数理解析研究所講究録 1238 巻 2001 年 12-19

CONSTRUCTION OF IRREDUCIBLE RELATIVE INVARIANT OF THE PREHOMOGENEOUS VECTOR SPACE $(SL_5 \times GL_4, \Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4)$

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ABSTRACT. We explicitly construct the irreducible relative invariant of the prehomogeneous vector space $(SL_5 \times GL_4, \Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4)$. This prehomogeneous vector space has been known as the "most difficult" case in irreducible regular prehomogeneous vector spaces.

1. INTRODUCTION

The prehomogeneous vector space $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4)$ is known as the classification number (11) in [SK, Theorem 54 I)]. It has been known that the irreducible relative invariant of this prehomogeneous vector space should be a homogeneous polynomial in degree 40 and it would be computed from the determinant of certain 40×40 matrix (see [SK, Section 4, proposition 16]). However, even with a computer, it is too hard to compute such determinant, and even if we could computed by this method, we would not be able to understand the result easily.

We try to construct the relative invariant by the another method which is treated in [O, Section 3]. The idea of the method is to construct an equivariant surjection from $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \Lambda^2(\mathbb{C}) \otimes \mathbb{C}^4)$ to $(GL_4, 2\Lambda_1, S^2(\mathbb{C}^4))$. After that, the irreducible relative invariant of the former prehomogeneous vector space is obtained as the composition of the surjection and the relative invariant of the latter. The merit in this construction is that we can reach the explicit form of the relative invariant very easily and we may make use for the research on the former prehomogeneous vector space $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \Lambda^2(\mathbb{C}) \otimes \mathbb{C}^4)$ by studying the latter $(GL_4, 2\Lambda_1, S^2(\mathbb{C}^4))$.

In addition, the more another construction of this irreducible relative invariant has been obtained by A. Yukie [Y, Section 16]. This prehomogeneous vector space is one of the Dynkin-Kostant types defined by A. Gyoja [G2] in which there is mentioned about the relative invariant

2. Notations

Let $\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and E_ε be the following:

$$\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) := \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}, \quad E_\varepsilon := \begin{pmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{S}_4$ be the 4-th symmetric group. In $\mathfrak{S}_4$, a transposition between i and j is denoted by $(i\ j)$. One sees each permutation $\sigma \in \mathfrak{S}_4$ is considered as the 4×4 matrix such that its (i, j)-element is 1 or 0 with respect to $i = \sigma(j)$ or not. So we may apply for regarding one as the other.

The set of all $n \times n$ complex matrices is denoted by M_n. Let Alt_n be the set of all skew-symmetric matrices in M_n (i.e. $\mathrm{Alt}_n = \{X \in M_n \mid {}^tX = -X\}$) and $\mathrm{Alt}_n^{\oplus 4}$ the direct sum of four Alt_ns. One sees that the $\mathbb{C}$-vector space $\Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4$ is isomorphic to $\mathrm{Alt}_5^{\oplus 4}$. Then we think on $\mathrm{Alt}_5^{\oplus 4}$ instead of on $\Lambda^2(\mathbb{C}^5) \otimes \mathbb{C}^4$. The triplet $(SL_5 \times GL_4, \rho = \Lambda_2 \otimes \Lambda_1, \mathrm{Alt}_5^{\oplus 4})$ denotes the prehomogeneous vector space that the action ρ is

$$\rho(A, B) : (X_1, X_2, X_3, X_4) \longmapsto (AX_1{}^tA, AX_2{}^tA, AX_3{}^tA, AX_4{}^tA)^tB$$

for $(X_1, X_2, X_3, X_4) \in \mathrm{Alt}_5^{\oplus 4}$ and $(A, B) \in SL_5 \times GL_4$. Our purpose is to construct the irreducible relative invariant of this prehomogeneous vector space explicitly.

3. Polynomials on $\mathrm{Alt}_5^{\oplus 4}$

To construct the equivariant surjection mentioned in section 1, we shall first define some polynomials on $\mathrm{Alt}_5^{\oplus 4}$ which are invariants with respect to the action of SL_5. The same type polynomials on $\mathrm{Alt}_5^{\oplus 3}$ are used in [O] and originally in [G1].

In the beginning, we define certain SL_5-equivariant map $\beta : \mathrm{Alt}_5 \times \mathrm{Alt}_5 \to \mathbb{C}^5$. Let Pf be the Pfaffian on Alt_4. For $X \in \mathrm{Alt}_5$ and $i = 1, \cdots, 5$, let $X^{(i)}$ denote the matrix in Alt_4 which is obtained by deleting i-th row and i-th column from X. For $X = (x_{ij}), Y = (y_{ij}) \in \mathrm{Alt}_5$,

$\beta(X,Y)$ is defined by

$$\beta(X,Y) := \begin{pmatrix} \mathrm{Pf}(X^{(1)}+Y^{(1)}) - \mathrm{Pf}(X^{(1)}) - \mathrm{Pf}(Y^{(1)}) \\ -(\mathrm{Pf}(X^{(2)}+Y^{(2)}) - \mathrm{Pf}(X^{(2)}) - \mathrm{Pf}(Y^{(2)})) \\ \mathrm{Pf}(X^{(3)}+Y^{(3)}) - \mathrm{Pf}(X^{(3)}) - \mathrm{Pf}(Y^{(3)}) \\ -(\mathrm{Pf}(X^{(4)}+Y^{(4)}) - \mathrm{Pf}(X^{(4)}) - \mathrm{Pf}(Y^{(4)})) \\ \mathrm{Pf}(X^{(5)}+Y^{(5)}) - \mathrm{Pf}(X^{(5)}) - \mathrm{Pf}(Y^{(5)}) \end{pmatrix}$$
$$= \begin{pmatrix} x_{23}y_{45} - x_{24}y_{35} + x_{25}y_{34} + y_{23}x_{45} - y_{24}x_{35} + y_{25}x_{34} \\ x_{34}y_{51} - x_{35}y_{41} + x_{31}y_{45} + y_{34}x_{51} - y_{35}x_{41} + y_{31}x_{45} \\ x_{45}y_{12} - x_{41}y_{52} + x_{42}y_{51} + y_{45}x_{12} - y_{41}x_{52} + y_{42}x_{51} \\ x_{51}y_{23} - x_{52}y_{13} + x_{53}y_{12} + y_{51}x_{23} - y_{52}x_{13} + y_{53}x_{12} \\ x_{12}y_{34} - x_{13}y_{24} + x_{14}y_{23} + x_{12}y_{34} - x_{13}y_{24} + x_{14}y_{23} \end{pmatrix}.$$

After that, for $i,j,k,l,m \in \{1,2,3,4\}$, we define a polynomial $[ijklm]$ on $\mathrm{Alt}_5^{\oplus 4}$ by

$$[ijklm](X_1,X_2,X_3,X_4) := {}^t\beta(X_i,X_j)X_k\beta(X_l,X_m)$$

for $X_1,X_2,X_3,X_4 \in \mathrm{Alt}_5$. They are 5-th multilinear forms, and satisfy the following lemmas:

Lemma 3.1 ([G1, Section 2, Lemma]). *For all $i,j,k,l,m \in \{1,2,3,4\}$, the polynomial $[ijklm]$ is invariant with respect to SL_5, i.e.*

$$\begin{aligned} &[ijklm](AX_1{}^tA, AX_2{}^tA, AX_3{}^tA, AX_4{}^tA, AX_5{}^tA) \\ &= [ijklm](X_1,X_2,X_3,X_4,X_5) \end{aligned}$$

for all $A \in SL_5$.

Lemma 3.2 ([G1, Section 2, (4)]). *If there are only one or two kinds of numbers among $\{i,j,k,l,m\}$, then $[ijklm]=0$.*

Lemma 3.3 (c.f. [O, Lemma 3.1]). *For each $i,j,k,l,m \in \{1,2,3,4\}$,*

(1) $[ijklm]=[jiklm]$, $[ijklm]=[ijkml]$ *(2)* $[ijklm]=-[lmkij]$
(3) $[ijklm]+[jkilm]+[kijlm]=0$ *(4)* $[iiklm]=-2[kiilm]$
(5) $[iikli]=-[iilki]=[iklii]=-[ilkii]$ *(6)* $[iiilm]=0$, $[ijkij]=0$.

Finally in this section, we shall consider the action of GL_4 on $[ijklm]$. GL_4 is generated by the following three types of matrices: $\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, permutation matrices, and E_ε. Then we only need to think on these types.

For $B \in GL_4$ and P a polynomial on $\mathrm{Alt}_5^{\oplus 4}$, let P^B denotes the polynomial such that $P^B(X) = P(X^tB)$. The actions of diagonal matrices $D = \mathrm{diag}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ and $\sigma \in \mathfrak{S}_4$ are

$$\begin{aligned} [ijklm]^D &= \alpha_i\alpha_j\alpha_k\alpha_l\alpha_m[jiklm], \\ [ijklm]^\sigma &= [\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)\sigma^{-1}(l)\sigma^{-1}(m)]. \end{aligned}$$

Since $[ijklm]$ is a multilinear form, the action of E_ε is, for example,

$$\begin{aligned}
[ijklm]^{E_\varepsilon} &= [ijklm],\\
[1ijkl]^{E_\varepsilon} &= [1ijkl]+\varepsilon[2ijkl],\\
[11ijk]^{E_\varepsilon} &= [11ijk]+2\varepsilon[12ijk]+\varepsilon^2[22ijk],\\
[11ij1]^{E_\varepsilon} &= [11ij1]+\varepsilon(2[12ij1]+[11ij2])\\
&\quad +\varepsilon^2(2[12ij2]+[22ij1])+\varepsilon^3[22ij2], \text{ etc.}
\end{aligned}$$

for $i,j,k,l,m \in \{2,3,4\}$.

4. Construction of the equivariant map

Now we shall define the equivariant map $\Phi : \mathrm{Alt}_5^{\oplus 4} \to S^2(\mathbb{C}^4)$ such that $\Phi(X) = (\varphi_{st}(X))$ and each φ_{st} is like $\varphi_{st} = \sum c_{stijklmi'j'k'l'm'}[ijklm][i'j'k'l'm']$ in which $[ijklm][i'j'k'l'm'](X) = [ijklm](X)[i'j'k'l'm'](X)$. Then we shall explain that we have the irreducible relative invariant in degree 40 as $\det \Phi(X)$. Furthermore, we shall prove that Φ is surjection.

First, we define the polynomials $\varphi_{11}, \varphi_{12}$ as

$$\begin{aligned}
\varphi_{11} &= 160[31114](3[24132]-2[21342]-2[23412])\\
&\quad +160[41112](3[32143]-2[34213]-2[31423])\\
&\quad +160[21113](3[43124]-2[41234]-2[42314])\\
&\quad +50([11233][11244]+[11322][11344]+[11422][11433])\\
&\quad -288([13241]^2+[14321]^2+[12431]^2)\\
&\quad +224([13241][14321]+[14321][12431]+[12431][13241]),\\
\varphi_{12} &= 400[31114][32224]\\
&\quad -100([21113][22344]+[21114][22433])\\
&\quad -100([12223][11344]+[12224][11433])\\
&\quad +20[13332](4[31423]-[34213]-[32143])\\
&\quad +20[14442](4[41324]-[43214]-[42134])\\
&\quad -25([22144][11233]+[11244][22133])\\
&\quad +368[13241][23142]\\
&\quad +112([13241]([21342]+[23412])+[23142]([12341]+[13421]))\\
&\quad +192([14321][23412]+[13421][24312])\\
&\quad -208([14321][21342]+[12431][23412]).
\end{aligned}$$

These polynomials satisfy the following properties:
(1) If $\sigma \in \mathfrak{S}_4$ and $\sigma(1) = 1$, then $\varphi_{11}^\sigma = \varphi_{11}$,
(2) If $\sigma \in \mathfrak{S}_4$ and $\{\sigma(1), \sigma(2)\} = \{1, 2\}$, then $\varphi_{12}^\sigma = \varphi_{12}$.

Then we define the map $\Phi : \mathrm{Alt}_5^{\oplus 4} \to S^2(\mathbb{C}^4)$ as $\Phi(X) = (\varphi_{st}(X))$ in which φ_{st} is

$$\varphi_{st} = \begin{cases} \varphi_{11}^{(1\ s)} & (s = t) \\ (\varphi_{12}^{(1\ s)})^{(2\ t)} & (s \neq t) \end{cases}$$

for $s, t \in \{1, 2, 3, 4\}$ and $(1\ s), (2\ t) \in \mathfrak{S}_4$. It is easily seen from (1), (2) that $\varphi_{st} = \varphi_{ts}$ and $\varphi_{st}^{\sigma} = \varphi_{\sigma^{-1}(s)\sigma^{-1}(t)}$ for all $\sigma \in \mathfrak{S}_4$.

Lemma 4.1. *For $X \in \mathrm{Alt}_5^{\oplus 4}$ and $(A, B) \in SL_5 \times GL_4$,*

$$\Phi(\rho(A, B)X) = (\det B)^2 B\Phi(X)\,{}^tB.$$

Proof. Let $D = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and let A be arbitrary element of SL_5. Since $\varphi_{st}^D = (\alpha_1\alpha_2\alpha_3\alpha_4)^2\alpha_s\alpha_t\varphi_{st}$ for all $s, t \in \{1, 2, 3, 4\}$, and each φ_{st} is invariant with respect to SL_5, we have

$$\Phi(\rho(A, D)X) = (\det D)^2 D\Phi(X)\,{}^tD.$$

From the remark after the definition of Φ,

$$\Phi(\rho(A, \sigma)X) = (\varphi_{\sigma^{-1}(s)\sigma^{-1}(t)}(X)) = \sigma\Phi(X)\,{}^t\sigma$$

for all $\sigma \in \mathfrak{S}_4$.

The rest of the proof is to show $\Phi(\rho(A, E_\varepsilon)X) = E_\varepsilon\Phi(X)\,{}^tE_\varepsilon$, i.e.

$$\begin{array}{ll} \text{(i)} & \varphi_{11}^{E_\varepsilon} = \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^2\varphi_{22}, \\ \text{(ii)} & \varphi_{1t}^{E_\varepsilon} = \varphi_{t1}^{E_\varepsilon} = \varphi_{1t} + \varepsilon\varphi_{2t} \text{ for } t = 2, 3, 4, \\ \text{(iii)} & \varphi_{st}^{E_\varepsilon} = \varphi_{ts}^{E_\varepsilon} = \varphi_{st} \text{ for } s, t = 2, 3, 4. \end{array}$$

First, we have directly

$$\varphi_{11}^{E_\varepsilon} = \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^2\varphi_{22}, \tag{4.1}$$
$$\varphi_{22}^{E_\varepsilon} = \varphi_{22}, \tag{4.2}$$
$$\varphi_{33}^{E_\varepsilon} = \varphi_{33}, \tag{4.3}$$
$$\varphi_{13}^{E_\varepsilon} = \varphi_{13} + \varepsilon\varphi_{23}, \tag{4.4}$$
$$\varphi_{34}^{E_\varepsilon} = \varphi_{34}. \tag{4.5}$$

From $E_\varepsilon^2 = E_{2\varepsilon}$ and (4.1),

$$\varphi_{11}^{E_\varepsilon^2} = \varphi_{11} + 4\varepsilon\varphi_{12} + 4\varepsilon^2\varphi_{22}.$$

Otherwise, from (4.1) and (4.2),

$$\begin{aligned} \varphi_{11}^{E_\varepsilon^2} &= \varphi_{11}^{E_\varepsilon} + 2\varepsilon\varphi_{12}^{E_\varepsilon} + \varepsilon^2\varphi_{22}^{E_\varepsilon} \\ &= \varphi_{11} + 2\varepsilon\varphi_{12} + 2\varepsilon\varphi_{12}^{E_\varepsilon} + 2\varepsilon^2\varphi_{22}. \end{aligned}$$

Therefore $\varphi_{12}^{E_\varepsilon} = \varphi_{12} + \varepsilon\varphi_{22}$.

Similarly from (4.4),

$$\begin{aligned}\varphi_{13}^{E_\varepsilon^2} &= \varphi_{13} + 2\varepsilon\varphi_{23} \\ &= \varphi_{13} + \varepsilon\varphi_{23} + \varepsilon\varphi_{23}^{E_\varepsilon}.\end{aligned}$$

Then we have $\varphi_{23}^{E_\varepsilon} = \varphi_{23}$.

From (4.4) and $E_\varepsilon(3\ 4) = (3\ 4)E_\varepsilon$, we have

$$\varphi_{14}^{E_\varepsilon} = \varphi_{13}^{(E_\varepsilon(3\ 4))} = \varphi_{13}^{((3\ 4)E_\varepsilon)} = (\varphi_{13}^{E_\varepsilon})^{(3\ 4)} = \varphi_{14} + \varepsilon\varphi_{24}.$$

Similarly from (4.3), we have $\varphi_{44}^{E_\varepsilon} = \varphi_{44}$. □

To prove that Φ is surjection, we only need to find five points in $\mathrm{Alt}_5^{\oplus 4}$ such that each image has rank $0, 1, 2, 3, 4$.

For

$$X_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{02} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$X_{03} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad X_{04} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$Y_{01} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_{02} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$Y_{03} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

we have

$$\Phi(X_{01},X_{02},X_{03},X_{04}) = \begin{pmatrix} 0 & 0 & -720 & 0 \\ 0 & -480 & 0 & 0 \\ -720 & 0 & 0 & 0 \\ 0 & 0 & 0 & -288 \end{pmatrix} \quad \text{(rank 4)},$$

$$\Phi(Y_{01},X_{02},X_{03},X_{04}) = \begin{pmatrix} -192 & 0 & -192 & -96 \\ 0 & -480 & 0 & 0 \\ -192 & 0 & -192 & -96 \\ -96 & 0 & -96 & -288 \end{pmatrix} \quad \text{(rank 3)},$$

$$\Phi(Y_{01},Y_{02},X_{03},X_{04}) = \begin{pmatrix} -192 & 0 & -192 & -96 \\ 0 & 0 & 0 & 0 \\ -192 & 0 & -192 & -96 \\ -96 & 0 & -96 & -288 \end{pmatrix} \quad \text{(rank 2)},$$

$$\Phi(Y_{01},Y_{02},Y_{03},X_{04}) = \begin{pmatrix} -192 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(rank 1)},$$

$$\Phi(0,0,0,0) = 0 \quad \text{(rank 0)}.$$

Therefore Φ is surjection and especially $\det\Phi(X) \not\equiv 0$. This fact and lemma 4.1 implies that $\det\Phi(X)$ is the relative invariant in degree 40.

Theorem 4.2. *(i) The map* $\Phi : \mathrm{Alt}_5^{\oplus 4} \to S^2(\mathbb{C}^4)$ *is surjection.*

(ii) $f(X) = \det\Phi(X)$ *is the irreducible relative invariant of the prehomogeneous vector space* $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, \mathrm{Alt}_5^{\oplus 4})$ *in degree* 40 *corresponding to the rational character* $(\det B)^4$.

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