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ON THE ANTIMAXIMUM PRINCIPLE FOR THE P-LAPLACIAN WITH INDEFINITE WEIGHT

T. GODOY
FAMAF, Univ. Nacional Córdoba,
Ciudad Universitaria,
5000 Córdoba, Argentina.
e-mail :godoy@mate.uncor.edu

J.-P. GOSSEZ
Département de Mathématique, C. P. 214,
Université Libre de Bruxelles,
1050 Bruxelles, Belgium.
e-mail :gossez@ulb.ac.be

S. PACZKA
FAMAF, Univ. Nacional Córdoba,
Ciudad Universitaria,
5000 Córdoba, Argentina.
e-mail :paczka@mate.uncor.edu

Abstract

This paper is concerned with the antimaximum principle for the quasilinear problem $-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x)$, under Dirichlet or Neumann boundary conditions. Here Δ_p is the p-laplacian and $m(x)$ is a weight function which may change sign. We will in particular investigate the question of the uniformity of this principle and provide a variational characterization for the interval of uniformity. An identity of Picone's type for the p-laplacian plays an important role in our approach.

KEY WORDS AND PHRASES. Antimaximum principle, indefinite weight, prinal eigenvalue, Picone's identity, p-laplacian.

1991 MATHEMATICS SUBJECT CLASSIFICATIONS : 35J20, 35P05, 35B50, 35J60.

1 Introduction

This paper is concerned with the study of the antimaximum principle (in brief AMP) for the problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, whose smoothness will be specified later, $\Delta_p u :=$ div $((|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, is the p-laplacian and $Bu = 0$ represents either the Dirichlet or the Neumann homogeneous boundary conditions.

The original form of the AMP concerns the case where $p = 2$ (linear operator) and $m \equiv 1$ (no weight). It reads as follows : given $h \underset{\not\equiv}{\geq} 0$ there exists $\delta = \delta(h) > 0$ such that if $\lambda_1 < \lambda < \lambda_1 + \delta$, then any solution u of (1.1) (with $p = 2$ and $m \equiv 1$) satisfies $u < 0$ in Ω (cf. [8]). We will refer to such a situation by saying that "the AMP holds at the right of λ_1". Here λ_1 denotes the principal eigenvalue of $-\Delta$ under the corresponding boundary condition. It is also shown in [8] that δ can be taken independently of h for the Neumann problem in dimension $N = 1$. In such a situation we will say that "the AMP holds uniformly at the right of λ_1".

The AMP was extended in [17] to the case of a linear operator with weight, i.e. (*) $p = 2$ and m indefinite in (1.1). The proof in [17] involves as in [8] estimating the projections of the solution onto the eigenspace associated to the principal eigenvalue and onto one of its complementary subspaces. The AMP was also extended in [13] to the case of a nonlinear operator without weight, i.e. (**) $1 < p < \infty$ and $m \equiv 1$ in (1.1). The argument here is quite different. It goes by contradiction and involves a preliminary nonexistence result whose proof uses Diaz-Saa's inequality. Further investigations in each of the two cases (*) and (**) were carried out recently in [16] and [5] respectively.

It is our purpose in this paper to study the general situation of a nonlinear operator with weight, i.e. $1 < p < \infty$ and m indefinite in (1.1). We will in particular investigate the question of the uniformity of the AMP and provide a variational characterization for the interval of uniformity.

To give an idea of our results, let us consider in (1.1) the Neumann problem with a weight m which changes sign in Ω. Suppose first $\int_\Omega m \neq 0$, say $\int_\Omega m < 0$. It is then known that there are two principal eigenvalues : 0 and a positive one which we denote by λ^* (cf. [19], [10] as well as section 2 below). We show that the AMP holds at the right of λ^* and at the left of 0. Moreover it is non uniform when $p \leq N$ and uniform when $p > N$. In the latter case, the intervals of uniformity are exactly $\lambda^* < \lambda \leq \bar{\lambda}(m)$ and $-\bar{\lambda}(-m) \leq \lambda < 0$, where

$$\bar{\lambda}(m) := \inf\{\int_\Omega |\nabla u|^p \ : \ u \in W^{1,p}(\Omega), \quad \int_\Omega m|u|^p = 1 \text{ and } u \text{ vanishes somewhere in } \bar{\Omega}\}. \tag{1.2}$$

We also show in this latter case that the AMP still holds at the right of $\bar{\lambda}(m)$ and at the left of $-\bar{\lambda}(-m)$, of course now non uniformly. Suppose now $\int_\Omega m = 0$. In this singular case, 0 is the unique principal eigenvalue. We show that the AMP holds at the right and at the left of 0. Moreover it is non uniform when $p \leq N$ and uniform when $p > N$. In the latter case the intervals of uniformity are exactly $0 < \lambda \leq \bar{\lambda}(m)$ and $-\bar{\lambda}(-m) \leq \lambda < 0$, with $\bar{\lambda}(m)$ as in (1.2). In this latter case also the AMP still holds (non uniformly) at the right of $\bar{\lambda}(m)$ and at the left of $-\bar{\lambda}(-m)$. We will also see that the AMP cannot hold far away to the right of $\bar{\lambda}(m)$ or to the left of $-\bar{\lambda}(-m)$. This is true for all p, with a suitable extension of definition (1.2) for $p \leq N$ (cf. formula (3.1)).

In each of the two particular cases (*) and (**) considered above, our present results reduce to those in [16] and [5] respectively. Some our arguments of course are inspired from [16], [5]. The main difference occurs in the proof of the non uniformity and, in case of uniformity, in the proof of the sharpness of $\bar{\lambda}$. Indeed, in the case (**) considered in [5], the proof of these facts was based on some properties of the asymptotic behaviour of the first curve of the corresponding Fučik spectrum. But it was observed recently that these properties are not valid anymore in the presence of a general weight (cf. [1], [4]). This difficulty was bypassed in the case (*) considered in [16] through some argument which involves "completing a square" (cf. formula (2.9) in [16]). This latter argument of course does not extend to the nonlinear case. But it turns out that one of its consequences can be suitably adapted and derived for the p-laplacian, which suffices for our purposes. This is the inequality provided by Lemmas 2.5 and 2.12 (as well as 4.2). Its proof uses an identity of Picone's type for the p-laplacian which was established recently in [2]. We observe that this inequality also enters the proof of the preliminary nonexistence results which are used to derive the AMP itself (cf. Propositions 2.4 and 2.7, as well as 4.1 and 4.3).

Our results relative to the Neumann problem, as briefly described above, are given in details in section 3. The case of the Dirichlet problem is considered in section 5. We show in particular that for the Dirichlet problem, whatever the weight and whatever p, the AMP is always nonuniform. This should be compared with the recent result of [4] which says that for the Dirichlet problem, if the weight has compact support in Ω and if $p > N$, then the first curves in the corresponding Fučik spectrum are not asymptotic to the trivial horizontal and vertical lines of that spectrum (cf. also [1] when $p = 2$ and $N = 1$).

In sections 2 and 4 we collect some preliminary results on the principal eigenvalues for the Neumann and Dirichlet problems respectively. Less regularity on the domain is needed in parts of these two sections.

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2 Principal eigenvalues in the Neumann case

Part of this paper is concerned with the Neumann problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x) \text{ in } \Omega, \quad \partial u/\partial \nu = 0 \text{ on } \partial\Omega. \tag{2.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $\partial/\partial\nu$ represents, at least formally, the derivative of u in the direction of the unit exterior normal to $\partial\Omega$. The real-valued functions m and h will always be assumed to belong to $L^\infty(\Omega)$, with, unless otherwise stated, the assumption that m changes sign in Ω, i.e.

$$\text{meas}\{x \in \Omega : m(x) > 0\} > 0 \text{ and } \text{meas}\{x \in \Omega : m(x) < 0\} > 0. \tag{2.2}$$

Also, without loss of generality, we can assume

$$|m(x)| < 1 \quad \text{a. e. in } \Omega. \tag{2.3}$$

Note that more regularity on Ω will be required later.

Solutions of (2.1) (or of (2.6) below) are always understood in the weak sense : $u \in W^{1,p}(\Omega)$ with

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \lambda \int_\Omega m|u|^{p-2}u\varphi + \int_\Omega h\varphi \quad \forall\varphi \in W^{1,p}(\Omega). \tag{2.4}$$

Adapting to the Neumann problem the L^∞ estimates of [3] and using the regularity results of [11], one has that any solution of (2.1) (or (2.6) below) belongs to $L^\infty(\Omega) \cap C^1(\Omega)$.

Our purpose in this preliminary section is to collect some results relative to the principal eigenvalues of

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \text{ in } \Omega, \quad \partial u/\partial\nu = 0 \text{ on } \partial\Omega. \tag{2.5}$$

Some of these results can be found in [19], [10], although not with the same approach nor with the same degree of generality. For the sake of completeness and for later references, some proofs will be sketched.

The fundamental tool is the following form of the maximum principle.

Proposition 2.1 *Let u be a solution of*

$$-\Delta_p u + a_0(x)|u|^{p-2}u = h \text{ in } \Omega, \quad \partial u/\partial\nu = 0 \text{ on } \partial\Omega, \tag{2.6}$$

where $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$, $h \in L^\infty(\Omega)$, $h \underset{\not\equiv}{\geq} 0$. Then

$$u > 0 \text{ in } \Omega \tag{2.7}$$

Proof. As observed above $u \in C^1(\Omega)$ and so (2.7) makes sense in the usual way. Writing $u = u^+ - u^-$ with $u^\pm = \max\{\pm u, 0\}$ and taking $-u^-$ as testing function in (2.6), one deduces $u \geq 0$ in Ω. The maximum principle of [24] then implies $u > 0$ in Ω. Q. E. D.

We are thus interested in the principal eigenvalues of (2.5). Clearly 0 is a principal eigenvalue, with the nonzero constants as eigenfunctions. We also observe that if $u \underset{\not\equiv}{\geq} 0$ is a solution of (2.1) with $h \geq 0$ (for instance an eigenfunction of (2.5) associated to a principal eigenvalue), then $u > 0$ in Ω. (This follows from Proposition 2.1 by writing equation (2.1) as $-\Delta_p u \pm \lambda |u|^{p-2}u = \lambda(m \pm 1)|u|^{p-2}u + h$ and using (2.3); here $+$ is used if $\lambda \geq 0$, $-$ if $\lambda < 0$).

The following expression will play a central role in our approach :

$$\lambda^*(m) := \inf\{\int_\Omega |\nabla u|^p : u \in W^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p = 1\}. \tag{2.8}$$

Proposition 2.2 *(i) Suppose $\int_\Omega m < 0$. Then $\lambda^*(m) > 0$ and $\lambda^*(m)$ is the unique nonzero principal eigenvalue; moreover the interval $]0, \lambda^*(m)[$ does not contain any eigenvalue. (ii) Suppose $\int_\Omega m \geq 0$. Then $\lambda^*(m) = 0$; moreover, if $\int_\Omega m = 0$, then 0 is the unique principal eigenvalue.*

Proposition 2.2 of course also applies to the weight $-m$. In particular, if $\int_\Omega m > 0$, then $-\lambda^*(-m)$ is the unique nonzero principal eigenvalue of (2.5).

The statements relative to the unicity of the principal eigenvalues in Proposition 2.2 follow from Proposition 2.4 below. The proof of the remaining parts of Proposition 2.2 can be easily adapted from that of an analogous result in [16]. It uses the following lemma, whose proof is also easily adapted from that of a corresponding lemma in [16].

Lemma 2.3 *Assume $\int_\Omega m < 0$. Then there exists a constant $c > 0$ such that $\int_\Omega |\nabla u|^p \geq c\int_\Omega |u|^p$ for all $u \in W^{1,p}(\Omega)$ with $\int_\Omega m|u|^p > 0$.*

Proposition 2.4 *Suppose $\int_\Omega m \leq 0$. If $\lambda \notin [0, \lambda^*(m)]$, then problem (2.1) with $h \geq 0$ has no solution $u \underset{\not\equiv}{\geq} 0$.*

Proof. Assume that there exists a solution $u \underset{\not\equiv}{\geq} 0$ of (2.1) for some $\lambda \in \mathbb{R}$ and some $h \geq 0$. Applying Proposition 2.1, we get $u > 0$ in Ω. So Lemma 2.5 below can be applied, which gives

$$\lambda \int_\Omega m|\varphi|^p \leq \int_\Omega |\nabla \varphi|^p$$

for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega)$ with $\varphi \geq 0$. By density this inequality still holds for all $\varphi \in W^{1,p}(\Omega)$. This implies $\lambda \leq \lambda^*(m)$ as well as $-\lambda \leq \lambda^*(-m)$. Since $\int_\Omega(-m) \geq 0$, one has $\lambda^*(-m) = 0$ by Proposition 2.2, and we conclude $\lambda \in [0, \lambda^*(m)]$. Q. E. D.

Lemma 2.5 *Let u be a solution of (2.1) with $h \geq 0$ and $u > 0$ in Ω. Then, for any $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega)$ with $\varphi \geq 0$, one has $h\varphi^p/u^{p-1} \in L^1(\Omega)$ and*

$$\lambda \int_\Omega m\varphi^p + \int_\Omega h\varphi^p/u^{p-1} \leq \int_\Omega |\nabla \varphi|^p. \tag{2.9}$$

Moreover equality holds in (2.9) if and only if φ is a multiple of u.

Proof. It is inspired from [2] (which deals with the Dirichlet problem). For $u, \varphi \in C^1(\Omega)$ with $u > 0$ and $\varphi \geq 0$ in Ω, denote

$$\begin{aligned} R(\varphi, u) &:= |\nabla\varphi|^p - |\nabla u|^{p-2}\nabla u \nabla\left(\varphi^p/u^{p-1}\right), \\ L(\varphi, u) &:= |\nabla\varphi|^p + (p-1)\frac{\varphi^p}{u^p}|\nabla u|^p - p\frac{\varphi^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla u \nabla\varphi. \end{aligned}$$

The following version of Picone's identity is proved in [2] : $R(\varphi, u) = L(\varphi, u) \geq 0$ in Ω, with moreover $L(\varphi, u) = 0$ in Ω if and only if φ is a multiple of u. (The equality of $R(\varphi, u)$ with $L(\varphi, u)$ follows by direct calculation, and the rest can be deduced from Minkowski's inequality). Let now u and φ be as in the statement of Lemma 2.5. Applying the above to $u + \epsilon$ with $\epsilon > 0$ and to φ, we obtain, for Ω_0 a domain with compact closure in Ω,

$$\begin{aligned} 0 &\leq \int_{\Omega_0} L(\varphi, u+\epsilon) \leq \int_\Omega L(\varphi, u+\epsilon) = \int_\Omega R(\varphi, u+\epsilon) \\ &= \int_\Omega |\nabla\varphi|^p - \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\varphi^p/(u+\epsilon)^{p-1}\right) \\ &= \int_\Omega |\nabla\varphi|^p - \lambda\int_\Omega m\left(\frac{u}{u+\epsilon}\right)^{p-1}\varphi^p - \int_\Omega h\frac{\varphi^p}{(u+\epsilon)^{p-1}}, \end{aligned}$$

where we have used that $\varphi^p/(u+\epsilon)^{p-1}$ belongs to $W^{1,p}(\Omega)$ and consequently is an admissible testing function in (2.1). Letting $\epsilon \downarrow 0$, one applies first the dominated convergence theorem to $\int_{\Omega_0} L(\varphi, u+\epsilon)$ and to $\int_\Omega m(u/(u+\epsilon))^{p-1}\varphi^p$, and then the monotone convergence theorem to $\int_\Omega h\varphi^p/(u+\epsilon)^{p-1}$. This yields $h\varphi^p/u^{p-1} \in L^1(\Omega)$ and

$$0 \leq \int_{\Omega_0} L(\varphi, u) \leq \int_\Omega |\nabla\varphi|^p - \lambda\int_\Omega m\varphi^p - \int_\Omega h\varphi^p/u^{p-1}. \tag{2.10}$$

So (2.9) follows. Moreover, if equality holds in (2.9), then, by (2.10), $L(\varphi, u) = 0$ on Ω_0, and so on Ω since Ω_0 is arbitrary. The conclusion that φ is a multiple of u then follows. Q. E. D.

Remark 2.6 Taking φ^p/u^{p-1} as testing function in the study of the p-laplacian is a well-known technical device (cf. e.g. [12]). This device is already present for $p = 2$ in [18], although in a non explicit way.

Proposition 2.7 *Suppose $\int_\Omega m \leq 0$. Then problem (2.1) with $h \gneqq 0$ does not admit any solution if $\lambda = 0$ or $\lambda = \lambda^*(m)$. It admits an unique solution, which is > 0 in Ω, if $0 < \lambda < \lambda^*(m)$.*

Proof. Nonexistence when $\lambda = 0$ immediately follows by taking $\varphi = 1$ as testing function in (2.1). Nonexistence in the case $\lambda = \lambda^*(m)$ requires more care. Assume by contradiction that (2.1) with $\lambda = \lambda^*(m)$ has a solution u. We first show that $u \geq 0$ in Ω. Indeed if $u^- \not\equiv 0$, then taking $-u^-$ as testing function in (2.1) gives

$$\int_\Omega |\nabla u^-|^p = \lambda^*(m) \int_\Omega m|u^-|^p - \int_\Omega hu^-$$

and consequently, since $h \geq 0$, u^- is a minimizer in the definition of $\lambda^*(m)$ and $\int_\Omega hu^- = 0$. But then, by Lagrange multipliers, u^- solves

$$-\Delta_p u^- = \lambda^*(m) m |u^-|^{p-2} u^- \text{ in } \Omega, \quad \partial u^-/\partial\nu = 0 \text{ on } \partial\Omega,$$

and consequently, by Proposition 2.1 applied to $-\Delta_p u^- + \lambda^*(m)|u^-|^{p-2}u^- = \lambda^*(m)(m+1)|u^-|^{p-2}u^-$, u^- is > 0 in Ω, which contradicts $\int_\Omega hu^- = 0$. So $u \geq 0$ in Ω, and applying once more Proposition 2.1, one gets $u > 0$ in Ω. Lemma 2.5 can thus be applied, which gives

$$\lambda^*(m) \int_\Omega m\varphi^p + \int_\Omega h\varphi^p/u^{p-1} \leq \int_\Omega |\nabla\varphi|^p$$

for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cup C^1(\Omega)$ with $\varphi \geq 0$. Taking for φ a positive eigenfunction associated to $\lambda^*(m)$, we deduce $\int_\Omega h\varphi^p/u^{p-1} \leq 0$, which is impossible since $\varphi > 0$ in Ω and $h \underset{\not\equiv}{\geq} 0$.

We now consider (2.1) with $0 < \lambda < \lambda^*(m)$ and first prove the existence of a solution. This can be done for instance by minimization of the functional

$$\Phi(u) := \int_\Omega |\nabla u|^p - \lambda \int_\Omega m|u|^p - p \int_\Omega hu.$$

The existence of a minimum for Φ (and consequently of a solution to (2.1)) will follow by standard arguments if we show that Φ is coercive. For that purpose first note that Proposition 2.2 implies $\int_\Omega m < 0$ (since $\lambda^*(m) > 0$). We will distinguish two cases : $u \in A$ or $u \in B$, where A (resp. B) denotes the set of those $u \in W^{1,p}(\Omega)$ such that $\int_\Omega m|u|^p > 0$ (resp. ≤ 0). For $u \in A$ one has, using $0 < \lambda < \lambda^*$ and Lemma 2.3,

$$\begin{aligned} \Phi(u) &\geq \left(1 - \frac{\lambda}{\lambda^*(m)}\right) \int_\Omega |\nabla u|^p - p \int_\Omega hu \\ &\geq c_1 \int_\Omega |\nabla u|^p + c_2 \int_\Omega |u|^p - p \int_\Omega hu \end{aligned}$$

for some constants $c_1, c_2 > 0$. So Φ is coercive on A. For $u \in B$ one has, using $\lambda > 0$ and Lemma 2.8 below,

$$\Phi(u) \geq c_3 \int_\Omega |\nabla u|^p + c_4 \int_\Omega |u|^p - p \int_\Omega hu$$

for some constants $c_3, c_4 > 0$. So Φ is also coercive on B. The existence of at least one solution to (2.1) is thus proved. Now if u is a solution of (2.1), taking as before $-u^-$ as testing function and applying Proposition 2.1, one gets $u > 0$ in Ω. We will now prove unicity. Suppose that v is another solution of (2.1). Applying Lemma 2.5 to $-\Delta_p u = \lambda m|u|^{p-2}u + h$ with $\varphi = v$ gives

$$\lambda \int_\Omega mv^p + \int_\Omega hv^p/u^{p-1} \leq \int_\Omega |\nabla v|^p = \lambda \int_\Omega mv^p + \int_\Omega hv. \tag{2.11}$$

Consequently

$$\int_\Omega hv\left(1 - \frac{v^{p-1}}{u^{p-1}}\right) \geq 0.$$

Interchanging u and v and adding, we get

$$\int_\Omega h\left[v(1 - \frac{v^{p-1}}{u^{p-1}}) + u\left(1 - \frac{u^{p-1}}{v^{p-1}}\right)\right] \geq 0. \tag{2.12}$$

But the bracket $[\ldots]$ in (2.12) is ≤ 0, which implies that equality holds in (2.12). It follows that equality also holds in (2.11). Lemma 2.5 then yields that $v = cu$ in Ω for some constant c. Using in (2.1) the fact that $h \not\equiv 0$ finally gives $c = 1$, i.e. $v = u$. Q. E. D.

Lemma 2.8 *Assume $\int_\Omega m \neq 0$ and let $\lambda > 0$. Then there exists a constant $c > 0$ such that $\int_\Omega |\nabla u|^p - \lambda \int_\Omega m|u|^p \geq c \int_\Omega |u|^p$ for all $u \in B := \{u \in W^{1,p}(\Omega) : \int_\Omega m|u|^p \leq 0\}$.*

Proof. Assume by contradiction that for each $k = 1, 2, \ldots,$ there exists $u_k \in B$ such that $\int_\Omega |\nabla u_k|^p - \lambda \int_\Omega m|u_k|^p \leq 1/k \int_\Omega |u_k|^p$. Considering $v_k := u_k/||u_k||_p$, one has

$$0 \leq \int_\Omega |\nabla v_k|^p \leq \int_\Omega |\nabla v_k|^p - \lambda \int_\Omega m|v_k|^p \to 0.$$

It follows that for a subsequence, v_k converges in $W^{1,p}(\Omega)$ to a nonzero constant function v, which satisfies $-\lambda \displaystyle\int_\Omega m|v|^p = 0$. This contradicts $\displaystyle\int_\Omega m \neq 0$. Q. E. D.

Proposition 2.9 *Suppose $\int_\Omega m \leq 0$. The principal eigenvalues* 0 *and* $\lambda^*(m)$ *are simple.*

Proof. This is clearly true for $\lambda = 0$. So let us consider $\lambda^*(m)$. If u is an eigenfunction associated to $\lambda^*(m)$, then standard arguments as above based on Proposition 2.1 give that if $u^- \equiv 0$ then $u > 0$ in Ω and if $u^- \not\equiv 0$ then $u < 0$ in Ω. Similarly for another eigenfunction v associated to $\lambda^*(m)$. So replacing if necessary u or v by $-u$ or $-v$, we can assume $u > 0$ and $v > 0$. Applying Lemma 2.5 to $-\Delta_p u = \lambda^*(m)|u|^{p-2}u$ with $\varphi = v$ then gives

$$\lambda^*(m) \int_\Omega mv^p \leq \int_\Omega |\nabla v|^p. \tag{2.13}$$

In fact equality holds in (2.13) since v is an eigenfunction associated to $\lambda^*(m)$. Consequently, by Lemma 2.5, v is a multiple of u. Q. E. D.

Remark 2.10 The above results can easily be adapted to the simpler case where m does not change sign in Ω, say $m \gneqq 0$. In this case 0 is the unique principal eigenvalue. Problem (2.1) with $h \gneqq 0$ has no solution $u \geq 0$ if $\lambda > 0$, and no solution at all if $\lambda = 0$; its (unique) solution is > 0 in Ω if $\lambda < 0$.

More regularity on Ω will be required to study the AMP. In the final part of this section, we assume Ω of class $C^{1,1}$ and indicate briefly how some of the previous results should be modified.

Under this stronger assumption on Ω, any solution u of (2.4) belongs to $C^{1,\gamma}(\bar{\Omega})$ for some $\gamma = \gamma(N, p, M) \in]0, 1[$, where M is a bound for $|\lambda|$, $||m||_\infty$ and $||h||_\infty$; moreover the following estimate holds :

$$||u||_{C^{1,\gamma}(\bar{\Omega})} \leq C = C(\Omega, N, p, M, M') \tag{2.14}$$

where M' is a bound for $||u||_\infty$ (cf. [20]). One also has that if u solves (2.4), then

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \text{ in the usual pointwise sense.} \tag{2.15}$$

The proof of (2.15) is given in the annex. These above considerations on the regularity of the solutions and on the meaning of the boundary condition of course also apply to solutions of (2.6).

The maximum principle of Proposition 2.1 can be strengthened in the following way.

Proposition 2.11 *Let u be a solution of (2.6) with $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$, $h \in L^\infty(\Omega)$, $h \gneqq 0$. Then $u > 0$ in $\bar{\Omega}$.*

Proof. Arguing as in the proof of Proposition 2.1, one deduces from [24] that $u > 0$ in Ω with $\partial u/\partial \nu < 0$ at the points of $\partial\Omega$ where $u = 0$ (since a $C^{1,1}$ domain satisfies the interior ball condition). But by (2.15) above, $\partial u/\partial \nu = 0$ on $\partial\Omega$ in the usual pointwise sense. Consequently u cannot vanish on $\partial\Omega$ and so $u > 0$ on $\bar{\Omega}$. Q. E. D.

It follows as before from Proposition 2.11 that any solution $u \gneqq 0$ of (2.1) with $h \geq 0$ satisfies $u > 0$ in $\bar{\Omega}$.

The inequality of Lemma 2.5 remains valid without any restriction on the sign of h, which will be useful later in the proof of Theorem 3.5. More precisely we have

Lemma 2.12 *Let u be a solution of (2.1) with $u > 0$ in $\bar{\Omega}$. Then*

$$\lambda \int_\Omega m|\varphi|^p + \int_\Omega h|\varphi|^p/u^{p-1} \leq \int_\Omega |\nabla\varphi|^p \tag{2.16}$$

for any $\varphi \in W^{1,p}(\Omega)$.

Proof. One first derives (2.16) for $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega)$ with $\varphi \geq 0$. The argument here can in fact be slightly simplified with respect to that in the proof of Lemma 2.5 since $\varphi^p/u^{p-1} \in W^{1,p}(\Omega)$ and so there is no need to introduce $u+\epsilon$ nor Ω_0. One then deduces (2.16) for $\varphi \in W^{1,p}(\Omega)$ by a standard density argument. Q. E. D.

Finally, in Proposition 2.7, the solution u is > 0 in $\bar{\Omega}$ when $0 < \lambda < \lambda^*(m)$.

3 Antimaximum Principle in the Neumann case

We consider in this section problem (2.1) with Ω of class $C^{1,1}$ and m, h as before, i.e. in $L^\infty(\Omega)$ with (2.2), (2.3). The following expression will play an important role in our study of the AMP :

$$\bar{\lambda}(m) := \inf\{\int_\Omega |\nabla u|^p : u \in W^{1,p}(\Omega),\ \int_\Omega m|u|^p = 1 \text{ and } u \text{ vanishes on some ball in } \Omega\}. \tag{3.1}$$

It is easily seen that when $p > N$, this definition coincides with that given in (1.2). (This follows from the easily verified fact that if $p > N$ and $u \in W^{1,p}(\Omega)$ is ≥ 0 and vanishes at $x_0 \in \bar{\Omega}$, then $(u-\epsilon)^+$ vanishes on some ball in Ω and converges to u in $W^{1,p}(\Omega)$). Clearly $\lambda^*(m) \leq \bar{\lambda}(m)$. Whether these two numbers differ or are equal depends on p and N, as is seen from the following

Lemma 3.1 *If $p \leq N$, then $\lambda^*(m) = \bar{\lambda}(m)$. If $p > N$, then $\lambda^*(m) < \bar{\lambda}(m)$. Moreover, in the latter case, there is no eigenvalue in $]\lambda^*(m), \bar{\lambda}(m)]$.*

As in section 2 we can limit ourselves without loss of generality in the study of (2.1) to the case where

$$\int_\Omega m \leq 0 \tag{3.2}$$

We recall that if $\int_\Omega m < 0$ and $0 < \lambda < \lambda^*(m)$, then the solution u of (2.1) with $h \gneqq 0$ is > 0 in $\bar{\Omega}$. If $\int_\Omega m = 0$, then no result of the type "$h \gneqq 0$ implies $u \geq 0$" holds. The following four theorems concern the AMP. Theorem 3.2 states its validity in general and its non uniformity when $p \leq N$. Theorem 3.3 characterizes the interval of uniformity when $p > N$. Theorem 3.4 shows that some form of the AMP still holds outside this interval of uniformity. Finally Theorem 3.5 makes precise the statement in the introduction that the AMP cannot hold far away to the right of $\bar{\lambda}(m)$ or to the left of $-\bar{\lambda}(-m)$.

Theorem 3.2 *Assume (3.2). (i) Given $h \gneqq 0$, there exists $\delta = \delta(h) > 0$ such that if $\lambda^*(m) < \lambda < \lambda^*(m) + \delta$ or $-\delta < \lambda < 0$, then any solution u of (2.1) satisfies $u < 0$ in $\bar{\Omega}$. (ii) If $p \leq N$, then no such δ independent of h exists (either at the right of $\lambda^*(m)$ or at the left of 0).*

Theorem 3.3 *Assume (3.2) and $p > N$. (i) If $\lambda^*(m) < \lambda \leq \bar{\lambda}(m)$ or $-\bar{\lambda}(-m) \leq \lambda < 0$, then any solution u of (2.1) with $h \underset{\not\equiv}{\geq} 0$ satisfies $u < 0$ in $\bar{\Omega}$. (ii) $\bar{\lambda}(m)$ and $-\bar{\lambda}(-m)$ are respectively the largest and the smallest numbers such that the preceding implications hold.*

Theorem 3.4 *Assume (3.2) and $p > N$. (i) Given $h \underset{\not\equiv}{\geq} 0$, there exists $\delta = \delta(h) > 0$ such that if $\bar{\lambda}(m) < \lambda < \bar{\lambda}(m) + \delta$ or $-\bar{\lambda}(-m) - \delta < \lambda < -\bar{\lambda}(-m)$, then any solution u of (2.1) satisfies $u < 0$ in $\bar{\Omega}$. (ii) No such δ independent of h exists (either at the right of $\bar{\lambda}(m)$ or at the left of $-\bar{\lambda}(-m)$).*

Theorem 3.5 *Assume (3.2). (i) Given $\epsilon > 0$, there exists $h \underset{\not\equiv}{\geq} 0$ such that for any $\lambda \geq \bar{\lambda}(m) + \epsilon$, (2.1) has no solution u satisfying $u < 0$ in $\bar{\Omega}$. (ii) Given $\epsilon > 0$, there exists $h \underset{\not\equiv}{\geq} 0$ such that for any $\lambda \leq -\bar{\lambda}(-m) - \epsilon$, (2.1) has no solution u satisfying $u < 0$ in $\bar{\Omega}$.*

We thus see that if (3.2) holds, then the following four numbers

$$-\bar{\lambda}(-m) \leq -\lambda^*(-m) = 0 \leq \lambda^*(m) \leq \bar{\lambda}(m)$$

control the domain of validity of the maximum principle and of the antimaximum principle.

We now turn to the proof of the preceding results.

Proof of Lemma 3.1. The proof that $\lambda^*(m) = \bar{\lambda}(m)$ in the case $p \leq N$ can be easily adapted from that of a similar result in [16]. We thus turn to the proof that if $p > N$, then

$$\lambda^*(m) < \bar{\lambda}(m). \tag{3.3}$$

As observed at the beginning of this section, when $p > N$, $\bar{\lambda}(m)$ is equivalently defined by (1.2). Since $p > N$, $W^{1,p}(\Omega)$ is compactly imbedded into $C(\bar{\Omega})$, and consequently the infimum in (1.2) is achieved. Replacing u by $|u|$ if necessary, we can assume that this infimum is achieved at some u with $u \geq 0$.

Claim. *u vanishes at exactly one point x_0 in $\bar{\Omega}$.*

The proof of this claim can easily be adapted from that of a similar result in [5], [16].

The idea of the proof of (3.3) is now the following. Define, for $\epsilon > 0$, $u_\epsilon(x) := max\{u(x), \epsilon\}$. Clearly $u_\epsilon \to u$ in $W^{1,p}(\Omega)$ as $\epsilon \to 0$. We will show that for $\epsilon > 0$ sufficiently small

$$\int_\Omega |\nabla u_\epsilon|^p \Big/ \int_\Omega m|u_\epsilon|^p < \int_\Omega |\nabla u|^p \Big/ \int_\Omega m|u|^p. \tag{3.4}$$

This will imply (3.3) since the left-hand side is $\geq \lambda^*(m)$ (because $\int_\Omega m|u_\epsilon|^p > 0$ for ϵ small) and the right-hand side is equal to $\bar{\lambda}(m)$.

To prove (3.4) we write the difference between the two sides of (3.4) as

$$\frac{\int_\Omega |\nabla u_\epsilon|^p}{\int_\Omega m|u_\epsilon|^p} - \frac{\int_\Omega |\nabla u|^p}{\int_\Omega mu^p} = \frac{-\int_{B_\epsilon} |\nabla u|^p \int_\Omega mu^p + \int_\Omega |\nabla u|^p (\int_{B_\epsilon} mu^p - \epsilon^p \int_{B_\epsilon} m)}{\left(\int_\Omega mu^p - \int_{B_\epsilon} mu^p + \epsilon^p \int_{B_\epsilon} m\right)\left(\int_\Omega mu^p\right)} \tag{3.5}$$

where $B_\epsilon := \{u < \epsilon\}$. Since u vanishes only at x_0, B_ϵ decreases to $\{x_0\}$ as $\epsilon \downarrow 0$, and consequently the measure of $B_\epsilon \to 0$. The denominator in (3.5) thus goes to $(\int_\Omega mu^p)^2 = 1$, while the second and third terms of the numerator are $o(\epsilon^p)$. We will show that the first term in the numerator of (3.5) satisfies

$$-\frac{1}{\epsilon}\int_{B_\epsilon} |\nabla u|^p \int_\Omega mu^p \to \text{ some } r < 0 \tag{3.6}$$

as $\epsilon \to 0$. Combining these informations, one deduces that (3.5) is < 0 for $\epsilon > 0$ sufficiently small, and consequently (3.4) holds.

To prove (3.6), we first observe that the minimizer u of (1.2) is also a minimizer for

$$\inf\{\int_\Omega |\nabla\varphi|^p : \varphi \in W_{x_0} \text{ and } \int_\Omega m|\varphi|^p = 1\}$$

where $W_{x_0} := \{\varphi \in W^{1,p}(\Omega) : \varphi(x_0) = 0\}$. Applying Lagrange multipliers rule in the space W_{x_0}, we thus have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \bar\lambda(m)\int_\Omega m|u|^{p-2}u\varphi \quad \forall\varphi \in W_{x_0}. \tag{3.7}$$

This allows us to write

$$\begin{aligned}\int_{B_\epsilon} |\nabla u|^p &= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v_\epsilon = \bar\lambda(m)\int_\Omega m|u|^{p-2}uv_\epsilon \\ &= \bar\lambda(m)\int_{B_\epsilon} m|u|^p + \bar\lambda(m)\epsilon\int_{\Omega - B_\epsilon} m|u|^{p-2}u\end{aligned} \tag{3.8}$$

where $v_\epsilon := \min\{u, \epsilon\} \in W_{x_0}$. The first term in (3.8) is $o(\epsilon^p)$ and we will show that

$$\int_\Omega m|u|^{p-2}u > 0. \tag{3.9}$$

Relation (3.6) then clearly follows.

To prove (3.9), we first show that $\int_\Omega m|u|^{p-2}u = 0$ is impossible. Indeed if $\int_\Omega m|u|^{p-2}u = 0$, then (3.7) holds not only for $\varphi \in W_{x_0}$ but also for $\varphi \equiv 1$. Since any φ in $W^{1,p}(\Omega)$ can be written as the sum of $\varphi - \varphi(x_0) \in W_{x_0}$ and of the constant $\varphi(x_0)$, we conclude that (3.7) holds for all $\varphi \in W^{1,p}(\Omega)$. But this means that $u \underset{\not\equiv}{\geq} 0$ is a solution of the Neumann problem

$$-\Delta_p u = \bar\lambda(m)m|u|^{p-2}u \text{ in } \Omega, \quad \partial u/\partial\nu = 0 \text{ on } \partial\Omega,$$

and consequently, by Proposition 2.11, $u > 0$ in $\bar\Omega$, which contradicts the fact that u vanishes at x_0. We now show that $\int_\Omega m|u|^{p-2}u < 0$ is also impossible. Indeed if $\int_\Omega m|u|^{p-2}u < 0$, then the last integral in (3.8) converges to $\int_\Omega m|u|^{p-2}u < 0$ and consequently, by (3.8), $\int_{B_\epsilon} |\nabla u|^p < 0$ for $\epsilon > 0$ sufficiently small, which is clearly a contradiction.

To conclude the proof of Lemma 3.1, it remains to see that when $p > N$, there is no eigenvalue in $]\lambda^*(m), \bar\lambda(m)]$. The argument here can be easily adapted from the proof of a similar result in [16]. Q. E. D.

Remark 3.6 Lemma 3.1 still holds, with the same proof, if m does not change sign, with $m \gneqq 0$. In this case of course $\lambda^*(m) = 0$.

Proof of Theorem 3.2. We first prove part (i) at the right of $\lambda^*(m)$ (the argument at the left of 0 is similar). Assume by contradiction the existence for some $h \gneqq 0$ of sequences $\lambda_k > \lambda^*(m)$ and u_k such that $\lambda_k \to \lambda^*(m)$,

$$-\Delta_p u_k = \lambda_k m|u_k|^{p-2}u_k + h \text{ in } \Omega, \quad \partial u_k/\partial\nu = 0 \text{ on } \partial\Omega \tag{3.10}$$

and

$$u_k \geq 0 \text{ somewhere in } \bar{\Omega}. \tag{3.11}$$

We distinguish two cases : either $||u_k||_\infty$ remains bounded, or, for a subsequence, $||u_k||_\infty \to +\infty$. In the first case one derives from (3.10) and (2.14) that u_k remains bounded in $C^{1,\gamma}(\bar{\Omega})$. Consequently, for a subsequence, u_k converges to some u in $C^1(\bar{\Omega})$. Going to the limit in (3.10), one sees that u solves

$$-\Delta_p u = \lambda^*(m)m|u|^{p-2}u + h \text{ in } \Omega,\ \partial u/\partial\nu = 0 \text{ on } \partial\Omega,$$

which contradicts Proposition 2.7. In the second case, one considers $v_k := u_k/||u_k||_\infty$, and arguing in a way similar as above from

$$-\Delta_p v_k = \lambda_k m|v_k|^{p-2}v_k + h/||u_k||_\infty \text{ in } \Omega,\ \partial v_k/\partial\nu = 0 \text{ on } \partial\Omega,$$

one gets that, for a subsequence, v_k converges to some v in $C^1(\bar{\Omega})$ where $||v||_\infty = 1$ and

$$-\Delta_p v = \lambda^*(m)\, m|v|^{p-2}v \text{ in } \Omega,\ \partial v/\partial\nu = 0 \text{ on } \partial\Omega.$$

Consequently v is an eigenfunction associated to $\lambda^*(m)$ and so either $v > 0$ in $\bar{\Omega}$ or $v < 0$ in $\bar{\Omega}$. If $v > 0$ in $\bar{\Omega}$ we deduce $v_k > 0$ in $\bar{\Omega}$ for k sufficiently large, which leads to a contradiction with Proposition 2.4. If $v < 0$ in $\bar{\Omega}$ we deduce $v_k < 0$ in $\bar{\Omega}$ for k sufficiently large, which leads to a contradiction with (3.11). (This argument to derive the AMP is adapted from [13]).

Part (ii) of Theorem 3.2 is a consequence of Theorem 3.5 since (3.2) and $p \leq N$ imply $\bar{\lambda}(m) = \lambda^*(m)$ and $\bar{\lambda}(-m) = \lambda^*(-m) = 0$. Q. E. D.

Proof of Theorem 3.3. Part (i) is easily adapted from the proof of a similar result in [5] or [16]. Part (ii) is a consequence of Theorem 3.5. Q. E. D.

Proof of Theorem 3.4. The proof is easily adapted from that of a similar result in [5] or [16]. Q. E. D.

Proof of Theorem 3.5. We prove part (i) (part (ii) is proved similarly). Assume by contradiction that there exists $\epsilon > 0$ such that for any $h \gneqq 0$ there exists λ with $\lambda \geq$

$\bar{\lambda}(m)+\epsilon$ such that (2.1) has a solution $u<0$ in $\bar{\Omega}$. We start with $\varphi \in W^{1,p}(\Omega)$ satisfying $\int_\Omega m|\varphi|^p > 0$ and vanishing on some ball in Ω, as in the definition (3.1) of $\bar{\lambda}(m)$. Then we choose $h \gneqq 0$ with supp $h \cap$ supp $\varphi = \emptyset$, and finally we consider $\lambda = \lambda_\varphi$ and $u = u_\varphi$ as provided by the above contradictory hypothesis. So $v := -u > 0$ in $\bar{\Omega}$ solves

$$-\Delta_p v = \lambda m|v|^{p-2}v - h \text{ in } \Omega, \quad \partial u/\partial\nu = 0 \text{ on } \partial\Omega.$$

Applying to this equation Lemma 2.12 with the function φ above as testing function, we get

$$\lambda \int_\Omega m|\varphi|^p - \int_\Omega h|\varphi|^p/v^{p-1} \le \int_\Omega |\nabla\varphi|^p.$$

But the integral involving h vanishes since h and φ have disjoint supports. Consequently

$$\bar{\lambda}(m)+\epsilon \le \lambda_\varphi \le \int_\Omega |\nabla\varphi|^p / \int_\Omega m|\varphi|^p$$

for all φ as above. Taking the infimum with respect to φ yields $\bar{\lambda}(m)+\epsilon \le \bar{\lambda}(m)$, a contradiction. Q. E. D.

Remark 3.7 The above arguments can easily be adapted to the case where m does not change sign in Ω, say $m \gneqq 0$, as in Remark 2.10. In this case the AMP holds at the right of 0. It is non uniform when $p \le N$ and uniform when $p > N$. In this latter case the interval of uniformity is exactly $0 < \lambda \le \bar{\lambda}(m)$ with $\bar{\lambda}(m)$ given by (1.2); moreover the AMP still holds at the right of $\bar{\lambda}(m)$, in a non uniform way. Finally, as in Theorem 3.5, the AMP cannot hold far away to the right of $\bar{\lambda}(m)$.

4 Principal eigenvalues in the Dirichlet case

In this section, which as section 2 has a preliminary character, we briefly collect some results relative to the principal eigenvalues associated to the Dirichlet problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \tag{4.1}$$

Here m and h lies as before in $L^\infty(\Omega)$, with (2.2) and (2.3), and at the beginning we do not assume any regularity on the bounded domain Ω.

The basic spectral theory for (4.1) has been extensively studied in the last twenty years (cf. e.g. [22], [3], [21], [2],...). Solutions of (4.1) belong to $L^\infty(\Omega) \cap C^1(\Omega)$. There are two principal eigenvalues : $\lambda_1(m) > 0$ and $\lambda_{-1}(m) := -\lambda_1(-m)$, where

$$\lambda_1(m) := \inf\{\int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p = 1\}.$$

These eigenvalues are simple and the corresponding eigenfunctions can be taken > 0 in Ω.

Proposition 4.1 *If $\lambda \notin [\lambda_{-1}(m), \lambda_1(m)]$, then problem (4.1) with $h \geq 0$ has no solution $u \underset{\not\equiv}{\geq} 0$.*

The proof of this proposition follows the same lines as that of Proposition 2.4. It uses the following lemma whose proof is analogous to that of Lemmas 2.5 and 2.12.

Lemma 4.2 *Let u be a solution of (4.1) with $h \geq 0$ and $u > 0$ in Ω. Then for any $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega)$ with $\varphi \geq 0$ in Ω, one has that $h\varphi^p/u^{p-1} \in L^1(\Omega)$ and (2.9) holds. Moreover equality holds in (2.9) if and only if φ is a multiple of u. Finally the restriction that h is ≥ 0 is not needed to get (2.16) for all $\varphi \in C_c^1(\Omega)$.*

Proposition 4.3 *Problem (4.1) with $h \underset{\not\equiv}{\geq} 0$ does not have any solution if $\lambda = \lambda_{-1}(m)$ or $\lambda = \lambda_1(m)$. It admits an unique solution, which is > 0 in Ω, if $\lambda_{-1}(m) < \lambda < \lambda_1(m)$.*

The proof of this proposition follows the same lines as that of Proposition 2.7. In fact it is simpler since for instance, in the functional Φ, $\int_\Omega |\nabla u|^p$ is a norm on $W_0^{1,p}(\Omega)$. We observe that the nonexistence part in Proposition 4.3 was already derived in [2] (see also [13] when $m \equiv 1$ and Ω is regular). The unicity part was already derived in [15] when $m \geq 0$ and Ω is regular.

Let us now assume in the final part of this section that Ω is of class $C^{1,1}$. The solutions then belong to $C^{1,\gamma}(\bar{\Omega})$ and one has an estimate analogous to (2.14). Moreover, by a standard property of Sobolev spaces (cf. e.g. [7]), the boundary condition $u = 0$ is satisfied in the usual pointwise sense. Consequently the maximum principle of [24] implies that a solution $u \underset{\not\equiv}{\geq} 0$ of (4.1) with $h \geq 0$ satisfies $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$.

5 Antimaximum Principle in the Dirichlet Case

We assume in this section Ω of class $C^{1,1}$, and m and h as before.

Theorem 5.1 *(i) Given $h \underset{\not\equiv}{\geq} 0$, there exists $\delta = \delta(h) > 0$ such that if $\lambda_1(m) < \lambda < \lambda_1(m)+\delta$ or $\lambda_{-1}(m) - \delta < \lambda < \lambda_{-1}(m)$, then any solution u of (4.1) satisfies $u < 0$ in Ω and $\partial u/\partial \nu > 0$ on $\partial\Omega$. (ii) No such δ independent of h exists (either at the right of $\lambda_1(m)$ or at the left of $\lambda_{-1}(m)$).*

Theorem 5.2 *(i) Given $\epsilon > 0$ there exists $h \underset{\not\equiv}{\geq} 0$ such that for any $\lambda \geq \lambda_1(m) + \epsilon$, (4.1) has no solution u satisfying $u < 0$ in Ω. (ii) Similar statement at the left of $\lambda_{-1}(m)$.*

The proof of part (i) of Theorem 5.1 can be carried out by contradiction in a way similar to the proof of Theorem 3.2. Part (ii) of Theorem 5.1 follows from Theorem 5.2. Let us sketch the proof of the latter.

Proof of Theorem 5.2. We only consider part (i). Assume by contradiction that there exists $\epsilon > 0$ such that for any $h \gneqq 0$ there exists λ with $\lambda \geq \lambda_1(m) + \epsilon$ such that (4.1) has a solution u satisfying $u < 0$ in Ω. We start with $\varphi \in C_c^\infty(\Omega)$ satisfying $\int_\Omega m|\varphi|^p > 0$. Then we choose $h \gneqq 0$ with supp $h \cap$ supp $\varphi = \emptyset$, and finally we consider $\lambda = \lambda_\varphi$ and $u = u_\varphi$ as provided by the above contradictory hypothesis. So $v := -u > 0$ in Ω solves

$$-\Delta_p v = \lambda m|v|^{p-2}v - h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Applying to v the last part of Lemma 4.2, with the function φ as testing function, and using the fact that h and φ have disjoint supports, we get

$$\lambda_1(m) + \epsilon \leq \lambda_\varphi \leq \int_\Omega |\nabla\varphi|^p \Big/ \int_\Omega m|\varphi|^p$$

for all φ as above. Since the infimum of the right-hand side with respect to φ is equal to $\lambda_1(m)$, we reach a contradiction. Q. E. D.

Remark 5.3 The result of Theorem 5.2 holds (with the same proof) without assuming any regularity on Ω.

Remark 5.4 A result analogous to that of part (i) of Theorem 5.1 is stated in [2] for a general, even unbounded, domain Ω. This however cannot hold true as stated there since it is known that the AMP does not hold for $p = 2$, $m \equiv 1$ and $\Omega =$ a square in $\mathbb{R}^2$ (cf. [6]). We also observe that new difficulties arise in the unbounded case (cf. [23], [14]).

Remark 5.5 Results similar to those in sections 4 and 5 of course also hold when the weight does not change sign in Ω.

6 Annex

It is our purpose in this annex to prove (2.15). This equality will clearly follow by applying the local result of Proposition 6.1 below to the vector field $\mathbf{a} := |\nabla u|^{p-2}\nabla u$.

Proposition 6.1 *Let D be an open subset of $\mathbb{R}^N$ which is of class C^1 near $x_0 \in \partial D$. Let $\mathbf{a}$ be a continuous vector field on $\bar{D}$. Assume that for some $f \in L^1(D)$,*

$$\int_D < \mathbf{a}, \nabla\varphi >= \int_D f\varphi \tag{6.1}$$

for all $\varphi \in C^1(\bar{D})$ with bounded support, where $<,>$ denotes the scalar product in $\mathbb{R}^N$. Then $< \mathbf{a}(x_0), \nu(x_0) >= 0$, where $\nu(x_0)$ denotes the unit exterior normal to D at x_0.

Proof. We start by representing an open neighbourhood U of x_0 in $\mathbb{R}^N$ as $\psi(W\times]-\delta,\delta[)$, with $\psi(s,t)=\bar{x}(s)-t\nu(x_0)$ given by Lemma 6.2 below. We will write $\psi^{-1}(x)$ as $(\bar{s}(x),\bar{t}(x))$ where $x\in U$, $\bar{s}(x)\in W$ and $\bar{t}(x)\in]-\delta,\delta[$. Let us fix two functions g and h with $g\in C^\infty(\mathbb{R}^{N-1})$, $0\le g\le 1$, $g\equiv 1$ on the ball $B_1(0)$, $g\equiv 0$ outside $B_2(0)$, $h\in C^\infty(\mathbb{R})$, $0\le h\le 1$, $h\equiv 1$ on $]-1,1[$, $h\equiv 0$ outside $]-2,2[$. For $\epsilon>0$, $\eta>0$, we then define

$$\varphi_{\epsilon,\eta}(x):=g\left(\frac{\bar{s}(x)}{\eta}\right)\quad h\left(\frac{\bar{t}(x)}{\epsilon}\right)$$

where $x\in U$. Clearly $\varphi_{\epsilon,\eta}$ is a C^1 function on U, $0\le\varphi_{\epsilon,\eta}\le 1$, $\varphi_{\epsilon,\eta}\equiv 1$ on $\psi(B_\eta(0)\times]-\epsilon,\epsilon[)$ and $\varphi_{\epsilon,\eta}\equiv 0$ outside $\psi(B_{2\eta}(0)\times]-2\epsilon,2\epsilon[)$. So, for ϵ,η sufficiently small, $\varphi_{\epsilon,\eta}$ has compact support in U. Consequently its restriction to D is an admissible testing function in (6.1) :

$$\int_D <\mathbf{a}(x),\nabla\varphi_{\epsilon,\eta}(x)>\,dx=\int_D f(x)\,\varphi_{\epsilon,\eta}(x)\,dx. \tag{6.2}$$

Let us start by fixing $\eta>0$ and go to the limit in (6.2) as $\epsilon\to 0$. Since the measure of the support of $\varphi_{\epsilon,\eta}$ goes to zero as $\epsilon\to 0$, the right hand side of (6.2) goes to zero. Computing $\nabla\varphi_{\epsilon,\eta}$, one sees that the left hand side of (6.2) is a sum I+II, where

$$\begin{aligned} I &= \int_D g\left(\frac{\bar{s}(x)}{\eta}\right) <\mathbf{a}(x),\nabla\bar{t}(x)> \frac{1}{\epsilon}h'\left(\frac{\bar{t}(x)}{\epsilon}\right)\,dx,\\ II &= \int_D <\mathbf{a}(x),\frac{1}{\eta}\nabla g\left(\frac{\bar{s}(x)}{\eta}\right)\nabla\bar{s}(x)> h\left(\frac{\bar{t}(x)}{\epsilon}\right)\,dx. \end{aligned}$$

Since the measure of the support of the integrand in II goes to zero as $\epsilon\to 0$, we see that II goes to zero. To study I, we go from the x-coordinates to the (s,t) coordinates and apply Fubini's theorem to get

$$I=\int_0^{2\epsilon}\left[\int_{B_{2\eta}(0)} g\left(\frac{s}{\eta}\right) <\mathbf{a}(\psi(s,t)),\nabla\bar{t}(\psi(s,t))> |J(s,t)|\,ds\right] h'(\frac{t}{\epsilon})/\epsilon\,dt$$

where J denotes the Jacobian determinant of ψ. Since the bracket [...] in the above integrand is a continuous function of t, call it $G(t)$, the definition of h implies that $\int_0^{2\epsilon}G(t)h'(t/\epsilon)/\epsilon\,dt\to -G(0)$ as $\epsilon\to 0$. Consequently, for any $\eta>0$, we have

$$\int_{B_{2\eta}(0)} g(s/\eta) <\mathbf{a}(\bar{x}(s)),\ \nabla\bar{t}(\bar{x}(s))> |J(s,0)|\,ds=0.$$

Dividing by η^{N-1}, letting $\eta\to 0$, and using $|J(0,0)|\neq 0$, one gets

$$<\mathbf{a}(\bar{x}(0)),\nabla\bar{t}(\bar{x}(0))>=0.$$

This yields the conclusion of Proposition 6.1 since it is easily verified that $\nabla\bar{t}(x_0)=-\nu(x_0)$.
Q.E.D.

Lemma 6.2 *Let D be an open subset of $\mathbb{R}^N$ which is of class C^1 near $x_0 \in \partial D$. Then there exists a C^1 chart of D near x_0 : $\bar{x} : W \to \partial D$ with W an open neighbourhood of 0 in $\mathbb{R}^{N-1}$, $\bar{x}(0) = x_0$, and there exists $\delta > 0$ such that $\psi : W \times]-\delta, \delta[\ : (s,t) \to \bar{x}(s) - t\nu(x_0)$ is a C^1 diffeomorphism onto a neighbourhood U of x_0 in $\mathbb{R}^N$, with moreover $\psi(s,t) \in D$ (resp. $\in \partial D$, $\notin \bar{D}$) whenever $s \in W$ and $0 < t < \delta$ (resp. $t = 0$, $-\delta < t < 0$).*

Proof. Since D is of class C^1 near x_0, there exists an open neighbourhood V of x_0 in $\mathbb{R}^N$ of the form $V = X(W \times]-\delta, \delta[)$, where W is an open neighbourhood of 0 in $\mathbb{R}^{N-1}$, $\delta > 0$, X is a C^1 diffeomorphism from $W \times]-\delta, \delta[$ onto V, with the property that $X(0,0) = x_0$, $X(\sigma, \tau) \in D$ (resp. $\in \partial D$, $\notin \bar{D}$) whenever $\sigma \in W$ and $0 < \tau < \delta$ (resp. $\tau = 0$, $-\delta < \tau < 0$). Clearly we can assume $\partial X / \partial \tau (0,0) = -\nu(x_0)$. Let us define $\psi(s,t) := X(s,0) - t\nu(x_0)$ for $s \in W$ and $t \in \mathbb{R}$. Since the Jacobian matrix of ψ at $(0,0)$ is invertible, ψ is a C^1 diffeomorphism from $W_1 \times]-\delta_1, \delta_1[$ onto an open neighbourhood $V_1 \subset V$ of x_0 in $\mathbb{R}^N$, where $W_1 \subset W$ is an open neighbourhood of 0 in $\mathbb{R}^{N-1}$ and $\delta_1 > 0$. Clearly $\bar{x}(s) := X(s,0)$ is a C^1 chart of ∂D near x_0, $\psi(0,0) = x_0$ and $\psi(s,0) \in \partial D$ if $s \in W_1$. We have to show that, by diminishing W_1 and δ_1 if necessary, $\psi(s,t) \in D$ (resp. $\notin \bar{D}$) whenever $s \in W_1$ and $0 < t < \delta_1$ (resp. $-\delta_1 < t < 0$).

To prove this property of ψ, we first observe that for any $(s,t) \in W_1 \times]-\delta_1, \delta_1[$, there exists an unique $(\sigma, \tau) := (\bar{\sigma}(s,t), \bar{\tau}(s,t)) \in W \times]-\delta, \delta[$ such that $\psi(s,t) = X(\sigma, \tau)$.Applying the implicit function theorem to $F(s,t,\sigma,\tau) := \psi(s,t) - X(\sigma,\tau)$ and diminishing W_1 and δ_1 if necessary, one sees that $\bar{\sigma}(s,t)$ and $\bar{\tau}(s,t)$ are C^1 functions. Moreover, computing the derivative of $X(s,0) - t\nu(x_0) \equiv X(\bar{\sigma}(s,t), \bar{\tau}(s,t))$ with respect to t and using $\partial X / \partial \tau (0,0) = -\nu(x_0)$, one easily gets that $\partial \bar{\tau} / \partial t (0,0) = 1$. It follows that $\partial \bar{\tau} / \partial t (s,t) > 0$ for (s,t) in some convex open neighbourhood $W_2 \times]-\delta_2, \delta_2[$ of $(0,0)$. Since $\bar{\tau}(s,0) = 0$, we thus have $\bar{\tau}(s,t) > 0$ (resp. < 0) for $s \in W_2$ and $0 < t < \delta_2$ (resp. $-\delta_2 < t < 0$), and consequently the desired property of ψ follows from the corresponding property of X. Q. E. D.

Remark 6.3 If the bounded domain Ω is of class C^2, then (2.15) can be derived by applying a version of the divergence theorem given in [9]. Indeed first observe that (2.4) implies that $-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x)$ in $\mathcal{D}'(\Omega)$. It follows that, for a given $\varphi \in C^1(\bar{\Omega})$, the vector field $\mathbf{b} := \varphi |\nabla u|^{p-2} \nabla u$ belongs to $C(\bar{\Omega})$ and has a distributional derivative which belongs to $L^\infty(\Omega) \subset L^1(\Omega)$. Applying to $\mathbf{b}$ the divergence theorem of [9] and using (2.4), one gets $\int_\Omega |\nabla u|^{p-2} < \nabla u, \nu > \varphi = 0$. Since φ is arbitrary in $C^1(\bar{\Omega})$, the conclusion (2.15) follows.

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