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AN ELEMENTARY CONSTRUCTION OF COMPLEX PATTERNS IN ONE-DIMENSIONAL SINGULAR PERTURBATION PROBLEMS

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0. INTRODUCTION

This note is based on my joint works with M. del Pino and P. Felmer [DFT] and we introduce an elementary method to construct solutions with complex patterns for one-dimensional singular perturbation problems for nonlinear Schrödinger equation:

$$\begin{aligned} &-\epsilon^2 u_{xx} + V(x)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \\ &u \in H^1(\mathbf{R}), \end{aligned} \tag{0.1}$$

where $\epsilon > 0$ is a small parameter, $p > 1$ and $V(x)$ is a continuous positive function. We remark that our method is originally introduced for inhomogeneous phase transition problems in [NT].

For the nonlinear Schrödinger equation (0.1) and its higher dimensional version, the existence of single or multi-peaked solutions is widely studied since the works of Floer-Weinstein and Oh; in [FW, O1, O2, O3] Floer-Weinstein and Oh constructed solutions which concentrate around given set of non-degenerate critical points and their results has been extended in a variety of situations; including relaxing non-degeneracy assumption and more general nonlinearities. See [ABC, DF1, DF2, GW, G, L, R, W]. Among them, an interesting phenomenon was discovered by Kang and Wei [KW]; they find the existence of positive solutions with any prescribed number of peaks clustering around given local maximum point of the potential (in any space dimensions).

In this note we revisit one-dimensional problem (0.1) and introduce a new variational construction of multi-peaked solutions. We consider not only positive solutions but also solutions that may change sign. Our method allows us to glue clusters of any prescribed number of spikes associated to local maxima or minima. These clusters must be constituted of peaks of the same sign around a local maximum point and of the alternating sign around a local minimum point.

To state our result precisely, we introduce the following rescaled problem: For a solution u_ϵ of (0.1) with a local maximum $\xi_\epsilon \to \xi$, we set $v_\epsilon(y) = u_\epsilon(\epsilon y + \xi_\epsilon)$. Then $v_\epsilon(y)$ satisfies

$$-v_{yy} + V(\epsilon y + \xi_\epsilon)v = |v|^{p-1}v \qquad \text{in } \mathbf{R}$$

and $v_\epsilon(y)$ approaches to the unique solution $\omega(\xi; y)$ of

$$-\omega_{yy} + V(\xi)\omega = |\omega|^{p-1}\omega, \tag{0.2}$$

$$\omega_y(0) = 0, \ \ \omega(y) > 0. \tag{0.3}$$

Thus $u_\epsilon(x) \sim \omega(\xi; (x - \xi)/\epsilon)$ near ξ_ϵ as $\epsilon \sim 0$. We need the following definitions to state our existence result: We say that a solution $u_\epsilon(x)$ of (0.1) has a cluster of spikes of type $(n, +)$ with constant sign at ξ if there exists points $p_1^\epsilon < p_2^\epsilon < \cdots < p_n^\epsilon$ with $p_i^\epsilon \to \xi$ as $\epsilon \to 0$ so that for some $\delta > 0$

$$\sup_{|x-\xi|<\delta} |u_\epsilon(x) - \sum_{i=1}^{n} \omega(\xi; (x - p_i^\epsilon)/\epsilon)| \to 0 \quad \text{as } \epsilon \to 0.$$

Similarly we say that $u_\epsilon(x)$ has a cluster of spikes of $(n, +)$ with alternating sign at ξ if there exist points $p_1^\epsilon < p_2^\epsilon < \cdots < p_n^\epsilon$ with $p_i^\epsilon \to \xi$ as $\epsilon \to 0$ so that for some $\delta > 0$

$$\sup_{|x-\xi|<\delta} |u_\epsilon(x) - \sum_{i=1}^{n} (-1)^{i-1}\omega(\xi; (x - p_i^\epsilon)/\epsilon)| \to 0 \quad \text{as } \epsilon \to 0.$$

We say that $u_\epsilon(x)$ has a cluster of spikes of type $(n, -)$ if $-u_\epsilon(x)$ has a cluster of type $(n, +)$ (with constant or alternating sign) at ξ.

Theorem 0.1 ([DFT]). *Let us consider* m *critical points of* $V(x)$, $x_1 < \cdots < x_m$ *such that for some* $h > 0$, $(x - x_i)V'(x) \neq 0$ *whenever* $0 < |x - x_i| < h$. *Then for a given collection of pairs* (n_i, r_i), $i = 1, 2, \cdots, m$ *with* $n_i \in \mathbf{N}$ *and* $r_i \in \{+, -\}$ *there exists* $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0]$, (0.1) *has a solution* $u_\epsilon(x)$ *with a cluster of type* (n_i, r_i) *at* x_i *with alternating sign if* x_i *is a local minimum and with constant sign if* x_i *is a local maximum.*

We can extend the above result to the construction of solutions with infinitely many clusters of spikes and we can show the presence of chaotic patterns of clusters of spikes. See Theorem 1.2 in [DFT] for detail.

Chaotic patterns with finite or infinitely many spikes in related problems have been studied via variational techniques in Coti Zelati-Rabinowitz [CZR], Séré [S], Allessio-Montecchiari [AM]. We also mention the works by Kath [K] and Gedeon-Kokubu-Mischaikow-Oka [GKMO], where slowly varying planar Hamiltonian systems are studied. In particular in [GKMO] the existence of complex dynamical systems are constructed by means of the Conley index theory.

REMARK 0.2. An existence result related to our Theorem 0.1 also can be obtained for 1-dimensional inhomogeneous phase transition problem. A typical example is the Allen-Cahn equation:

$$-\epsilon^2 u_{xx} + h(x)(1-u^2)u = 0 \quad \text{in } (0,1),$$
$$u_x(0) = u_x(1),$$

where $h(x)$ is a continuous positive function on $[0,1]$. We can construct solutions with clusters of transition layers (instead of spikes). See [NT] for a precise statement. In [NT], we also construct solutions with boundary layers.

In the following sections, we give an explanation of our approach for the construction of solutions with complex patterns. A main feature of our approach is its elementary nature. It exhibits resemblance with the broken geodesics method in Riemannian geometry.

1. BASIC SOLUTIONS AND VARIATIONAL FORMULATION

For a given numbers $a < b$, we consider the following boundary value problems:

$$-\epsilon^2 u_{xx} + V(x)u = |u|^{p-1}u \quad \text{in } (a,b), \tag{1.1}$$
$$u_x(a) = u_x(b) = 0, \tag{1.2}$$
$$-\epsilon^2 u_{xx} + V(x)u = |u|^{p-1}u \quad \text{in } (-\infty,a), \tag{1.3}$$
$$u(-\infty) = u_x(a) = 0, \tag{1.4}$$
$$-\epsilon^2 u_{xx} + V(x)u = |u|^{p-1}u \quad \text{in } (b,\infty), \tag{1.5}$$
$$u_x(b) = u(\infty) = 0. \tag{1.6}$$

The following result concerns the existence and uniqueness of solutions with spikes at the end points of the intervals of the above problems.

PROPOSITION 1.1. *Let* $M > 0$ *and* $\sigma = (s_1, s_2) \in \{+,-\}^2$ *be given. Then there exist positive numbers* δ_0, ϵ_0 *and* ℓ_0 *such that for any* $0 < \epsilon < \epsilon_0$ *and any* $a,\ b \in [-M, M]$ *with* $\frac{b-a}{\epsilon} \geq \ell_0$, *the problem* (1.1)--(1.2) *has a unique solution* $u_\epsilon(x) = u_{\epsilon,\sigma}(a, b; x)$ *satisfying*

$$\|u_\epsilon(x) - s_1\omega(a; (x-a)/\epsilon) - s_2\omega(b; (x-b)/\varepsilon)\|_{L^\infty[a,b]} < \delta_0.$$

Similarly, for given $M > 0$, $\sigma \in \{+,-\}$ *there exist* ϵ_0, $\delta_0 > 0$ *such that for any* $a \in [-M, M]$ *and* $0 < \epsilon < \epsilon_0$ *the problem* (1.3)--(1.4) *has a unique solution* $u_\epsilon(x) = u_{\epsilon,\sigma}(-\infty, a; x)$ *satisfying*

$$\|u_\epsilon(x) - \sigma\omega(a; (x-a)/\epsilon)\|_{L^\infty(-\infty,a)} < \delta_0.$$

A similar statement holds for (1.5)--(1.6). *We denote the corresponding solution by* $u_{\epsilon,\sigma}(b, \infty; x)$.

As we mentioned in the Introduction, our approach is variational. Solutions of (0.1) are characterized as critical points of the following functional:

$$I_\epsilon(u) = \int_{-\infty}^{\infty} \frac{\epsilon}{2}|u_x|^2 + \frac{1}{\epsilon}G(x,u)\,dx : H^1(\mathbf{R}) \to \mathbf{R},$$

where $G(x,u) = \frac{1}{2}V(x)u^2 - \frac{1}{p+1}|u|^{p+1}$. And we try to find critical points using basic soltuions.

Let us consider m critical points $x_1 < \cdots < x_m$ of the potential $V(x)$, a number $h > 0$ and prescribed pairs (n_i, r_i) as in Theorem 0.1. Let $n = \sum_{i=1}^m n_i$. We introduce a functional $f_\epsilon(\mathbf{t})$ of the n tuple $\mathbf{t} = (t_1, \cdots, t_n)$ $(t_1 < \cdots < t_n)$, whose critical points are corresponding to solutions with spikes (positive maxima or negative minima) precisely at points $t_1, \cdots, t_n$.

To define $f_\epsilon(\mathbf{t})$ precisely, we define $\nu_1 = 0$, $\nu_j = \sum_{i=1}^{j-1} n_i$, $j = 2, 3, \cdots, m$. To have a cluster of size n_j at x_j, we impose that

$$x_j - h \leq t_{\nu_j+1} < \cdots < t_{\nu_j+n_j} \leq x_j + h \quad \text{for each j.} \tag{1.7}$$

We also consider signs $s_1, s_2, \cdots, s_n$ determined so that $s_{\nu_j+1} = r_j$ and

$$s_{\nu_j+1} = -s_{\nu_j+2} = \cdots = (-1)^{n_j-1}s_{\nu_j+n_j} \quad \text{if } x_j \text{ is a local minimum of } V(x),$$
$$s_{\nu_j+1} = s_{\nu_j+2} = \cdots = s_{\nu_j+n_j} \quad \text{if } x_j \text{ is a local maximum of } V(x).$$

Choosing $M > 0$ so that $-M \le x_1 - h < \cdot x_m + h \le M$ and we consider numbers δ_0, ϵ_0 and ℓ_0 as in Proposition 1.1. Additionally we assume

$$(t_{i+1} - t_i)/\epsilon \ge \ell_0. \tag{1.8}$$

We define $u_\epsilon(\mathbf{t}; x)$ by

$$u_\epsilon(\mathbf{t}; x) = \begin{cases} u_{\epsilon,s_1}(-\infty, t_1; x) & \text{for } x \in (-\infty, t_1), \\ u_{\epsilon,(s_i,s_{i+1})}(t_i, t_{i+1}; x) & \text{for } x \in (t_i, t_{i+1}), \\ u_{\epsilon,s_n}(t_n, \infty; x) & \text{for } x \in (t_n, \infty). \end{cases}$$

We can easily see that the function $u_\epsilon(\mathbf{t}; x)$ is a solution of (0.1) if and only if it is continuous. We have a characterization of such $\mathbf{t}$ by means of the following ''broken energy'':

$$f_\epsilon(\mathbf{t}) = \sum_{j=0}^{n} I_{\epsilon,(t_j,t_{j+1})}(u_\epsilon(\mathbf{t}; x)),$$

where $I_{\epsilon,(a,b)}(u) = \int_a^b \frac{\epsilon}{2}|u_x|^2 + \frac{1}{\epsilon}G(x,u)\,dx$ and $t_0 = -\infty$, $t_{n+1} = \infty$.

Proposition 1.2. *Assume that* $\epsilon \in (0, \epsilon_0)$ *and* $t_1 < \cdots < t_n$ *satisfy* (1.7) *and* (1.8). *Then* $f_\epsilon(\mathbf{t})$ *is of class* C^1 *within this range and* $u_\epsilon(\mathbf{t}; x)$ *is a solution of* (0.1) *if and only if* $\nabla f_\epsilon(\mathbf{t}) = 0$. ∎

Thus to prove our Theorem 0.1, we will find a critical point of the function $f_\epsilon(\mathbf{t})$ on the set $\Delta^\epsilon = \Delta_1^\epsilon \times \cdots \times \Delta_m^\epsilon$, where

$$\Delta_j^\epsilon = \{(t_{\nu_j+1}, \cdots, t_{\nu_j+n_j});\ x_j - h \le t_{\nu_j+1} < \cdots < t_{\nu_j+n_j} \le x_j + h, \\ \frac{t_{\nu_j+i+1} - t_{\nu_j+i}}{\epsilon} \ge \ell_0 \quad \text{for all } i = 1, 2, \cdots, n_j\}.$$

More precisely, we will show

$$\deg(\nabla f_\epsilon, \Delta^\epsilon, 0) \ne 0. \tag{1.9}$$

2. Estimates of ∇f_ϵ

To show (1.9), we simplify notation and write $\tau_i = t_{\nu_j+i}$, $s_i = s_{\nu_j+i}$ $(i = 0, 1, 2, \cdots, n_j + 1)$, $\sigma_i = (s_i, s_{i+1})$ and introduce

$$g_\epsilon^j(\tau_1, \cdots, \tau_{n_j}) = \sum_{i=1}^{n_j} m_{\epsilon,\sigma_i}(\tau_i, \tau_{i+1}) : \Delta_j^\epsilon \to \mathbf{R}, \tag{2.1}$$

$$m_{\epsilon,\sigma}(a,b) = I_{\epsilon,(a,b)}(u_{\epsilon,\sigma}(a,b;x)),$$
$$\Delta_j^\epsilon = \{(\tau_1,\cdots,\tau_{n_j});\ x_j - h \le \tau_1 < \cdots < \tau_{n_j} \le x_j + h,\ \frac{\tau_{i+1}-\tau_i}{\epsilon} \ge \ell_0 \text{ for all } i = 1,2,\cdots,n_j\}.$$

We regard that points $\tau_0 = t_{\nu_j}$, $\tau_{n_j+1} = t_{\nu_{j+1}}$ are fixed.

We observe that $\tau_0 = -\infty$ if $j = 1$ or $|x_{j-1} - \tau_0| \le h$ otherwise. Similarly $\tau_{n_j+1} = \infty$ if $j = m$ or $|x_{j+1} - \tau_{n_j+1}| \le h$ otherwise. Thus τ_0, τ_{n_j+1} are relatively far away from τ_i's if h is chosen sufficiently small.

PROPOSITION 2.1. *There exists numbers $\kappa > 0$, $\epsilon_0 > 0$ such that for all j, $\epsilon \in (0,\epsilon_0)$, τ_0, τ_{n_j+1}*

$$|\nabla g_\epsilon^j(\tau)| \ge \kappa \quad \textit{for all } \tau \in \partial\Delta_j^\epsilon$$

and

$$\deg(\nabla g_\epsilon^j, \Delta_j^\epsilon, 0) \ne 0.$$

From the above Proposition 2.1, we can deduce our Theorem 0.1.

PROOF OF THEOREM 0.1. From the definition of f_ϵ and those of the g_ϵ^j's, a direct computation shows that

$$\nabla_{t^j} f_\epsilon(t) = \nabla_{t^j} g_\epsilon^j(t^j) + o(1),$$

where $t^j = (t_{\nu_j+1},\cdots,t_{\nu_j+n_j})$ and $o(1) \to 0$ as $\epsilon \to 0$ uniformly on $t \in \Delta^\epsilon$. Thus by Proposition 2.1 we have

$$\deg(\nabla f_\epsilon, \Delta^\epsilon, 0) = \prod_{j=1}^m \deg(\nabla_{t^j} g_\epsilon^j, \Delta_j^\epsilon, 0) \ne 0.$$

Thus we get (1.9) and $f_\epsilon(t)$ has a critical point in Δ^ϵ. ∎

3. ∇g_ϵ ON $\partial\Delta_j^\epsilon$

In this section, we study the behavior of ∇g_ϵ^j on $\partial\Delta_j^\epsilon$. In what follows we fix $\tau_0 \in [-\infty, x_j - 2h]$, $\tau_{n_j+1} \in [x_j + 2h, \infty]$. By the definition of Δ_j^ϵ, we have

$$\partial\Delta_j^\epsilon = \{\tau;\ \tau_1 = x_j - h\} \cup \{\tau;\ \tau_{n_j+1} = x_j + h\} \cup \{\tau;\ \frac{\tau_{i+1}-\tau_i}{\epsilon} = \ell_0 \text{ for some } i\}.$$

We will give estimates of ∇g_ϵ^j on each of the above 3 sets and show that

(i) ∇g_ϵ^j and $\nabla\Phi_\epsilon$ are homotopic in Δ_j^ϵ if x_j is a local maximum of $V(x)$, (3.1)

(ii) ∇g_ϵ^j and $-\nabla\Phi_\epsilon$ are homotopic in Δ_j^ϵ if x_j is a local minimum of $V(x)$, (3.2)

where

$$\Phi_\epsilon(\tau) = \frac{1}{2}(\tau_1 - x_j)^2 + \frac{1}{2}(\tau_{n_j} - x_j)^2 + \sum_{i=1}^{n_j-1} \exp\left(-\frac{\tau_{i+1}-\tau_i}{\epsilon}\right).$$

That is

$$\theta\nabla g_\epsilon^j(\tau_1,\cdots,\tau_{n_j}) \pm (1-\theta)\nabla\Phi_\epsilon(\tau_1,\cdots,\tau_{n_j}) \neq 0 \tag{3.3}$$

for all $\theta \in [0,1]$ and $(\tau_1,\cdots,\tau_{n_j}) \in \partial\Delta_j^\epsilon$.

From (3.3) it follows that

$$\deg(\nabla g_\epsilon^j, \Delta_j^\epsilon, 0) = \deg(\theta\nabla g_\epsilon^j \pm (1-\theta)\nabla\Phi_\epsilon, \Delta_j^\epsilon, 0) = \pm 1.$$

We recall (2.1) and we give just estimates for $m_{\epsilon,\sigma}(a,b)$ which are essential to derive (3.1) and (3.2). We need the following notation:

$$H(\xi) = \int_0^\infty \frac{1}{2}|\omega_y(\xi;y)|^2 + G(\xi;\omega(\xi;y))\,dy,$$

where $\omega(\xi;y)$ is the unique solution of (0.2)--(0.3). We can easily see that

$$H(\xi) = C_0 V(\xi)^{\frac{p+3}{2(p-1)}},$$

where $C_0 > 0$ is independent of $\xi \in \mathbf{R}$.

To deal with ∇g_ϵ^j on $\{\tau;\, \tau_1 = x_j - h\} \cup \{\tau;\, \tau_{n_j+1} = x_j + h\}$, we need the following estimates of $m_{\epsilon,\sigma}(a,b)$ which treat the case where the distance between a, b are relatively large.

LEMMA 3.1. *For any $\delta > 0$ there exists $\epsilon_1 = \epsilon_1(\delta) \in (0,\epsilon_0]$ and $L = L(\delta) > \ell_0$ such that*

(i) If $\epsilon \in (0,\epsilon_1]$ and a, $b \in [-M,M]$ satisfy $(b-a)/\epsilon \geq 3L|\log\epsilon|$, then

$$\left|\frac{\partial}{\partial a} m_{\epsilon,\sigma}(a,b) - H'(a)\right| \leq \delta,$$

$$\left|\frac{\partial}{\partial b} m_{\epsilon,\sigma}(a,b) - H'(b)\right| \leq \delta.$$

(ii) If $\epsilon \in (0, \epsilon_1]$ *and* a, $b \in [-M, M]$ *satisfy* $(b-a)/\epsilon \geq \ell_0$, *then*

$$\left|\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right) m_{\epsilon,\sigma}(a,b) - (H'(a) + H'(b))\right| \leq \delta.$$

Using Lemmas 3.1 and 3.2, we can show $\nabla g_\epsilon^j(\tau) \neq 0$ on $\{\tau; \tau_1 = x_j - h\} \cup \{\tau; \tau_{n_j+1} = x_j + h\}$. More precisely, if $\tau \in \{\tau; \tau_1 = x_j - h\}$, we can find $k \in \{1, 2, \cdots, n\}$ such that

$$x_j - h = \tau_1 < \cdots < \tau_k \leq x_j - \frac{1}{2}h,$$
$$\frac{\tau_{k+1} - \tau_k}{\epsilon} \geq 3L|\log \epsilon|.$$

Then we have

$$\sum_{i=1}^{k} \frac{\partial}{\partial \tau_i} g_\epsilon^j(\tau_1, \cdots, \tau_{n_j}) \sim 2 \sum_{i=1}^{k} H'(\tau_i) \begin{cases} < 0 & \text{if } x_j \text{ is a local minimum,} \\ > 0 & \text{if } x_j \text{ is a local maximum.} \end{cases} \tag{3.4}$$

A similar estimate holds also on $\{\tau; \tau_{n_j+1} = x_j + h\}$.

To estimate ∇g_ϵ^j on $\{\tau; \frac{\tau_{i+1} - \tau_i}{\epsilon} = \ell_0$ for some $i\}$, we need to estimate $m_{\epsilon,\sigma}(a,b)$ for relatively small $|b-a|$.

First we consider the following homogenuous problem:

$$-\omega_{yy} + G'(a, \omega(y)) = 0 \quad \text{in } (0, \ell), \tag{3.5}$$
$$\omega_y(0) = \omega_y(\ell) = 0. \tag{3.6}$$

LEMMA 3.2. *There exist* $\ell_0 > 0$ *and* $\delta_0 > 0$ *such that if* $\ell \geq \ell_0$ *and* $\sigma = (s_1, s_2) \in \{+,-\}^2$, *then* (3.5)--(3.6) *has a unique solution satisfying*

$$\|\omega(y) - s_1\omega(a; y) - s_2\omega(a; \ell - y)\|_{L^\infty(0,\ell)} \leq \delta_0.$$

In what follows, we denote the unique solution by $\omega_\sigma(a, \ell; y)$. The following fact will be observed easily:

For any $a \in \mathbf{R}$, $\ell \geq \ell_0$ and $\sigma \in \{+,-\}^2$

$$E_\sigma(a, \ell) \equiv \frac{1}{2}\left|\frac{d}{dy}\omega_\sigma(a, \ell; y)\right|^2 - G(a, \omega_\sigma(a, \ell; y))$$

is independent of y. Moreover

(i) if $\sigma = (+,+)$ or $(-,-)$, then $E_\sigma(a, \ell) < 0$. (3.7)

(ii) if $\sigma = (+,-)$ or $(-,+)$, then $E_\sigma(a, \ell) > 0$. (3.8)

Our basic solution $u_{\epsilon,\sigma}(a, b; x)$ can be approximated by $\omega_\sigma(a, \frac{b-a}{\epsilon}; y)$.

LEMMA 3.3. *For any $\ell \geq \ell_0$ and $\delta > 0$, there exists $\epsilon_1 = \epsilon_1(\ell,\delta) > 0$ independent of* a, $b \in [-M, M]$ *such that for $\epsilon \in (0,\epsilon_1]$, $(b-a)/\epsilon \in [\ell_0, \ell]$ and $\sigma \in \{+,-\}^2$*

$$\|u_{\epsilon,\sigma}(a,b;a+\epsilon y) - \omega_\sigma(a, \frac{b-a}{\epsilon}; y)\|_{C^2(0,(b-a)/\epsilon)} \leq \delta.$$

From the above Lemma we have

LEMMA 3.4. *(i) For any $\ell \geq \ell_0$ there exist $\rho(\ell) > 0$ and $\epsilon_2(\ell) > 0$ such that for $(b-a)/\epsilon \in [\ell_0, \ell]$ and $\epsilon \in (0, \epsilon_2]$*

$$\epsilon \frac{\partial}{\partial a} m_{\epsilon,\sigma}(a,b) \begin{cases} \leq -\rho(\ell) & \text{if } \sigma = (+,+) \text{ or } (-,-), \\ \geq \rho(\ell) & \text{if } \sigma = (+,-) \text{ or } (+,-), \end{cases} \tag{3.9}$$

$$\epsilon \frac{\partial}{\partial b} m_{\epsilon,\sigma}(a,b) \begin{cases} \geq \rho(\ell) & \text{if } \sigma = (+,+) \text{ or } (-,-), \\ \leq -\rho(\ell) & \text{if } \sigma = (+,-) \text{ or } (+,-). \end{cases} \tag{3.10}$$

(ii) For any $\delta > 0$ there exists $\ell(\delta) \geq \ell_0$ and $\epsilon_2 > 0$ such that for $(b-a)/\epsilon \geq \ell(\delta)$ and $\epsilon \in (0, \epsilon_2]$

$$\epsilon \left| \frac{\partial}{\partial a} m_{\epsilon,\sigma}(a,b) \right| \leq \delta \quad \text{and} \quad \epsilon \left| \frac{\partial}{\partial b} m_{\epsilon,\sigma}(a,b) \right| \leq \delta. \tag{3.11}$$

We give just an idea of the proof of Lemma 3.3. (i) A direct computation gives us for $\frac{b-a}{\epsilon} \in [\ell_0, \ell]$ and $\epsilon \in (0, \epsilon_0]$

$$\begin{aligned} \epsilon \frac{\partial}{\partial a} m_{\epsilon,\sigma}(a,b) &= -G(a, u_{\epsilon,\sigma}(a,b;a)) \\ &\sim -G(a, \omega_\sigma(a, \frac{b-a}{\epsilon}; 0)) \\ &\equiv E_\sigma(a, \frac{b-a}{\epsilon}). \end{aligned}$$

Thus (3.9) follows from (3.7)--(3.8). We can show (3.10) in a similar way. Observing that $\omega(a;y)$ satisfies

$$\frac{1}{2}|\omega_y(a;y)|^2 - G(a;\omega(a;y)) \equiv 0.$$

We can deduce (ii) from Proposition 1.1.

Let $\tau = (\tau_1, \cdots, \tau_{n_j})$ satisfies $(\tau_{i+1} - \tau_i)/\epsilon = \ell_0$. Then we can find $k \in \{2, 3, \cdots, n\}$ such that

$$\frac{\tau_k - \tau_{k-1}}{\epsilon} \text{ is small relatively } \frac{\tau_{k+1} - \tau_k}{\epsilon}$$

so that Lemma 3.4 implies

$$\left|\frac{\partial}{\partial \tau_k} m_{\epsilon,\sigma_{k-1}}(\tau_{k-1},\tau_k)\right| > \left|\frac{\partial}{\partial \tau_k} m_{\epsilon,\sigma_k}(\tau_k,\tau_{k+1})\right|.$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial \tau_k} g_\epsilon^j(\tau_1,\cdots,\tau_{n_j}) &= \frac{\partial}{\partial \tau_k} m_{\epsilon,\sigma_{k-1}}(\tau_{k-1},\tau_k) + \frac{\partial}{\partial \tau_k} m_{\epsilon,\sigma_k}(\tau_k,\tau_{k+1}) \\ &= \begin{cases} > 0 & \text{if } (s_1,\cdots,s_n) = (+,+,\cdots) \text{ or } (-,-,\cdots), \\ < 0 & \text{if } (s_1,\cdots,s_n) = (+,-,\cdots) \text{ or } (-,+,\cdots). \end{cases}\end{aligned} \tag{3.12}$$

The estimates (3.4) and (3.12) are enough to show (3.1) and (3.2).

References

[AM] F. Alessio and P. Montecchiari, Multibump solutions for a class of Lagrangian systems slowly oscillating at infinity, *Ann. IHP, Analyse Nonlineaire* 16 (1999), 107-135

[ABC] A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schrödinger equations. *Arch. Rational Mech. Anal.* 140 (1997), 285--300.

[CZR] V. Coti Zelati and P. Rabinowitz. Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. *J. Amer. Math. Soc.* 4, (1992), 693-727.

[DF1] M. del Pino and P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. PDE* 4 (1996) 121--137.

[DF2] M. del Pino and P. Felmer, Multi-peak bound states of nonlinear Schrödinger equations. *Ann. IHP, Analyse Nonlineaire*, 15, (1998), 127-149.

[DFT] M. del Pino, P. Felmer and K. Tanaka, An elementary construction of complex patterns in nonlinear Schrödinger equations, *preprint*.

[FW] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* 69 (1986), 397--408.

[GKMO] T. Gedeon, H. Kokubu, K. Mischaikow and H. Oka, Chaotic solutions in slowly varying perturbations of Hamiltonian systems with applications to shallow water sloshing, *preprint Kyoto University*, 1999.

[GW] C. Gui and J. Wei, Multiple interior peak solutions for some singularly perturbed Neumann problems, *J. Diff. Eq.* 158 (1999), 1--27.

[G] C. Gui, Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method. *Comm. Partial Differential Equations* 21 (1996), 787--820.

[KW] X. Kang and J. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations, *Advances Diff. Eq.* 5 (2000), 899--928.

[K] W. L. Kath, Slowly varying phase planes and boundary-layer theory, *stud. Appl. Math.* 72 (1985), 221--239.

[L] YanYan Li, On a singularly perturbed elliptic equation. *Adv. Differential Equations* 2 (1997), 955--980.

[NT] K. Nakashima and K. Tanaka, Clustering layers and boundary layers in spatially inhomogeneous phase transition problems, *preprint*.

[O1] Y.-G. Oh, Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$. *Comm. Partial Differential Equations* 13 (1988), 1499--1519.

[O2] Y.-G. Oh, Stability of semiclassical bound states of nonlinear Schrödinger equations with potentials. *Comm. Math. Phys.* 121 (1989), 11--33.

[O3] Y.-G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential. Comm. Math. Phys. 131 (1990), 223--253.

[R] P. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew Math Phys* 43, (1992), 270--291.

[S] E. Séré Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.* 209, (1991), 27--42.

[W] X. Wang On concentration of positive bound states of nonlinear Schrödinger equations. *Comm. Math. Phys.* 153, 229--244 (1993).