Title: Koecher-Maass Dirichlet series for various liftings of Siegel modular forms (Theory of Prehomogeneous Vector Spaces)

Author(s): Katsurada, Hidenori

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Kyoto University
Siegel 保型形式の様々な持ち上げに付随する
Koecher-Maaß級数 (Koecher-Maaß Dirichlet
series for various liftings of Siegel modular
forms)

桂田 英典 (Hidenori Katsurada)
室蘭工業大学 (Muroran Institute of Technology)

1 Introduction

Let \( f(Z) \) be a Siegel modular form of weight \( k \) belonging to the symplectic
group \( \Gamma_n = Sp_n(\mathbb{Z}) \). Then \( f(Z) \) has the following Fourier expansion:

\[
    f(Z) = \sum_A a_f(A) \exp(2\pi i \text{tr}(AZ)),
\]

where \( A \) runs over all semi-positive definite half-integral matrices over \( \mathbb{Z} \) of
degree \( n \) and \( \text{tr}(X) \) denotes the trace of a matrix \( X \). We then define the
Koecher-Maaß Dirichlet series \( L(f, s) \) by

\[
    L(f, s) = \sum_A \frac{a_f(A)}{e(A)(\det A)^s},
\]

where \( A \) runs over a complete set of representatives of \( SL_n(\mathbb{Z}) \)-equivalence
classes of positive definite half-integral matrices of degree \( n \), and \( e(A) = \# \{ A \in SL_n(\mathbb{Z}); \hspace{1em}{}^tXAX = A \} \). We remark that in case \( n = 1 \), \( L(f, s) \) is
nothing but the Hecke L-series attached to \( f \).

Now let \( F(W) \) be a certain lifting of \( f(Z) \). Namely let \( F(W) \) be a modular
form with respect to \( \Gamma_m \) with some integer \( m \geq n \) whose standard zeta
function or spinor L-function is expressed by the standard zeta function or
the spinor L-function of \( f(Z) \). Then we present the following problem:
Problem 1. Express $L(F,s)$ in terms of Dirichlet series attached to $f$.

In this note, we consider the following two types of liftings, one the Klingen-Eisenstein lifting, and the other the Ikeda lifting. This work was partly collaborated with T. Ibukiyama.

2 Koecher-Maaß Dirichlet series for the Klingen-Eisenstein lifting

Let $r, n$ and $k$ be non-negative integers such that $0 \leq r \leq n \leq k - r - 2$ and $k \equiv 0 \mod 2$. For a cusp form $f$ of weight $k$ belonging to $\Gamma_r$, define $[f]^n_r(Z)$ as

$$[f]^n_r(Z) = \sum_{M \in \Delta_n,r \backslash \Gamma_n} f(M \prec Z \succ) j(M,Z)^{-k},$$

where $\Delta_n,r = \{(O_{n-r,n+r}^*, *) \in \Gamma_n \}$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M \prec Z \succ$ denote the upper left $(r \times r)$-block of the matrix $(AZ+B)(CZ+D)^{-1}$ and $j(M,Z) = \det(CZ+D)$. We note that $[1]^0_0(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$ of weight $k$. In [B], among others, Böcherer gave an explicit form of $L([f]^1_1,s)$ and $L(E_{2,k},s)$. In [I-K1] we gave an explicit form of $L(E_{n,k},s)$ for arbitrary $n$. We note that $L(E_{n,k},s)$ is also regarded as the zeta function of prehomogeneous vector space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to the above Problem 1 we should add one remark; in the explicit formula for $L([f]^1_1,s)$ by [B], a certain Dirichlet series attached to $f$ appears. Böcherer obtained a functional equation for it from the general theory of the Koecher-Maaß Dirichlet series. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K-Z], and is of importance in its own right. Hence the following problem seems very interesting.

Problem 2. Investigate the analytic and arithmetic properties of the Dirichlet series related to $f$ appearing in an explicit formula for $L([f]^n_r,s)$.

In this section, we give a resonable formula for $[f]^n_r$ when $f$ is a cuspidal Hecke eigenform belonging to $\Gamma_1$ and $n$ even. This also gives a certain generalization of Böcherer’s result in [B].
Now to state our main result in this section, for the fundamental discriminant \(d\) of a quadratic field, let \(\psi_d\) denote the Kronecker character associated with \(d\). Here we understand that \(\psi_1 = 1\). For \(l = \pm 1\), put

\[\mathcal{F}_l = \{D_0 \in \mathbb{Z}_{>0}; lD_0\text{ is the fundamental discriminant of a quadratic field or } 1\}\]

For an integer \(D\) such that \(lD > 0\) and \(D \equiv 1\) or \(\equiv 0 \mod 4\), write \(D = lD_0m^2\) with \(D_0 \in \mathcal{F}_l, m > 0\), and put

\[L_D(s) = L(s, \psi_{lD_0}) \sum_{d|m} \mu(d) \psi_{lD_0}(d) d^{-s} \sum_{c|m} c^{1-2s},\]

where \(L(s, \psi_{lD_0})\) is the Dirichlet L-function attached to \(\psi_{lD_0}\), and \(\mu\) is the Möbius function. Write \(L_D(s)\) as

\[L_D(s) = \sum_{\epsilon=1}^{\infty} \epsilon_D(\epsilon) \epsilon^{-s},\]

and for a cusp form \(f(z) = \sum_{e=1}^{\infty} b(e) \exp(2\pi i e z)\) of weight \(k\) with respect to \(\Gamma_1\) put

\[L(f, D, s) = \sum_{\epsilon=1}^{\infty} \epsilon_D(\epsilon) b(\epsilon) \epsilon^{-s}.\]

We note that

\[L(f, 1, s) = L(f, s).\]

Further for \(l = \pm 1\)

\[\mathcal{L}_l(f; \lambda, s) = \sum_D L(f, lD, \lambda) D^{-s},\]

where \(D\) runs over all positive integers such that \(D \equiv l, 0 \mod 4\). This type of Dirichlet series was originally introduced by Kohnen and Zagier [K-Z]. Assume that \(f\) is a Hecke eigenform. Then we note that

\[\mathcal{L}_l(f; \lambda, s) = \frac{\zeta^{st}(f, 2s + 2\lambda - k) \zeta(2s)}{\zeta(2s + 2\lambda - k)} \sum_{D_0 \in \mathcal{F}_l} D_0^{-s} L(f, lD_0, \lambda) \prod\{(1 + \psi_{lD_0}(p)^2 p^{-2s+k-1-2\lambda})(1 + p^{-2s+k-2\lambda}) - \psi_{lD_0}(p) b(p) p^{-2s-\lambda}(1 + p^{k-2\lambda})\},\]
where $\zeta(s)$ is Riemann's zeta function and $\zeta^*(f, s)$ is the standard zeta function of $f$.

**Theorem 1.** Let $n$ be an even positive integer. Then, under the above assumption, we have

$$L([f]^n_1, s) = 2^n \alpha_{n,k} \left[ L(f, k - n/2) \frac{\zeta(2s - 2i - 1)\zeta(2s - 2k + 2i + 2)}{\zeta^*(f, k - 1)} \prod_{i=1}^{n/2-1} \zeta(2s - 2i) \zeta(2s - 2k + 2i + 2) \mathcal{L}_{(-1)^{n/2}}(f; k-1, s-k+3/2) \right]$$

$$+\frac{(-1)^{n(n-2)/8} L(f, k-1)}{\zeta^*(f, k-1)} \frac{\zeta(2s - n + 1) \prod_{i=1}^{n/2-1} \zeta(2s - 2i) \zeta(2s - 2k + 2i + 1)}{\zeta(2s - 2k + 2i + 1)} \mathcal{L}_{(-1)^{n/2}}(f; k-n/2, s-k+(n+1)/2),$$

where $\alpha_{n,k}$ is a constant depending only on $n$ and $k$.

As for the proof, see [I-K2]. By the above theorem combined with the general theory of $L([f]^n_1, s)$ obtained by [M], we obtain

**Corollary.** Assume that $n \equiv 2 \mod 4$. Put

$$L_{-1}(f; \lambda, s) = \pi^{(2\lambda-2k)(s+\lambda-1/2)} \zeta(2s+4\lambda-2k) \Gamma(s+\lambda-1/2) \Gamma(s+\lambda-1) \mathcal{L}_{-1}(f; \lambda, s).$$

Then $L_{-1}(f; k-n/2, s)$ can be continued analytically to a meromorphic function of $s$ in the whole complex plane, and has the following functional equation:

$$L_{-1}(f; k-n/2, n+1-s-k) = L_{-1}(f; k-n/2, s).$$

**Remark.** If $n = 2$, the two terms in the above formula coincide with each other, and unify in one term. This is nothing but Böcherer's result [B,
3 Koecher-Maaß Dirichlet series for the Ikeda lifting

Let $f(z)$ be a normalized cuspidal Hecke eigenform of weight $2k - n$ with respect to $\Gamma_1$. Assume that $n$ and $k - n/2$ are even positive integers. Then Duke and Imamoglu conjectured that there exists a cuspidal Hecke eigenform $I(f)^n(Z)$ of weight $k$ with respect to $\Gamma_n$ such that

$$\zeta^s(I(f)^n, s) = \zeta(s) \prod_{i=1}^{n} L(f, s + k - i).$$

In [I], Ikeda constructed such a Hecke eigenform explicitly. Thus we call $I(f)^n(Z)$ the Ikeda lifting of $f$ to $\Gamma_n$. Let $\tilde{f}$ be the modular form of weight $k - n/2 + 1/2$ belonging to the Kohnen plus-space corresponding to $f$, and $E_{n/2+1/2}$ be the Cohen Eisenstein series of weight $n/2 + 1/2$. Let $L(\tilde{f}, s)$ and $L(E_{n/2+1/2}, s)$ be the Mellin transforms of $\tilde{f}$ and $E_{n/2+1/2}$, respectively, and $L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s)$ be the convolution product. Let

$$\tilde{f}(z) = \sum_{d_0} c(d_0) \exp(2\pi i |d_0|z),$$

where $d_0$ runs over all integers such that $(-1)^{k-n/2}d_0 \equiv 0, 1 \pmod{4}$. Then we note that $L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s)$ can be expressed as

$$L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s) = L(f, 2s)L(f, 2s - n + 1)$$

$$\times \sum_{d_0} c(d_0)d_0^{-s+(n-1)/2} \prod_p \{(1 + p^{-2s+k-1})(1 + \chi_p((-1)^{n/2}d_0)^2p^{-2s+k-2})$$

$$- \chi_p((-1)^{n/2}d_0)p^{-2s+k-3/2}\alpha_p(1 + p^{1/2-n/2}\alpha_p^{-1})(1 + p^{-1/2+n/2}\alpha_p^{-1})\} ,$$

where $\alpha_p$ denotes the Satake $p$-parameter determined by $f$.

**Theorem 2.** Under the above notation and assumption, we have

$$L(I(f)^n, s)$$

$$= 2^{ns}\beta_{n,k}[L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s) \prod_{i=1}^{n/2-1} L(f, 2s - 2i)$$
\[ +((-1)^{n/2} + 1)(-1)^{(n-2)/8} \prod_{i=1}^{n/2} L(f, 2s - 2i + 1), \]

where \( \beta_{n,k} \) is a constant depending only on \( n \) and \( k \).

As for the proof, see [I-K3].

**References**


[I] T. Ikeda, On the lifting elliptic modular forms to Siegel cusp forms of degree \( 2n \), preprint.


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Muroran Institute of Technology, 27-1 Mizumoto, Muroran, 050-8585 Japan

e-mail: hidenori@mmm. muroran-it.ac.jp