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Missing terms in Hardy's inequalities and its applications

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Abstract

この論文の主な目的は、様々な古典的ハーディーの不等式をミッシングタームを見つけて改良することである。その応用として、重複ラプラシアンに関する半線形楕円型方程式の特異解の解析をする。さらに、重み付きのハーディー不等式の改良も行われる。

1 イントロダクション

N を正整数、Ω を $\mathbb{R}^N$ の有界開集合とする。l を正整数として、$H^l(\Omega)$ で通常のソボレフ空間を表す。ノルムは次で定義する。

$$||u||_l = \sum_{|\gamma|\le l} \left(\int_\Omega |\partial^\gamma u(x)|^2\, dx \right)^{1/2} < +\infty. \tag{1.1}$$

$H^l_0(\Omega)$ を $C_0^\infty(\Omega)$ のこのノルムに関する完備化とする。
先ずハーディーの不等式を思い出そう。

Theorem 1.1 *If $l < \frac{N}{2}$, then it holds that for any $u \in H^l_0(\Omega)$*

$$\int_\Omega |\nabla^l u|^2\, dx \ge C_l \int_\Omega \frac{|u(x)|^2}{|x|^{2l}}\, dx. \tag{1.2}$$

Here $\nabla^l = \{D^\gamma\}$, where $|\gamma| = l$ and $\nabla = \nabla^1$, namely

$$|\nabla^l u|^2 = \sum_{|\gamma|=l} |D^\gamma u(x)|^2, \tag{1.3}$$

where $\gamma = (\gamma_1, \gamma_2, \cdots \gamma_N)$ is a multi-index as usual, and then $D^\gamma = \left(\frac{\partial}{\partial x_1}\right)^{\gamma_1} \cdot \left(\frac{\partial}{\partial x_2}\right)^{\gamma_2} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\gamma_N}$. C_l is a positive number independent of each u.

この論文では主に次のタイプのハーディー不等式を調べることになる。

任意の $u \in H_0^{2l}(\Omega)$,

$$\int_\Omega |\Delta^l u|^2\,dx \geq H(N,\Delta^l)\int_\Omega \frac{|u(x)|^2}{|x|^{4l}}\,dx \quad \text{for} \quad l=1,2. \tag{1.4}$$

$H(N,\Delta^l)$ $(l=1,2)$ は次の変分問題で与えられる最良定数である。

$$\inf\left[\int_\Omega |\Delta^l u|^2\,dx : u \in H_0^{2l}(\Omega), \int_\Omega \frac{|u(x)|^2}{|x|^{4l}}\,dx = 1\right], \quad l=1,2. \tag{1.5}$$

もし$0 \in \Omega$ かつ $N > 4l$であれば, $H(N,\Delta^l)$ $(l=1,2)$ は次で与えられることが良く知られている。

$$\begin{cases} H(N,\Delta) = \left(\frac{N(N-4)}{4}\right)^2, \\ H(N,\Delta^2) = \left(\frac{N(N-4)(N+4)(N-8)}{16}\right)^2. \end{cases} \tag{1.6}$$

詳しくは参考文献 [1] と [4]を参照せよ. さらに、極値を実現する関数が$H_0^{2l}(\Omega)$の中には存在しない ことが知られており、これは荒く言って候補者たちが原点で特異であることに原因しているのである。従って、この変分問題はエネルギークラス$H_0^{2l}(\Omega)$においては適切でないともいえる。このようにして、このハーディの不等式にはまだ "missing terms " が隠れていると考えるのが自然となる。我々はこの精神でハーディの不等式の**missing terms**を見つけ、古典的な不等式を改良することを目的とする.

その応用として、最終章において重複ラプラシアンに関する半線形楕円型境界値問題:

$$\begin{cases} \Delta^2 u = \lambda f(u,r) & \text{in } B \\ u = \Delta u = 0 & \text{on } \partial B, \end{cases} \tag{1.7}$$

ここで $r = |x|$, $B = \{x \in \mathbb{R}^N : |x| < 1\}$ 、λ は非負値パラメーターである。

Nonlinearity $f(u,r)$ としては下の f_p と f_eを採用することになる。

$$\begin{cases} f_p(u,r) = (1+u+Q_p(r))^p, \\ f_e(u,r) = e^{u+Q_e(r)}. \end{cases} \tag{1.8}$$

ここで$Q_p(r)$ と $Q_e(r)$ B 上の非負球対称関数で§7で定義されている. これらの問題に関して、爆発解の基本的な性質を研究することになる。また p-harmonic equations ([5])に関する同様の問題を扱うときに基本的な重み付きハーディーの不等式も改良されることになる。.

この論文の構成は以下のようである。§2では ハーディーの不等式に関する主要な結果が述べられている. §3 には§2で述べられた結果を証明するのに必要な基本的な補題が述べられる. §4 と §5 において Theorems 2.1 と 2.2 が証明のスケッチがされる。§6 Theorems 2.3 と 2.4 が簡単に証明される. §7 においては偏微分方程式の境界値問題への応用が述べられている.

2 Main results

Let $r > 0$ and let M be an arbitraly positive integer. We set

$$B_r^M = \{x \in \mathbb{R}^M : |x| < r\}. \tag{2.1}$$

By $|\Omega|$ and ω_N we denote the N-dimensional measure of the domain Ω and that of a unit ball B_1^N respectively. Further, we set

$$\begin{cases} \Delta_M = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_M^2}, \\ \nabla_M = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_M}\right). \end{cases} \tag{2.2}$$

Conventionally we set $\Delta = \Delta_N$ and $\nabla = \nabla_N$. In the next we introduce the first eigenvalues for various elliptic problems.

Definition 2.1 *Let us set*

$$\begin{cases} \lambda_1 = \inf\left[\int_{B_1^2} |\nabla_2 v|^2\,dx : v \in H_0^1(B_1^2), \int_{B_1^2} |v|^2\,dx = 1\right], \\ \lambda_2 = \inf\left[\int_{B_1^4} |\Delta_4 v|^2\,dx : v \in H_0^2(B_1^4), \int_{B_1^4} |v|^2\,dx = 1\right], \\ \lambda_3 = \inf\left[\int_{B_1^6} \left|\nabla_6(\Delta_6 v)\right|^2 dx : v \in H_0^3(B_1^6), \int_{B_1^6} |v|^2\,dx = 1\right], \\ \lambda_4 = \inf\left[\int_{B_1^8} |\Delta_8^2 v|^2\,dx : v \in H_0^4(B_1^8), \int_{B_1^8} |v|^2\,dx = 1\right], \\ \lambda_2^* = \inf\left[\int_{B_1^4} |\Delta_4 v|^2\,dx : v \in H^2(B_1^4) \cap H_0^1(\Omega), \int_{B_1^4} |v|^2\,dx = 1\right]. \end{cases} \tag{2.3}$$

Then the numbers $\lambda_k\,(k = 1, 2, 3, 4)$ and λ_2^* are characterized as follows:

Proposition 2.1 *The numbers $\lambda_k\,(k = 1, 2, 3, 4)$ and λ_2^* are the first eigenvalues of the elliptic boundary value problems below. Namely there exist pos-*

itive smooth functions v_k $(k = 1, 2, 3, 4)$ and v_2^* in B such that they satisfy

$$\begin{cases} -\Delta_2 v_1 & = \lambda_1 v_1, \ in\ B_1^2, \quad v_1 = 0 \ on\ \partial B_1^2 \\ \Delta_4^2 v_2 & = \lambda_2 v_2, \ in\ B_1^4, \quad v_2 = \frac{d}{dn} v_2 = 0 \ on\ \partial B_1^4 \\ -\Delta_6^3 v_3 & = \lambda_3 v_3, \ in\ B_1^6, \quad v_3 = \frac{d}{dn} v_3 = \frac{d^2}{dn^2} v_3 = 0 \ on\ \partial B_1^6 \\ \Delta_8^4 v_4 & = \lambda_4 v_4, \ in\ B_1^8, \quad v_4 = \frac{d}{dn} v_4 = \frac{d^2}{dn^2} v_4 = \frac{d^3}{dn^3} v_4 = 0 \ on\ \partial B_1^8 \\ \Delta_4^2 v_2^* & = \lambda_2^* v_2^*, \ in\ B_1^4, \quad v_2^* = \Delta_4 v_2^* = 0 \ on\ \partial B_1^4. \end{cases} \tag{2.4}$$

Here by n we denote the unit outer normal on ∂B.

Now we are in a position to state our results:

Theorem 2.1 *Suppose $N > 4$. Let Ω be a bounded domain of $\mathbb{R}^N$. Then we have the following two inequalities.*

(1) *For any $u \in H_0^2(\Omega)$, it holds that*

$$\int_\Omega |\Delta u|^2\, dx \geq H(N, \Delta) \int_\Omega \frac{|u|^2}{|x|^4}\, dx \tag{2.5}$$
$$+ \lambda_1 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \frac{N(N-4)}{2} \int_\Omega \frac{|u|^2}{|x|^2}\, dx + \lambda_2 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega |u|^2\, dx$$

(2) *For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$, it holds that*

$$\int_\Omega |\Delta u|^2\, dx \geq H(N, \Delta) \int_\Omega \frac{|u|^2}{|x|^4}\, dx \tag{2.6}$$
$$+ \lambda_1 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \frac{N(N-4)}{2} \int_\Omega \frac{|u|^2}{|x|^2}\, dx + \lambda_2^* \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega |u|^2\, dx$$

where

$$H(N, \Delta) = \left(\frac{N(N-4)}{4}\right)^2. \tag{2.7}$$

Theorem 2.2 *Suppose $N > 8$. Let Ω be a bounded domain of $\mathbb{R}^N$. Then it holds that for any $u \in H_0^4(\Omega)$*

$$\int_\Omega |\Delta^2 u|^2\, dx \geq H(N, \Delta^2) \int_\Omega \frac{|u|^2}{|x|^8}\, dx \tag{2.8}$$
$$+ a_1 \cdot \lambda_1 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega \frac{|u|^2}{|x|^6}\, dx + a_2 \cdot \lambda_2 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega \frac{|u|^2}{|x|^4}\, dx$$
$$+ a_3 \cdot \lambda_3 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{6}{N}} \int_\Omega \frac{|u|^2}{|x|^2}\, dx + \lambda_4 \cdot \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{8}{N}} \int_\Omega |u|^2\, dx.$$

$$H(N,\Delta^2) = \Big(\frac{N(N-4)(N+4)(N-8)}{16}\Big)^2 \tag{2.9}$$

By a_1, a_2 and a_3 we denote positive constants defined by

$$\begin{cases} a_1 = \frac{1}{16}N^2(N-4)^2(N+4)(N-8), \\ a_2 = \frac{3}{8}N(N-4)(N+4)(N-8), \\ a_3 = (N+4)(N-8). \end{cases} \tag{2.10}$$

In the next we state the results concerned with the weighted Hardy inequalities.

Theorem 2.3 *Suppose that a positive integer N and a real number α satisfy $N+\alpha>2$. Then it holds that for any $u \in H_0^1(\Omega)$*

$$\int_\Omega |\nabla u|^2|x|^\alpha\,dx \geq H(N,\nabla,\alpha)\int_\Omega |u|^2|x|^{\alpha-2}\,dx \tag{2.11}$$
$$+\lambda_1\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}\int_\Omega |u|^2|x|^\alpha\,dx,$$

where

$$H(N,\nabla,\alpha) = \Big(\frac{n-2+\alpha}{2}\Big)^2. \tag{2.12}$$

Remark 2.1 *When $\alpha = 0$, this result was initially established in [3] by H. Brezis and J.L. Vázquez. They also investigated in [3] fundamental properties of blow-up solutions of some nonlinear elliptic problems.*

We also note that when one linearlizes the p-laplacian at the singular function such as $\log|x|$, the weighted Hardy inequalities appear in a natural way.

A similar result can be expected for Δ. In fact, the following weighted inequality hold.

Theorem 2.4 *Suppose that a positive integer N and a real number α satisfy $N+\alpha>4$. Then it holds that for any $u \in H_0^2(\Omega)$*

$$\int_\Omega |\Delta u|^2|x|^\alpha\,dx + \frac{\alpha(\alpha-4)}{2}\int_\Omega \Big(|\nabla u|^2 - 2\Big(\frac{x}{|x|}\cdot\nabla u\Big)^2\Big)|x|^{\alpha-2}\,dx \tag{2.13}$$
$$\geq I(N,\Delta,\alpha)\int_\Omega |u|^2|x|^{\alpha-4}\,dx + \lambda_1\frac{N(N-4)}{2}\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}\int_\Omega |u|^2|x|^{\alpha-2}\,dx$$
$$+\lambda_2\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{4}{N}}\int_\Omega |u|^2|x|^\alpha\,dx,$$

$$I(N,\Delta,\alpha)=\Big(\frac{N(N-4)}{4}\Big)^2-\frac{\alpha(\alpha-4)(\alpha+2N-4)(\alpha+2N-8)}{16}. \quad (2.14)$$

If we further assume either $\alpha\le 0$ or $\alpha\ge 4$, we have the following.

Corollary 2.1 *Suppose that the same assumptions as in the previous theorem 2.4. Moreover we assume either $\alpha\le 0$ or $\alpha\ge 4$. Then it holds that for any $u\in H_0^2(\Omega)$*

$$\begin{aligned}&\int_\Omega |\Delta u|^2|x|^\alpha\,dx+\alpha(\alpha-4)\int_\Omega\Big(|\nabla u|^2-\Big(\frac{x}{|x|}\cdot\nabla u\Big)^2\Big)|x|^{\alpha-2}\,dx \quad (2.15)\\ &\ge H(N,\Delta,\alpha)\int_\Omega |u|^2|x|^{\alpha-4}\,dx+b_1\lambda_1\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}\int_\Omega |u|^2|x|^{\alpha-2}\,dx\\ &+\lambda_2\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{4}{N}}\int_\Omega |u|^2|x|^\alpha\,dx,\end{aligned}$$

where

$$\begin{cases} H(N,\Delta,\alpha)=\big(\frac{N(N-4)}{4}-\frac{\alpha(\alpha-4)}{4}\big)^2\\ b_1=\frac{N(N-4)}{2}+\frac{\alpha(\alpha-4)}{2}.\end{cases} \quad (2.16)$$

In a similar way we have the following.

Corollary 2.2 *Suppose that the same assumptions as in the previous theorem 2.4. Moreover we assume that $0\le\alpha\le 4$. Then it holds that for any $u\in H_0^2(\Omega)$*

$$\begin{aligned}&\int_\Omega |\Delta u|^2|x|^\alpha\,dx+\frac{\alpha(4-\alpha)}{2}\int_\Omega |\nabla u|^2|x|^{\alpha-2}\,dx \quad (2.17)\\ &\ge I(N,\Delta,\alpha)\int_\Omega |u|^2|x|^{\alpha-4}\,dx+\lambda_1\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}\frac{N(N-4)}{2}\int_\Omega |u|^2|x|^{\alpha-2}\,dx\\ &+\lambda_2\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{4}{N}}\int_\Omega |u|^2|x|^\alpha\,dx.\end{aligned}$$

Remark 2.2 *In Theorem 2.4 and its corollaries, we can replace the admissible space $H_0^2(\Omega)$ by $H^2(\Omega)\cap H_0^1(\Omega)$. Then the same results hold if we replace λ_2 by λ_2^* as before.*

3 Lemmas

For a domain Ω we define the ball having the same measure as Ω by

$$\Omega^* = \{x \in \mathbb{R}^N : \omega_N |x|^N < |\Omega|\}, \tag{3.1}$$

where by ω_N we denote the measure of a unit ball. If $|\Omega| = +\infty$, we put $\Omega^* = \mathbb{R}^N$. For a measurable function u, we denote by $u^*(x)$ the spherically symmetric decreasing rearrangemant of u (the Schwarz symmetrization of u). Namely,

$$\begin{cases} u^*(x) = \inf\{t \geq 0 : \mu(t) < \omega_N |x|^N\} & \text{in } \Omega^* \\ \mu(t) = |\{x \in \Omega : |u(x)| > t\}|. \end{cases} \tag{3.2}$$

Then it is well-known that

Lemma 3.1 *Under these notations we have for every $p > 0$*

$$\begin{cases} \int_\Omega |u(x)|^p \, dx = \int_{\Omega^*} u^*(x)^p \, dx, \\ \int_\Omega |\nabla u(x)|^p \, dx \geq \int_{\Omega^*} |\nabla u^*(x)|^p \, dx, \end{cases} \tag{3.3}$$

Let $g \in C^0((0,\infty))$ be a nonnegative decreasing function. Then we have

$$\int_\Omega |u(x)|^p g(|x|) \, dx \leq \int_{\Omega^*} u^*(x)^p g(|x|) \, dx. \tag{3.4}$$

From this we see in particular that the symmetric rearrangement does not change the L^2-norm and increases the integral $\int_\Omega (|u^2|/|x|^l) \, dx$. The following is due to G. Talenti (See [9]).

Lemma 3.2 (Talenti) *Let Ω be a domain of $\mathbb{R}^N$. Assume that $N \geq 3$ and $f \in L^p(\Omega)$, where $p = \frac{2N}{N+2}$.*

If a measurable function u is the weak solution to the Dirichlet problem $-\Delta u = f$ in $\Omega, u|_{\partial\Omega} = 0$; v is the weak solution to the Dirichlet problem $-\Delta v = f^*$ in $\Omega^*, u|_{\partial\Omega^*} = 0$; then

(1) *$v \geq u^*$ pointwise.*

(2)

$$\int_{\Omega^*} |\nabla v|^q \, dx \geq \int_\Omega |\nabla u|^q \, dx, \qquad \text{if } 0 < q \leq 2. \tag{3.5}$$

Let us set

$$\begin{cases} I^l(u;\Omega) = \int_\Omega |\Delta^l u|^2\,dx, u \in C_0^\infty(\Omega) \\ I^l = \inf\left[I^l(u;\Omega) : u \in C_0^\infty(\Omega), \int_\Omega \frac{|u|^2}{|x|^{2l}}\,dx = 1\right], \\ I^l_r = \inf\left[I^l(u;\Omega^*) : u \in C^\infty_{0,rad}(\Omega^*), \int_{\Omega^*} \frac{|u|^2}{|x|^{2l}}\,dx = 1\right]. \end{cases} \tag{3.6}$$

By $C^\infty_{0,rad}(\Omega^*)$, we denote the set of all spherically symmetric functions $u \in C_0^\infty(\Omega^*)$. Under these preparations, we can show the following:

Lemma 3.3 (Reduction) *Under these notations, it holds that $I^l \geq I^l_r$ for every positive integer l. If Ω is a ball with its center being the origin, then it holds that $I^l = I^l_r$.*

Sketch ofProof. Without a loss of generality, we can assume $u \in C_0^\infty(\Omega)$. It suffices to show that there is a function $v \in C^{2l}_{0,rad}(\Omega^*)$ such that

$$\frac{I^l(u;\Omega)}{\int_\Omega |u|^2/|x|^{2l}\,dx} \geq \frac{I^l_r(v;\Omega^*)}{\int_{\Omega^*} |v|^2/|x|^{2l}\,dx}. \tag{3.7}$$

We shall prove this assuming $l = 1$.

We put $-\Delta u = f \in C_0^\infty(\Omega)$. From the definition of the decreasing rearrangement, we see that f^* is spherically symmetric in Ω^* and Lipschitz continuous. Let $v \in C^2(\overline{\Omega^*}) \cap C^1_0(\Omega^*)$ be the unique solution of the Dirichilet problem defined by

$$-\Delta v = f^*, \quad \text{in } \Omega^*, \quad v = 0 \quad \text{on } \partial\Omega^* \tag{3.8}$$

Then we see from Lemma 3.2 that $u^* \leq v$ in Ω^* and

$$\int_\Omega |\Delta u|^2\,dx = \int_\Omega |f|^2\,dx = \int_{\Omega^*} |f^*|^2\,dx = \int_{\Omega^*} |\Delta v|^2\,dx. \tag{3.9}$$

Further we see that

$$\int_\Omega \frac{|u|^2}{|x|^4}\,dx \leq \int_{\Omega^*} \frac{|u^*|^2}{|x|^4}\,dx \leq \int_{\Omega^*} \frac{|v|^2}{|x|^4}\,dx. \tag{3.10}$$

Therefore we see $I^1 \geq I^1_r$, and this proves the assertion when $l = 1$.

4 Proof of Theorems 2.1 and 2.2

Definition 4.1 (m **Laplacian**) *For $m \in \mathbb{R}$ and $v \in C^2((0,\infty))$, we set*

$$\delta_m v(r) = r^{1-m}\frac{\partial}{\partial r}\left(r^{m-1}\frac{\partial}{\partial r}v(r)\right) = \frac{\partial^2 v(r)}{\partial r^2} + \frac{m-1}{r}\frac{\partial v(r)}{\partial r} \tag{4.1}$$

Lemma 4.1 *Let M and m be positive integers. Let us set $r=|x|$ for $x\in\mathbb{R}^M$. For $\alpha\in\mathbb{R}$ and $v\in C^\infty((0,\infty))$ it holds that*

$$\Delta_M v(r)=\delta_M v(r) \tag{1}$$

$$\Delta_M^m(r^\alpha v(r))=r^\alpha\Big(\delta_{M+2\alpha}+\frac{\alpha(M+\alpha-2)}{r^2}\Big)^m v(r) \tag{2}$$

Proof of Theorem2.1.

Since the assertion (2) follows in a quite similar way, we prove the assertion (1) only. From Lemma 3.3, it is enough to prove the result in the symmetric case. To this end we set

$$\omega_N R^N=|\Omega| \tag{4.2}$$

and replace Ω by Ω^*. In addition to this fact, since $C_0^\infty(\Omega)$ is densely contained in $H_0^2(\Omega)$, we also replace the function space $H_0^2(\Omega)$ by $C_{0,rad}^\infty(\Omega^*)$. Moreover, a simple scaling allows to consider the case $R=1$.

Let us set for $B=B_1^N(0)$ and $u\in C_{0,rad}^\infty(B)$

$$u=r^{2-\frac{N}{2}}v,\quad v\in C_{0,rad}^\infty(B). \tag{4.3}$$

Here we note that v vanishes at the origin, if $N>4$. We see from Lemma 4.1 with $\alpha=2-\frac{N}{2}$ that

$$\Delta(r^{2-\frac{N}{2}}v(r))=r^{2-\frac{N}{2}}\Big(\delta_4 v(r)+Q\frac{v(r)}{r^2}\Big),\qquad Q=-\frac{N(N-4)}{4} \tag{4.4}$$

Then

$$\int_B|\Delta u|^2\,dx=\int_B|\Delta(r^{2-\frac{n}{2}}v)|^2\,dx$$
$$=|S^{N-1}|\int_0^1\Big(\delta_4 v+\frac{Q}{r^2}v\Big)^2 r^3\,dr\quad(\text{ Polar coordinate })$$
$$=|S^{N-1}|\int_0^1\Big(|\delta_4 v|^2-\frac{2Q}{r^2}|\partial_r v|^2+\frac{Q^2}{r^4}v^2\Big)r^3\,dr$$
$$=\frac{|S^{N-1}|}{|S^3|}\int_{B_1^4}|\Delta v(|y|)|^2\,dy-\frac{2Q|S^{N-1}|}{|S^1|}\int_{B_1^2}|\nabla_2 v(|y|)|^2\,dy+Q^2\int_B\frac{v(|y|)^2}{r^N}\,dy$$

Here by $|S^{M-1}|$ we denote the measure of the M-dimensional unit sphere. Then it holds that

$$\begin{aligned}
&\int_B |\Delta u|^2\,dx = \int_B |\Delta(r^{2-\frac{N}{2}}v)|^2\,dx && (4.5)\\
&\geq \lambda_2\frac{|S^{N-1}|}{|S^3|}\int_{B_1^4}|v(|y|)|^2\,dy - 2Q\lambda_1\frac{|S^{N-1}|}{|S^1|}\int_{B_1^2}|v(|y|)|^2\,dy + Q^2\int_B \frac{v(|y|)^2}{r^N}\,dy\\
&\geq H(N,\Delta)\int_B \frac{|u|^2}{|x|^4}\,dx + \lambda_1\cdot\frac{N(N-4)}{2}\int_B\frac{|u|^2}{|x|^2}\,dx + \lambda_2\cdot\int_B |u|^2\,dx,
\end{aligned}$$

where λ_1 and λ_2 are defined in (2.3). This proves the assertion.

Remark 4.1 *To prove the assertion (1), it suffices to replace $C_0^\infty(\Omega)$ by $H^2(\Omega)\cap C_0^1(\Omega)$.*

5 Proof of Theorem2.2.

Again from Lemma 3.2 and Lemma 3.3, it is enough to prove the result in the symmetric case. Let us set for $B = B_1^N(0)$ and $u \in C_{0,rad}^\infty(B)$

$$u = r^{4-\frac{N}{2}}v, \quad v \in C_{0,rad}^\infty(B). \tag{5.1}$$

Here we note that v vanishes at the origin, if $N > 8$. We see from Lemma 4.1 with $\alpha = 4 - \frac{N}{2}$ that

$$\Delta(r^{4-\frac{N}{2}}v(r)) = r^{4-\frac{N}{2}}\Big(\delta_8 v(r) + P\frac{v(r)}{r^2}\Big), \qquad P = -\frac{(N+4)(N-8)}{4} \tag{5.2}$$

As before we see

$$\int_B |\Delta^2 u|^2\,dx = |S^{N-1}|\int_0^1 \Big(\delta_8^2 v(r) + \frac{2P}{r^2}\delta_6 v(r) + \frac{S}{r^4}v(r)\Big)^2 r^7\,dr,$$

where

$$S = \frac{N(N-4)(N+4)(N-8)}{16} = H(N,\Delta^2)^{\frac{1}{2}}. \tag{5.3}$$

Integration by parts gives

Lemma 5.1 *For any $v \in C_0^\infty((0,1))$, we have*

$$\int_0^1 \left(\delta_8^2 v + \frac{2P}{r^2}\delta_6 v + \frac{S}{r^4}v\right)^2 r^7\,dr \tag{5.4}$$
$$= \int_0^1 |\delta_8^2 v|^2 r^7\,dr + S^2 \int_0^1 \frac{v^2}{r}\,dr$$
$$+ a_1 \int_0^1 |\partial_r v|^2 r\,dr + a_2 \int_0^1 |\delta_4 v|^2 r^4\,dr + a_3 \int_0^1 |\partial_r \delta_6 v|^2 r^5\,dr.$$

Here a_1, a_2 and a_3 are defined by (2.10).

The proof is omitted. The end of proof of Theorem 2.2

From the previous lemma, we see

$$\int_B |\Delta^2 u|^2\,dx = S^2 \int_B \frac{v(|y|)^2}{|y|^N}\,dy + a_1 \frac{|S^{N-1}|}{|S^1|}\int_{B_1^2} |\nabla_2 v(|y|)|^2\,dy$$
$$+ a_2 \frac{|S^{N-1}|}{|S^3|}\int_{B_1^4} |\Delta_4 v(|y|)|^2\,dy + a_3 \frac{|S^{N-1}|}{|S^5|}\int_{B_1^6} |\nabla_6 \Delta_6 v(|y|)|^2\,dy$$
$$+ \frac{|S^{N-1}|}{|S^7|}\int_{B_1^8} |\Delta_8^2 v(|y|)|^2\,dy$$

$$\geq S^2 \int_B \frac{v(|y|)^2}{|y|^N}\,dy + a_1\lambda_1 \frac{|S^{N-1}|}{|S^1|}\int_{B_1^2} |v(|y|)|^2\,dy$$
$$+ a_2\lambda_2 \frac{|S^{N-1}|}{|S^3|}\int_{B_1^4} |v(|y|)|^2\,dy + a_3\lambda_3 \frac{|S^{N-1}|}{|S^5|}\int_{B_1^6} |v(|y|))|^2\,dy$$
$$+ \lambda_4 \frac{|S^{N-1}|}{|S^7|}\int_{B_1^8} |v(|y|)|^2\,dy$$
$$= H(N,\Delta^2)\int_B \frac{u^2}{|x|^8}\,dx + a_1\lambda_1 \int_B \frac{|u|^2}{|x|^6}\,dx$$
$$+ a_2\lambda_2 \int_B \frac{|u|^2}{|x|^4}\,dx + a_3\lambda_3 \int_B \frac{|u|^2}{|x|^2}\,dy + \lambda_4 \int_B |u|^2\,dx$$

This proves the assertion.

6 Sketch of Proofs of Theorem 2.3 and Theorem 2.4

Theorems easily follow from the next lemmas:

Lemma 6.1 *Let Ω be a domain of $\mathbb{R}^N$. Assume that $u \in C_0^\infty(\Omega)$ and $f \in C^2(\Omega)$. Then it holds that*

$$\int_\Omega |\nabla(uf)|^2\,dx = \int_\Omega |\nabla u|^2 f\,dx - \frac{1}{2}\int_\Omega u^2(\Delta(f^2) - 2|\nabla f|^2)\,dx. \tag{6.1}$$

Lemma 6.2 *Let Ω be a domain of $\mathbb{R}^N$. Assume that $u \in C_0^\infty(\Omega)$ and $f \in C^4(\Omega)$. Then it holds that*

$$\begin{aligned}\int_\Omega |\Delta(uf)|^2\,dx &= \int_\Omega (|\Delta u|^2 f^2 + \int_\Omega u^2 f \Delta^2 f)\,dx \\ &\quad + 2\int_\Omega \left(|\nabla u|^2|\nabla f|^2 - 2f\sum_{j,k=1}^N \frac{\partial^2 f}{\partial x_j \partial x_k}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}\right)dx.\end{aligned} \tag{6.2}$$

7 Applications

Let Ω be a bounded domain of $\mathbb{R}^N$. In connection with combustion theory and other applications, many authors have been studied positive solutions of the semi-linear elliptic boundary value problem defined by

$$-\Delta u = \lambda f(u), \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{7.1}$$

Here λ is a nonnegative parameter, and the nonlinearity f is, roughly speaking, continuous, positive, increasing, superlinear and convex function. A typical example is $f(u) = e^u$. It is well-known that there is a finite number λ^* such that (7.1) has a classical positive solution $u \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*$. On the other hand no solution exists, even in the weak sense, for $\lambda > \lambda^*$. This value λ^* is often called the extremal value and solutions for this extremal value are called extremal solutions. It has been a very interesting problem to find and study the properties of these extremal solutions. In this section we shall consider a similar problem for the fourth order equations.

Let B be a unit ball of $\mathbb{R}^N$. Let $f(t, r)$ be a continuous positive function defined for $t \geq 0$ and $r \geq 0$. Moreover we assume that $f(\cdot, r)$ is increasing and strictly convex with

$$f(0, r) > 0 \quad \text{and} \lim_{t\to\infty}\frac{f(t,r)}{t} = 0 \qquad \text{for any } r \geq 0. \tag{7.2}$$

Now we consider the boundary value problem: For $r = |x|$

$$\begin{cases} \Delta^2 u = \lambda f(u,r) & \text{in } B \\ u = \Delta u = 0, & \text{on } \partial B \end{cases} \tag{7.3}$$

This problem is a generalization of (7.1). First we define a weak solution of the problem (7.3).

Definition 7.1 *(Weak solution of (7.3))*
Let us set $\delta(x) = dist(x, \partial B)$ (the distance to the boundary from x). A function $u \in L^1(B)$ is called a weak solution of (7.3) if $f(u, |x|)$ satisfy

$$\delta(x) f(u, |x|) \in L^1_{loc}(B) \tag{7.4}$$

and u satisfies (7.3) in the following weak sense:

$$\int_B (u\Delta^2\varphi - \lambda f(u,r)\varphi)\, dx = 0 \tag{7.5}$$

for all $\varphi \in C^4(\overline{B})$ with $\varphi = \Delta\varphi = 0$ on ∂B.

From the standard elliptic regularity theory it follows that bounded weak solutions for this problem are classical solutions. Moreover u satisfies the boundary conditions $u = \Delta u = 0$ in this case. Now we consider unbounded solutions. To this end we introduce an energy solution and a singular energy solution.

Definition 7.2 (Energy solution, singular energy solution)
A weak solution u of (7.3) is said to be an energy solution if $u \in H^2(B) \cap H^1_0(B)$. If an energy solution u is not bounded, u is said to be singular.

Remark 7.1 *Later we shall specify the nonlinearity $f(u,r)$ in order to study singular extremal solutions precisely. From the definition, an energy solution u satisfies*

$$\int_B (\Delta u \Delta\varphi - \lambda f(u, |x|)\varphi)\, dx = 0 \tag{7.6}$$

for all $\varphi \in C^2(\overline{B})$ with $\varphi = \Delta\varphi = 0$ on ∂B.
If $u \in H^4(B)$ and u is an energy solution of (7.3), then u satisfies the boundary conditions $u = \Delta u = 0$.

It is not difficult to see that the maximum principle works in this boundary value problem,even if the operator is of the fourth order. Therefore we can show that there exists a solution to (7.3) for sufficiently small $\lambda > 0$. In fact we can construct so-called supersolution and subsolution as follows.

Lemma 7.1 *Under these assumptions, there exist a supersolution and a subsolution for a sufficiently small $\lambda > 0$. Moreover there exists at least one classical solution u of (7.3).*

The proof is omitted.

By virtue of this, we can define the mimimal solution $u_\lambda \in C^4(\overline{B})$ which is minimal among all possible solutions. Then we define the extremal value λ^* as a upper bound of λ for which the minimal solution exists. The family of such solutions depends smoothly and monotonically on λ. Then the following property is well known.

Lemma 7.2 *Minimal solutions are stable. More precisely, the linearized operator*

$$L_\lambda \varphi = \Delta^2 \varphi - \lambda f'(u_\lambda, r)\varphi \tag{7.7}$$

has a positive first eigenvalue for all $0 < \lambda < \lambda^*$.

From the properties $\lim_{t\to\infty} \frac{f(t,r)}{t} = \infty$ and $\frac{f(t,r)}{t} \leq f'(t,r)$, we can show the following:

Lemma 7.3 *As $\lambda \to \lambda^*$, a finite limit $u^*(x) = \lim_{\lambda\to\lambda^*} u_\lambda(x)$ and u^* is a weak solution of (7.3) with $\lambda = \lambda^*$.*

The proof is omitted.

The limit u^* can be classical or singular. Assume that u^* is a classical solution. From the implicit function theorem, it is clear that the linearized operator

$$L_{\lambda^*} \varphi = \Delta^2 \varphi - f'(u^*, r)\varphi \tag{7.8}$$

has zero first eigenvalue.

If u^* is singular, then we have the following characterizations:

Proposition 7.1 *Assume that $u \in H^2(B) \cap H_0^1(B)$ is an unbounded weak solution of (7.3) for some $\lambda > 0$. Assume that*

$$\lambda \int_B f'(u, r)\varphi^2 \, dx \leq \int_B |\Delta \varphi|^2 \, dx \tag{7.9}$$

for all $\varphi \in C_0^2(B)$. *Then* $\lambda = \lambda^*$ *and* $u = u^*$.

Conversely, if $\lambda = \lambda^*$ *and* $u = u^*$, *then* (7.13) *holds.*

The proof is omitted.

Remark 7.2 *If* $f(u,r)$ *satisfies*

$$\liminf_{t\to\infty} \frac{f'(t,r)t}{f(t,r)} > 1 \qquad (\text{ uniformly in } r \in [0,1]), \tag{7.10}$$

then any extremal solution u^* *lies in the energy class (c.f. §3 in [3]).*

Now we consider the concrete example for which we can apply our refined Hardy inequalities. For $1 < p < \infty$ and $r = |x|$, we adopt as the nonlinearity $f(u,r)$ the following f_p and f_e, that is,

$$\begin{cases} f_p(u,r) = (1 + u + Q_p(r))^p \\ f_e(u,r) = e^{u+Q_e(r)}. \end{cases} \tag{7.11}$$

Here

$$\begin{cases} Q_p(r) = \beta(1 - r^2), \\ \lambda_N(p) = \alpha(\alpha - 2)(N + \alpha - 2)(N + \alpha - 4), \\ \alpha = -\frac{4}{p-1}, \quad \beta = \frac{2(N-2)}{N(p-1)^2}\left(p - \frac{N+2}{N-2}\right). \end{cases} \tag{7.12}$$

We define the function U_p as follows:

$$U_p(r) = r^\alpha - 1 - Q_p(r), \quad \alpha = -\frac{4}{p-1}. \tag{7.13}$$

Under these notations, we have the following.

Lemma 7.4 *Assume that* $\lambda = \lambda_N(p)$ *and* $f = f_p$. *Then it holds that:*

1. *If* $p > \frac{N}{N-4}$, *then* U_p *is a weak solution of (7.3).*
2. *If* $p > \frac{N+4}{N-4}$, *then* U_p *is a singular energy solution of (7.3) .*
3. *If* $p > \frac{N}{N-8}$, *then* $U_p \in H^4(B)$.

Now we define

$$H(p) = p\lambda_N(p) \tag{7.14}$$

Since it holds that

$$\lim_{p\to+\infty} H(p) = 8(N-2)(N-4), \tag{7.15}$$

we see $\lim_{p\to+\infty} H(p) \le \left(\frac{N(N-4)}{4}\right)^2$ if and only if $N \ge 13$. For $N > 4$ we also note that $H(\frac{N-4}{N+4}) > \left(\frac{N(N-4)}{4}\right)^2$ and that $H(p)$ is monotonously decreasing for $p \ge \frac{N-4}{N+4}$. Then the results of Section 2 allow us to study the singular energy solutions.

Theorem 7.1 (Polynomial case) *Assume that $N \ge 13$.*

(1) There exists a number $p^* \in (\frac{N+4}{N-4}, \infty)$ such that U_p is a singular extremal solution with $\lambda^* = \lambda_N(p)$ for any $p \ge p^*$.

(2) If $p \in (\frac{N+4}{N-4}, p^*)$, the U_p is not a singular extremal solution and $\lambda_N(p) < \lambda^*$. Here p^* is the same number in (1).

(3) If $p \in (\frac{4}{N-4}, \frac{N+4}{N-4}]$, U_p is not an energy solution but a weak solution. Therefore U_p is not singular extremal and $\lambda_N(p) < \lambda^*$.

Remark 7.3 *In the case that $N \ge 13$ and $p > p^*$, the linealized operator L^p_λ defined by*

$$\begin{aligned} L^p_\lambda \varphi &= \Delta^2\varphi - \lambda f_p'(U_p, r)\varphi \\ &= \Delta^2\varphi - p\lambda\frac{\varphi}{r^4} \end{aligned} \tag{7.16}$$

has a positive first eigenvalue $\mu(\lambda)$ for any $\lambda \in (0, \lambda_N(p)]$ corresponding to an eigenfunction $\varphi \in H^2(B) \cap H^1_0(B)$. In order to characterize the first eigenvalue we may consider the variational inequality

$$\begin{aligned} &\int_B |\Delta\varphi|^2\,dx - \lambda_N(p)\int_B f_p'(U_p, r)\varphi^2\,dx \\ &= \int_B \left(|\Delta\varphi|^2 - H(p)\frac{\varphi^2}{r^4}\right)dx \\ &\ge \left(1 - \frac{16H(p)}{(N(N-4))^2}\right)\int_B |\Delta\varphi|^2\,dx \end{aligned} \tag{7.17}$$

Therefore we see

$$\mu(\lambda_N(p)) \ge \left(1 - \frac{16H(p)}{(N(N-4))^2}\right)\mu_1, \tag{7.18}$$

where μ_1 is the first eigenvalue of Δ^2 with the boundary condition $\varphi = \Delta\varphi = 0$

If $p = p^*$, then $L^p_{\lambda_{N(p)}}$ does not have a first eigenfunction in $H^2(B) \cap H^1_0(B)$. However, the previous arguement gives a positive value for $\mu(\lambda_N(p))$ defined as

$$\mu(\lambda_{N(p)}) = \lim_{\lambda \to \lambda_{N(p)}} \mu(\lambda) \geq \lambda_2.$$

Remark 7.4 *We consider the case that $4 < N < 13$. Assume that $p > \frac{N-4}{N+4}$. Then U_p is not singular extremal, since the Hardy inequality (7.13) does not holds. In the next we assume that $p \leq \frac{N-4}{N+4}$. Then U_p is not an energy solution but a (singular) weak solution. Therefore we see that there exists a range of p where U_p is a weak solution and satisfies the Hardy inequality (7.13).*

In the next we consider the limit of this problem as $p \to +\infty$. Let us set

$$\begin{cases} Q_e(r) = \frac{2(N-2)}{N}(1 - r^2), \\ \lambda_N^e = 8(N-2)(N-4), \end{cases} \tag{7.19}$$

and we set

$$U_e = -4 \log r - Q_e(r) \tag{7.20}$$

As $p \to +\infty$ we see that

$$\Big(pQ_p(r), f_p(\frac{u}{p}, r), p\lambda_N(p), pU_p\Big) \longrightarrow \Big(Q_e(r), f_e(u,r), \lambda_N^e, U_e\Big) \tag{7.21}$$

for any $r \in (0,1)$.

Therefore the boundary value problem (7.3) with $\lambda = \lambda_N^e$ and $f = f_e$ is considered as a formal limit of the previous one.

Lemma 7.5 *Assume that $\lambda = \lambda_N^e$ and $f = f_e$. Then it holds that:*

1. *If $N > 4$, U_e is a singular energy solution of (7.3).*
2. *If $N > 8$ then $U_e \in H^4(B)$.*

Then we have the following:

Theorem 7.2 (Exponential case)
(1) If $N \geq 13$, then U_e is a singular extremal solution with $\lambda^* = \lambda_N^e$.
(2) If $N < 13$, then U_e is not a singular extremal solution and $\lambda_N^e < \lambda^*$.

Remark 7.5 *In the case that $N \geq 13$, the linealized operator $L^e_{\lambda^*}$ defined by*

$$L^e_{\lambda^*}\varphi = \Delta^2\varphi - \lambda^e_N f'_e(U_e, r)\varphi \tag{7.22}$$

$$= \Delta^2\varphi - \lambda^e_N \frac{\varphi}{r^4} \tag{7.23}$$

has a positive first eigenvalue $\mu(\lambda^e_N)$ as before.

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