frac{n-2}{2}} t(x_\lambda)}{k_0 \lambda^{-\frac{n-2}{2}} t(x_\lambda)} \right\|_{C^{1,\alpha}(B_R)} \leq C \left\| \frac{h_{x_\lambda} - t(x_\lambda)}{t(x_\lambda)} \right\|_{L^\infty(B_R)} = o(1)$$
because of Lemma 4.1. Especially, we obtain
\[ \left| \frac{\nabla w_{0}(0) - \nabla w_{\lambda,x_{\lambda}}(0)}{k_{0}\lambda^{-\frac{n-2}{2}}t(x_{\lambda})} \right| = o(1) \]
as \( \lambda \to \infty \). Using \( \nabla w_{0}(0) = \nabla w_{\lambda}(0) = 0 \) and the definition of \( \phi_{\lambda} \), we have
\[ |\nabla \phi_{\lambda}(0)| = o(1) \]
as \( \lambda \to \infty \). Especially, \( \nabla \psi_{\lambda}(0) = o(1) \) as \( \lambda \to \infty \).

Hence, we obtain \( \psi_{\lambda} = 0 \). It means \( \psi_{\lambda} \to 0 \) in \( C_{1\text{oc}}^{2}(\mathbb{R}^{n}) \) and contradicts to \( \|\psi_{\lambda}\|_{L^\infty(B_{R+\epsilon})} = 1 \).

Consequently, \( \phi_{\lambda} \) is uniformly bounded as \( \lambda \to \infty \).

Finally, we show \( \|\phi_{\lambda}\|_{L^\infty(\mathbb{R}^{n})} = o(1) \) as \( \lambda \to \infty \). If not, we can assume \( \|\phi_{\lambda}\|_{L^\infty(\mathbb{R}^{n})} = c + o(1) \) as \( \lambda \to \infty \) for some \( c > 0 \) by taking a subsequence if necessary. Noting \( h_{x_{\lambda}}/t(x_{\lambda}) - 1 = o(1) \) as \( \lambda \to \infty \) by Lemma 4.1, the above argument with \( \psi_{\lambda} = \phi_{\lambda} \) asserts \( \phi_{\lambda} \to 0 \) in \( C_{1\text{oc}}^{2}(\mathbb{R}^{n}) \). It contradicts to \( \|\phi_{\lambda}\|_{L^\infty(\mathbb{R}^{n})} = c + o(1) \) as \( \lambda \to \infty \).

\[ \square \]

Proposition 4.4 (Lower estimate). \( E_{\lambda}[u_{\lambda}] \) has the following asymptotic formula as \( \lambda \to \infty \):
\[ E_{\lambda}[u_{\lambda}] = \frac{I^{2}\lambda^{-\frac{n-2}{2}}}{2k_{0}}\{-1 + k_{0}t(x_{\lambda})\lambda^{-\frac{n-2}{2}} + o(t(x_{\lambda})\lambda^{-\frac{n-2}{2}})\}. \]

Proof. For the global minimizer \( u_{\lambda} \), put \( w_{\lambda} = (u_{\lambda}(\Gamma) - u_{\lambda})/u_{\lambda}(\Gamma) \) then we have
\[ w_{\lambda} = w_{0} - k_{0}\lambda^{-\frac{n-2}{2}}h_{x_{\lambda}} + t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} + o(1)) \]
because of Lemma 4.3. It follows from \( E_{\lambda}'[u_{\lambda}] = 0 \) that \( E_{\lambda}[u_{\lambda}] = -Iu_{\lambda}(\Gamma)/2 \) and we have
\[ \frac{I}{\lambda} = \int_{\Omega}(u_{\lambda})_{-} \, dx = \lambda^{-\frac{n}{2}}u_{\lambda}(\Gamma)\int_{\Omega_{\lambda}}(w_{\lambda}-1)_{+} \, dy. \]
So we obtain
\[ I^{-1}\lambda^{-\frac{n-2}{2}}u_{\lambda}(\Gamma) = \left\{ \int_{\Omega_{\lambda}}(w_{0} - k_{0}\lambda^{-\frac{n-2}{2}}h_{x_{\lambda}} + t(x_{\lambda})k_{0}\lambda^{-\frac{n-2}{2}}(w_{0} + o(1)) - 1)_{+} \, dy \right\}^{-1} \]
\[ = \left\{ \int_{\Omega_{\lambda}}(w_{0} - 1 + k_{0}\lambda^{-\frac{n-2}{2}}t(x_{\lambda})(\frac{t(x_{\lambda})-h_{x_{\lambda}}}{t(x_{\lambda})} + w_{0} - 1 + o(1)))_{+} \, dy \right\}^{-1} \]
\[ = \left\{ \int_{\Omega_{\lambda}}(w_{0} - 1)_{+}(k_{0}\lambda^{-\frac{n-2}{2}}t(x_{\lambda}) + 1) \, dy + o(\lambda^{-\frac{n-2}{2}}t(s_{\lambda})) \right\}^{-1} \]
\[ = \left\{ k_{0}(k_{0}\lambda^{-\frac{n-2}{2}}t(x_{\lambda}) + 1) + o(\lambda^{-\frac{n-2}{2}}t(s_{\lambda})) \right\}^{-1} \]
\[ = k_{0}^{-1}\left\{ 1 - k_{0}\lambda^{-\frac{n-2}{2}}t(x_{\lambda}) + o(\lambda^{-\frac{n-2}{2}}t(x_{\lambda})) \right\}. \]
It completes the proof of this lemma. \[ \square \]
Proposition 4.5. It holds that
\[ t(x_{\lambda}) \to \min_{x \in \Omega} t(x) \quad \text{as} \quad \lambda \to \infty. \]

Hence, \( \lim_{\lambda \to \infty} \text{dist}(x_{\lambda}, \Omega_{h}) = 0. \)

This proposition completes the proof of Theorem B.

Proof. Combining Proposition 3.1 and Proposition 4.4, we have
\[ k_{0} \lambda^{-\frac{n-2}{2}} t(x_{\lambda})(1 + o(1)) \leq k_{0} \lambda^{-\frac{n-2}{2}} \min_{x \in \Omega} t(x)(1 + o(1)). \]

Taking \( \lambda \to \infty \), it follows \( \min_{x \in \Omega} t(x) \leq \limsup_{\lambda \to \infty} t(x_{\lambda}) \leq \min_{x \in \Omega} t(x) \).

By continuity of \( t(x) \) and the definition of \( \Omega_{h} \), \( \text{dist}(x_{\lambda}, \Omega_{h}) = 0 \) holds and completes the proof. \( \square \)

5 Appendix

In this section, we give the proof of Lemma 4.2.

Proof of Lemma 4.2. For any \( \phi \in C_{0}^{\infty}(\mathbb{R}^{n}) \), we have \( \partial_{1}\phi \in C_{0}^{\infty}(\mathbb{R}^{n}) \) and
\[ \int_{\mathbb{R}^{n}} \nabla w_{0} \nabla \phi - (w_{0} - 1)_{+} \partial_{1}\phi \, dx = 0. \]
As \( w_{0}(x) = C|x|^{2-n} \) on \( \mathbb{R}^{n} \setminus B_{R} \), we obtain \( w_{0} \in H^{2} \) and
\[ \int_{\mathbb{R}^{n}} -\nabla(\partial_{1}w_{0}) \nabla \phi + \chi_{B_{R}} \partial_{1}w_{0} \phi \, dx = 0 \]
for any \( \phi \in C_{0}^{\infty}(\mathbb{R}^{n}) \). It means \( L\partial_{1}w_{0} = 0 \). Similarly, we have \( L\partial_{k}w_{0} = 0 \) for \( 1 \leq k \leq n \) and \( \ker L \supset \text{span} \{ \partial_{1}w_{0}, \ldots, \partial_{n}w_{0} \} \).

Let \( \mu_{k} \) be \( k \)th eigenvalue of \( -\Delta \) on \( \partial B_{1} \) and \( \phi_{k} \) be \( k \)th eigenfunction which orthonormalized in \( L^{2} \). It is well known that \( \mu_{0} = 0, \mu_{1} = \cdots = \mu_{n} = n - 1, \mu_{k} > n - 1 \) if \( k > n \). Fix any \( v \in \ker L \) and define \( v_{k} \) by
\[ v_{k}(r) = \int_{\partial B_{1}} v(r, \theta) \phi_{k}(\theta) \, d\theta. \]
By \( v \in \ker L, v \in W^{2,q}(\mathbb{R}^{n}) \) and the standard elliptic regularity theorem, we have \( v \in C^{1,\alpha}(\mathbb{R}^{n}) \cap C^{2,\alpha}(\mathbb{R}^{n} \setminus \overline{B}_{R}) \). It asserts \( v_{k} \in C^{1}([0, \infty)) \cap C^{2}((0, R)) \cap C^{2}((R, \infty)) \cap H_{1,0}^{2}((0, \infty)) \) and
\[
\begin{cases}
  v_{k}'' + \frac{n-1}{r} v_{k}' - \frac{\mu_{k}}{r^{2}} v_{k} + v_{k} = 0 & \text{on } (0, R), \\
  v_{k}'' + \frac{n-1}{r} v_{k}' - \frac{\mu_{k}}{r^{2}} v_{k} = 0 & \text{on } (R, \infty), \\
  v_{k}(0) = 0. 
\end{cases}
\]
We show $v_k \equiv 0$ if $k = 0$ or $k \geq n + 1$. For $k \geq n + 1$, taking $r^{n-1}w_0$ as a test function on $(r_1, r_2)$, one finds
\[
\left[\left\{v'_kw'_0 + \frac{n-1}{r}v_kw_0 + v_k(w_0-1)_+\right\}r^{n-1}\right]_{r_1}^{r_2} + (n-1-\mu_k) \int_{r_1}^{r_2} v_kw'_0r^{n-3} \, dx = 0. \tag{21}
\]
In the case $v_k$ has a zero point on $(0, \infty)$, we choose $r_1 \in (0, \infty)$ with $v(r_1) = 0$. If $v'(r_1) = 0$ then the uniqueness of ODE asserts $v_k \equiv 0$ on $(0, \infty)$. If $v'(r_1) \neq 0$ then linearity asserts we can assume $v'(r_1) > 0$. Put $r_2 = \sup\{r \in (0, \infty); v(t) > 0\}$ on $(r_1, t)$. If $r_2 < \infty$ then we have $v_k(r_1) = v_k(r_2) = 0$, $v_k > 0$ on $(r_1, r_2)$ and $v'(r_2) \leq 0$. It contradicts to (21) since $w'_0 < 0$ on $(0, \infty)$ and $\mu_k > n-1$. If $r_2 = \infty$ then $v_k(r) > 0$ on $(r_1, r_2)$. Since $v_k$ is subharmonic on $(\max\{r_1, R\}, \infty)$, we have $v_k(r)r^{n-2} = O(1)$ as $r \to \infty$. So we obtain (21) is a contradiction. In the case $v_k$ has no zero point, by linearity we can assume $v_k > 0$ on $(0, \infty)$. As above we obtain $v_k(r)r^{n-2} = o(1)$ as $r \to \infty$. Taking $r_1 = 0$ and $r_2 = \infty$ then (21) is a contradiction by $v_k'(r_1) = 0$. Consequently we obtain $v_k \equiv 0$ if $k \geq n - 1$. For $k = 0$, (20) asserts that $v_0$ is a solution of
\[
\Delta v_0 + \chi_{B_R} v_0 = 0 \quad \text{in } \mathbb{R}^n.
\]
Taking $(w_0 - 1)_+$ as a test function and integrating on $B_r$, we have
\[
\int_{B_r} \Delta v_0(w_0 - 1)_+ + v_0 \Delta w_0 \, dx = 0
\]
by noting $-\Delta w_0 = (w_0 - 1)_+$. Green's Theorem asserts
\[
\int_{\partial B_r} v'_0(r)(w_0(r) - 1)_+ - v_0(r)w'_0(r) \, dS(x).
\]
So we obtain $v'_0(r)(w_0(r) - 1)_+ = v_0(r)w'_0(r)$ if $r > 0$. Hence $v_0(R) = 0$. Since $v_0$ is harmonic in $\mathbb{R}^n \setminus B_R$ and $\lim_{r \to \infty} v_0(r) = 0$, we obtain $v \equiv 0$ on $(R, \infty)$. By uniqueness of the solution to ODE, we have $v \equiv 0$ on $(0, \infty)$. It completes the proof of this lemma. \qed

References


