**Title**

CONSTRUCTION OF IRREDUCIBLE RELATIVE INVARIANT OF THE PREHOMOGENEOUS VECTOR SPACE $(SL_5 \times GL_4, \Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4)$

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CONSTRUCTION OF IRREDUCIBLE RELATIVE INVARIANT OF THE PREHOMOGENEOUS VECTOR SPACE $(SL_5 \times GL_4, \Lambda^2(C^5) \otimes C^4)$

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ABSTRACT. We explicitly construct the irreducible relative invariant of the prehomogeneous vector space $(SL_5 \times GL_4, \Lambda^2(C^5) \otimes C^4)$. This prehomogeneous vector space has been known as the "most difficult" case in irreducible regular prehomogeneous vector spaces.

1. INTRODUCTION

The prehomogeneous vector space $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \Lambda^2(C) \otimes C^4)$ is known as the classification number (11) in [SK, Theorem 54 I]). It has been known that the irreducible relative invariant of this prehomogeneous vector space should be a homogeneous polynomial in degree 40 and it would be computed from the determinant of certain $40 \times 40$ matrix (see [SK, Section 4, proposition 16]). However, even with a computer, it is too hard to compute such determinant, and even if we could computed by this method, we would not be able to understand the result easily.

We try to construct the relative invariant by the another method which is treated in [O, Section 3]. The idea of the method is to construct an equivariant surjection from $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \Lambda^2(C) \otimes C^4)$ to $(GL_4, 2\Lambda_1, S^2(C^4))$. After that, the irreducible relative invariant of the former prehomogeneous vector space is obtained as the composition of the surjection and the relative invariant of the latter. The merit in this construction is that we can reach the explicit form of the relative invariant very easily and we may make use for the research on the former prehomogeneous vector space $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \Lambda^2(C) \otimes C^4)$ by studying the latter $(GL_4, 2\Lambda_1, S^2(C^4))$.

In addition, the more another construction of this irreducible relative invariant has been obtained by A. Yukie [Y, Section 16]. This prehomogeneous vector space is one of the Dynkin-Kostant types defined by A. Gyoja [G2] in which there is mentioned about the relative invariant...
2. Notations

Let $\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $E_\epsilon$ be the following:

$$
\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) := \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 \\
0 & 0 & \alpha_3 & 0 \\
0 & 0 & 0 & \alpha_4
\end{pmatrix},
E_\epsilon := \begin{pmatrix}
1 & \epsilon & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Let $S_4$ be the 4-th symmetric group. In $S_4$, a transposition between $i$ and $j$ is denoted by $(i, j)$. One sees each permutation $\sigma \in S_4$ is considered as the $4 \times 4$ matrix such that its $(i, j)$-element is 1 or 0 with respect to $i = \sigma(j)$ or not. So we may apply for regarding one as the other.

The set of all $n \times n$ complex matrices is denoted by $M_n$. Let $\text{Alt}_n$ be the set of all skew-symmetric matrices in $M_n$ (i.e. $\text{Alt}_n = \{X \in M_n \mid ^tX = -X\}$) and $\text{Alt}_n^{\oplus 4}$ the direct sum of four $\text{Alt}_5$s. One sees that the $\mathbb{C}$-vector space $\Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4$ is isomorphic to $\text{Alt}_5^{\oplus 4}$. Then we think on $\text{Alt}_5^{\oplus 4}$ instead of on $\Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4$. The triplet $(SL_5 \times GL_4, \rho = \Lambda_2 \otimes \Lambda_1, \text{Alt}_5^{\oplus 4})$ denotes the prehomogeneous vector space that the action $\rho$ is

$$
\rho(A, B) : (X_1, X_2, X_3, X_4) \mapsto (AX_1^tA, AX_2^tA, AX_3^tA, AX_4^tA)^tB
$$

for $(X_1, X_2, X_3, X_4) \in \text{Alt}_5^{\oplus 4}$ and $(A, B) \in SL_5 \times GL_4$. Our purpose is to construct the irreducible relative invariant of this prehomogeneous vector space explicitly.

3. Polynomials on $\text{Alt}_5^{\oplus 4}$

To construct the equivariant surjection mentioned in section 1, we shall first define some polynomials on $\text{Alt}_5^{\oplus 4}$ which are invariants with respect to the action of $SL_5$. The same type polynomials on $\text{Alt}_5^{\oplus 3}$ are used in [O] and originally in [G1].

In the beginning, we define certain $SL_5$-equivariant map $\beta : \text{Alt}_5 \times \text{Alt}_5 \rightarrow \mathbb{C}^5$. Let Pf be the Pfaffian on $\text{Alt}_4$. For $X \in \text{Alt}_5$ and $i = 1, \cdots, 5$, let $X^{(i)}$ denote the matrix in $\text{Alt}_4$ which is obtained by deleting $i$-th row and $i$-th column from $X$. For $X = (x_{ij}), Y = (y_{ij}) \in \text{Alt}_5,$
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$\beta(X,Y)$ is defined by

$$
\beta(X,Y) := \left( \begin{array}{c}
\text{Pf}(X^{(1)} + Y^{(1)}) - \text{Pf}(X^{(1)}) - \text{Pf}(Y^{(1)}) \\
-(\text{Pf}(X^{(2)} + Y^{(2)}) - \text{Pf}(X^{(2)}) - \text{Pf}(Y^{(2)}) \\
\text{Pf}(X^{(3)} + Y^{(3)}) - \text{Pf}(X^{(3)}) - \text{Pf}(Y^{(3)}) \\
-(\text{Pf}(X^{(4)} + Y^{(4)}) - \text{Pf}(X^{(4)}) - \text{Pf}(Y^{(4)}) \\
\text{Pf}(X^{(5)} + Y^{(5)}) - \text{Pf}(X^{(5)}) - \text{Pf}(Y^{(5)})
\end{array} \right)
$$

$$=
\begin{pmatrix}
x_{23}y_{45} - x_{24}y_{35} + x_{25}y_{34} + y_{23}x_{45} - y_{24}x_{35} + y_{25}x_{34}
x_{34}y_{51} - x_{35}y_{41} + x_{31}y_{45} + y_{34}x_{51} - y_{35}x_{41} + y_{31}x_{45}
x_{45}x_{12} - x_{41}y_{52} + x_{42}y_{51} + y_{45}x_{12} - y_{41}y_{52} + y_{42}y_{51}
x_{51}y_{23} - x_{52}y_{13} + x_{53}y_{12} + y_{51}x_{23} - y_{52}x_{13} + y_{53}x_{12}
x_{12}y_{34} - x_{13}y_{24} + x_{14}y_{23} + x_{12}y_{34} - x_{13}y_{24} + x_{14}y_{23}
\end{pmatrix}.
$$

After that, for $i,j,k,l,m \in \{1,2,3,4\}$, we define a polynomial $[ijklm]$ on $\text{Alt}_5^4$ by

$$[ijklm](X_1,X_2,X_3,X_4) := [ijklm](X_1,X_2,X_3,X_4)$$

for $X_1,X_2,X_3,X_4 \in \text{Alt}_5$. They are 5-th multilinear forms, and satisfy the following lemmas:

**Lemma 3.1** ([G1, Section 2, Lemma]). For all $i,j,k,l,m \in \{1,2,3,4\}$, the polynomial $[ijklm]$ is invariant with respect to $SL_5$, i.e.

$$[ijklm](AX_1^tA, AX_2^tA, AX_3^tA, AX_4^tA, AX_5^tA) = [ijklm](X_1,X_2,X_3,X_4,X_5)$$

for all $A \in SL_5$.

**Lemma 3.2** ([G1, Section 2, (4)]). If there are only one or two kinds of numbers among $\{i,j,k,l,m\}$, then $[ijklm] = 0$.

**Lemma 3.3** (c.f. [O, Lemma 3.1]). For each $i,j,k,l,m \in \{1,2,3,4\}$,

1. $[ijklm] = [ikljm]$, $[ijklm] = [ijklm]$  
2. $[ijklm] = -[mklij]$  
3. $[ijklm] + [jklm] + [ijklm] = 0$  
4. $[iiklm] = -[kiiil]$  
5. $[iikli] = -[iikli] = [ikllii] = -[iikli] = 0$  
6. $[iiilm] = 0$, $[ijik] = 0$.

Finally in this section, we shall consider the action of $GL_4$ on $[ijklm]$. $GL_4$ is generated by the following three types of matrices: $\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, permutation matrices, and $E_e$. Then we only need to think on these types.

For $B \in GL_4$ and $P$ a polynomial on $\text{Alt}_5^4$, let $P^B$ denotes the polynomial such that $P^B(X) = P(X'B)$. The actions of diagonal matrices $D = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\sigma \in \mathfrak{S}_4$ are

$$[ijklm]^D = \alpha_1\alpha_2\alpha_3\alpha_4[ijklm],$$

$$[ijklm]^\sigma = [\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)\sigma^{-1}(l)\sigma^{-1}(m)].$$
Since $[ijklm]$ is a multilinear form, the action of $E_{\epsilon}$ is, for example,

\[
[ijklm]^{E_{\epsilon}} = [ijklm],
\]
\[
[ijkl]^{E_{\epsilon}} = [ijkl] + \epsilon [2ijkl],
\]
\[
[11ijk]^{E_{\epsilon}} = [11ijk] + 2\epsilon [12ijk] + \epsilon^{2}[22ijk],
\]
\[
[11ijl]^{E_{\epsilon}} = [11ijl] + \epsilon(2[12ijl] + [11ijkl]) + \epsilon^{2}[22ijl] + \epsilon^{3}[33ijl],
\]

for $i, j, k, l, m \in \{2, 3, 4\}$.

4. CONSTRUCTION OF THE EQUIVARIANT MAP

Now we shall define the equivariant map $\Phi : \text{Alt}_{6}^{\mathbb{C}^{4}} \to S^{2}(\mathbb{C}^{4})$ such that $\Phi(X) = (\varphi_{st}(X))$ and each $\varphi_{st}$ is like $\varphi_{st} = \sum_{i,j,k,l,m} c_{ijkl'm} [ijklm][i'j'k'l'm']$ in which $[ijklm][i'j'k'l'm'](X) = [ijklm](X)[i'j'k'l'm'](X)$. Then we shall explain that we have the irreducible relative invariant in degree 40 as $\det \Phi(X)$. Furthermore, we shall prove that $\Phi$ is surjection.

First, we define the polynomials $\varphi_{11}, \varphi_{12}$ as

\[
\varphi_{11} = 160[31114](3[24132] - 2[21342] - 2[23412]) + 160[41112](3[32143] - 2[34213] - 2[31423]) + 160[21113](3[43124] - 2[41234] - 2[42314]) + 50([11233][11244] + [11322][11344] + [11422][11433]) - 288([13241]^2 + [14321]^2 + [12431]^2) + 224([13241][14321] + [14321][12431] + [12431][13241]),
\]
\[
\varphi_{12} = 400[31114][32224] - 100([21113][22344] + [21114][22433]) - 100([12223][11344] + [12224][11433]) + 20([33322][43143] - [34213] - [32143]) + 20([14442][41324] - [43214] - [42134]) - 25([22144][11233] + [11244][22133]),
\]
\[
+ 368[13241][23142] + 112([13241]([21342] + [23412]) + [23142]([12341] + [13421])) + 192([14321][23412] + [13421][24312]) - 208([13421][21342] + [12341][23412]).
\]

These polynomials satisfy the following properties:

1. If $\sigma \in \mathfrak{S}_{4}$ and $\sigma(1) = 1$, then $\varphi_{11}^{\sigma} = \varphi_{11}$,
2. If $\sigma \in \mathfrak{S}_{4}$ and $\{\sigma(1), \sigma(2)\} = \{1, 2\}$, then $\varphi_{12}^{\sigma} = \varphi_{12}$.
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Then we define the map $\Phi : \text{Alt}_{5}^{\oplus 4} \to S^{2}(\mathbb{C}^{4})$ as $\Phi(X) = (\varphi_{st}(X))$ in which $\varphi_{st}$ is

$$
\varphi_{st} = \begin{cases} 
\varphi_{11}^{(1)}(s) & (s = t) \\
(\varphi_{12}^{(1)})^{(2)}(t) & (s \neq t)
\end{cases}
$$

for $s, t \in \{1, 2, 3, 4\}$ and $(1 s), (2 t) \in \mathfrak{S}_{4}$. It is easily seen from (1), (2) that $\varphi_{st} = \varphi_{ts}$ and $\varphi_{st}^{\sigma} = \varphi_{\sigma^{-1}(s)\sigma^{-1}(t)}$ for all $\sigma \in \mathfrak{S}_{4}$.

**Lemma 4.1.** For $X \in \text{Alt}_{5}^{\oplus 4}$ and $(A, B) \in SL_{5} \times GL_{4}$,

$$
\Phi(\rho(A, B)X) = (\det B)^{2}B\Phi(X){}^{t}B.
$$

**Proof.** Let $D = \text{diag}(\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4})$ and let $A$ be arbitrary element of $SL_{5}$. Since $\varphi_{st}^{D} = (\alpha_{1}\alpha_{2}\alpha_{3}\alpha_{4})^{2}\alpha_{s}\alpha_{t}\varphi_{st}$ for all $s, t \in \{1, 2, 3, 4\}$, and each $\varphi_{st}$ is invariant with respect to $SL_{5}$, we have

$$
\Phi(\rho(A, D)X) = (\det D)^{2}D\Phi(X){}^{t}D.
$$

From the remark after the definition of $\Phi$,

$$
\Phi(\rho(A, \sigma)X) = (\varphi_{\sigma^{-1}(s)\sigma^{-1}(t)}(X)) = \sigma\Phi(X){}^{t}\sigma
$$

for all $\sigma \in \mathfrak{S}_{4}$.

The rest of the proof is to show $\Phi(\rho(A, E_{e})X) = E_{e}\Phi(X){}^{t}E_{e}$, i.e.

(i) $\varphi_{11}^{E_{e}} = \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^{2}\varphi_{22},$

(ii) $\varphi_{12}^{E_{e}} = \varphi_{12}^{E_{e}} = \varphi_{12} + \varepsilon\varphi_{22}$ for $t = 2, 3, 4,$

(iii) $\varphi_{st}^{E_{e}} = \varphi_{ts}^{E_{e}} = \varphi_{st}$ for $s, t = 2, 3, 4.$

First, we have directly

(4.1) $\varphi_{11}^{E_{e}} = \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^{2}\varphi_{22},$

(4.2) $\varphi_{22}^{E_{e}} = \varphi_{22},$

(4.3) $\varphi_{33}^{E_{e}} = \varphi_{33},$

(4.4) $\varphi_{13}^{E_{e}} = \varphi_{13} + \varepsilon\varphi_{23},$

(4.5) $\varphi_{34}^{E_{e}} = \varphi_{34}.$

From $E_{e}^{2} = E_{2e}$ and (4.1),

$$
\varphi_{11}^{E_{e}} = \varphi_{11} + 4\varepsilon\varphi_{12} + 4\varepsilon^{2}\varphi_{22}.
$$

Otherwise, from (4.1) and (4.2),

$$
\varphi_{11}^{E_{e}} = \varphi_{11}^{E_{e}} + 2\varepsilon\varphi_{12}^{E_{e}} + \varepsilon^{2}\varphi_{22}^{E_{e}}
$$

$$
= \varphi_{11} + 2\varepsilon\varphi_{12} + 2\varepsilon\varphi_{12}^{E_{e}} + 2\varepsilon^{2}\varphi_{22}.
$$

Therefore $\varphi_{12}^{E_{e}} = \varphi_{12} + \varepsilon\varphi_{22}.$
Similarly from (4.4),

\[
\varphi_{13}^{E_{e}^{2}} = \varphi_{13} + 2\epsilon \varphi_{23} = \varphi_{13} + \epsilon \varphi_{23} + \epsilon \varphi_{23}^{E_{e}}.
\]

Then we have \( \varphi_{23}^{E_{e}} = \varphi_{23} \).

From (4.4) and \( E_{e}(3\ 4) = (3\ 4)E_{e} \), we have

\[
\varphi_{14}^{E_{e}} = \varphi_{13}^{(E_{e}(3\ 4))} = \varphi_{13}^{((3\ 4)E_{e})} = (\varphi_{13}^{E_{e}})^{(3\ 4)} = \varphi_{14} + \epsilon \varphi_{24}.
\]

Similarly from (4.3), we have \( \varphi_{44}^{E_{e}} = \varphi_{44} \). \( \square \)

To prove that \( \Phi \) is surjection, we only need to find five points in \( \text{Alt}_{5}^{\otimes 4} \) such that each image has rank 0, 1, 2, 3, 4.

For

\[
X_{01} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
X_{02} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
X_{03} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
X_{04} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
Y_{01} = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
Y_{02} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
Y_{03} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix},
\]

we have
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\[ \Phi(X_{01}, X_{02}, X_{03}, X_{04}) = \begin{pmatrix} 0 & 0 & -720 & 0 \\ 0 & -480 & 0 & 0 \\ -720 & 0 & 0 & 0 \\ 0 & 0 & 0 & -288 \end{pmatrix} \quad \text{(rank 4),} \]

\[ \Phi(Y_{01}, X_{02}, X_{03}, X_{04}) = \begin{pmatrix} 0 & 0 & 0 & -288 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(rank 3),} \]

\[ \Phi(Y_{01}, Y_{02}, X_{03}, X_{04}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -192 & 0 & -192 & -96 \\ -192 & 0 & -192 & -96 \\ -96 & 0 & -96 & -288 \end{pmatrix} \quad \text{(rank 2),} \]

\[ \Phi(Y_{01}, Y_{02}, Y_{03}, X_{04}) = \begin{pmatrix} -720 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(rank 1),} \]

\[ \Phi(0, 0, 0, 0) = 0 \quad \text{(rank 0).} \]

Therefore \( \Phi \) is surjection and especially \( \det \Phi(X) \neq 0 \). This fact and lemma 4.1 implies that \( \det \Phi(X) \) is the relative invariant in degree 40.

**Theorem 4.2.** (i) The map \( \Phi : \text{Alt}^{n_{4}}_{5} \to S^{2}(\mathbb{C}^{4}) \) is surjection.

(ii) \( f(X) = \det \Phi(X) \) is the irreducible relative invariant of the prehomogeneous vector space \((SL_{5} \times GL_{4}, \Lambda_{2} \otimes \Lambda_{1}, \text{Alt}^{n_{4}}_{5})\) in degree 40 corresponding to the rational character \((\det B)^{4}\).

**References**


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