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<th>Koecher-Maass Dirichlet series for various liftings of Siegel modular forms (Theory of Prehomogeneous Vector Spaces)</th>
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<tr>
<td><strong>Author(s)</strong></td>
<td>Katsurada, Hidenori</td>
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<tr>
<td><strong>Citation</strong></td>
<td>数理解析研究所講究録 (2001), 1238: 63-68</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2001-11</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/2433/41574">http://hdl.handle.net/2433/41574</a></td>
</tr>
<tr>
<td><strong>Type</strong></td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td><strong>Textversion</strong></td>
<td>publisher</td>
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Kyoto University
Siegel 保型形式の様々な持ち上げに付随する Koecher-Maaß 級数 （Koecher-Maaß Dirichlet series for various liftings of Siegel modular forms）

桂田 英典 (Hidenori Katsurada)
室蘭工業大学 (Muroran Institute of Technology)

1 Introduction

Let $f(Z)$ be a Siegel modular form of weight $k$ belonging to the symplectic group $\Gamma_n = Sp_n(\mathbb{Z})$. Then $f(Z)$ has the following Fourier expansion:

$$f(Z) = \sum_{A} a_f(A) \exp(2\pi i \text{tr}(AZ)),$$

where $A$ runs over all semi-positive definite half-integral matrices over $\mathbb{Z}$ of degree $n$ and $\text{tr}(X)$ denotes the trace of a matrix $X$. We then define the Koecher-Maaß Dirichlet series $L(f, s)$ by

$$L(f, s) = \sum_{A} \frac{a_f(A)}{e(A)(\det A)^s},$$

where $A$ runs over a complete set of representatives of $SL_n(\mathbb{Z})$-equivalence classes of positive definite half-integral matrices of degree $n$, and $e(A) = \#\{A \in SL_n(\mathbb{Z}); \; ^{t}XAX = A\}$. We remark that in case $n = 1$, $L(f, s)$ is nothing but the Hecke L-series attached to $f$.

Now let $F(W)$ be a certain lifting of $f(Z)$. Namely let $F(W)$ be a modular form with respect to $\Gamma_m$ with some integer $m \geq n$ whose standard zeta function or spinor L-function is expressed by the standard zeta function or the spinor L-function of $f(Z)$. Then we present the following problem:
Problem 1. Express $L(F, s)$ in terms of Dirichlet series attached to $f$.

In this note, we consider the following two types of liftings, one the Klingen-Eisenstein lifting, and the other the Ikeda lifting. This work was partly collaborated with T. Ibukiyama.

2 Koecher-Maass Dirichlet series for the Klingen-Eisenstein lifting

Let $r$, $n$ and $k$ be non-negative integers such that $0 \leq r \leq n \leq k - r - 2$ and $k \equiv 0 \mod 2$. For a cusp form $f$ of weight $k$ belonging to $\Gamma_r$, define $[f]^n_r(Z)$ as

$$[f]^n_r(Z) = \sum_{M \in \Delta_{n,r} \setminus \Gamma_n} f(M < Z >^*) j(M, Z)^{-k},$$

where $\Delta_{n,r} = \left\{ \begin{pmatrix} * & * \\ O_{n-r,n+r} & * \end{pmatrix} \right\} \in \Gamma_n$, and for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ let $M < Z >^*$ denote the upper left $(r \times r)$-block of the matrix $(AZ + B)(CZ + D)^{-1}$ and $j(M, Z) = \det(CZ + D)$. We note that $[1]^n_0(Z)$ is nothing but the Siegel Eisenstein series $E_{n,k}(Z)$ of weight $k$. In [B], among others, Böcherer gave an explicit form of $L([f]^2_1, s)$ and $L(E_{2,k}, s)$. In [I-K1] we gave an explicit form of $L(E_{n,k}, s)$ for arbitrary $n$. We note that $L(E_{n,k}, s)$ is also regarded as the zeta function of prehomogeneous vector space. From this point of view, Saito gave a generalization of our result (cf. [Sa]). In relation to the above Problem 1 we should add one remark; in the explicit formula for $L([f]^2_1, s)$ by [B], a certain Dirichlet series attached to $f$ appears. Böcherer obtained a functional equation for it from the general theory of the Koecher-Maass Dirichlet series. This Dirichlet series is a modification of the Dirichlet series originally defined by Kohnen and Zagier [K-Z], and is of importance in its own right. Hence the following problem seems very interesting.

Problem 2. Investigate the analytic and arithmetic properties of the Dirichlet series related to $f$ appearing in an explicit formula for $L([f]^n_r, s)$.

In this section, we give a resonable formula for $[f]^n_r$ when $f$ is a cuspidal Hecke eigenform belonging to $\Gamma_1$ and $n$ even. This also gives a certain generalization of Böcherer's result in [B].
Now to state our main result in this section, for the fundamental discriminant \( d \) of a quadratic field, let \( \psi_d \) denote the Kronecker character associated with \( d \). Here we understand that \( \psi_1 = 1 \). For \( l = \pm 1 \), put
\[
\mathcal{F}_l = \{ D_0 \in \mathbb{Z}_{>0}; lD_0 \text{ is the fundamental discriminant of a quadratic field or } 1 \}
\]

For an integer \( D \) such that \( lD > 0 \) and \( D \equiv 1 \) or \( \equiv 0 \mod 4 \), write \( D = lD_0m^2 \) with \( D_0 \in \mathcal{F}_l, m > 0 \), and put
\[
L_D(s) = L(s, \psi_{lD_0}) \sum_{d|m} \mu(d) \psi_{lD_0}(d) d^{-s} \sum_{c|m} c^{1-2s},
\]
where \( L(s, \psi_{lD_0}) \) is the Dirichlet L-function attached to \( \psi_{lD_0} \), and \( \mu \) is the Möbius function. Write \( L_D(s) \) as
\[
L_D(s) = \sum_{e=1}^{\infty} \zeta_D(e)e^{-s},
\]
and for a cusp form \( f(z) = \sum_{e=1}^{\infty} b(e) \exp(2\pi i ez) \) of weight \( k \) with respect to \( \Gamma_1 \) put
\[
L(f, D, s) = \sum_{e=1}^{\infty} c_D(e)b(e)e^{-s}.
\]
We note that
\[
L(f, 1, s) = L(f, s).
\]
Further for \( l = \pm 1 \)
\[
\mathcal{L}_l(f; \lambda, s) = \sum_D L(f, lD, \lambda) D^{-s},
\]
where \( D \) runs over all positive integers such that \( D \equiv l, 0 \mod 4 \). This type of Dirichlet series was originally introduced by Kohnen and Zagier [K-Z]. Assume that \( f \) is a Hecke eigenform. Then we note that
\[
\mathcal{L}_l(f; \lambda, s) = \frac{\zeta(s, 2s + 2\lambda - k)\zeta(2s)}{\zeta(2s + 2\lambda - k)} \sum_{D_0 \in \mathcal{F}_l} D_0^{-s} L(f, lD_0, \lambda)
\]
\[
\times \prod \{(1 + \psi_{lD_0}(p)^2 p^{-2s+k-1-2\lambda}) (1 + p^{-2s+k-2\lambda}) - \psi_{lD_0}(p) b(p) p^{-2s-\lambda}(1 + p^{k-2\lambda})\},
\]
where $\zeta(s)$ is Riemann's zeta function and $\zeta^{st}(f, s)$ is the standard zeta function of $f$.

**Theorem 1.** Let $n$ be an even positive integer. Then, under the above assumption, we have

\[
L([f]_1^n, s) = 2^{n^2} \alpha_{n,k} \left\{ \frac{L(f, k-n/2)}{\zeta^{st}(f, k-1)} \zeta(2s-1) \prod_{i=1}^{n/2-1} \zeta(2s-2i-1) \zeta(2s-2k+2i+2) \right. \\
\left. \times \mathcal{L}_{(-1)^{n/2}}(f; k-1, s-k+3/2) \right.
\]

\[ + (-1)^{n(n-2)/8} \frac{L(f, k-1)}{\zeta^{st}(f, k-1)} \zeta(2s-n+1) \prod_{i=1}^{n/2-1} \zeta(2s-2i) \zeta(2s-2k+2i+1) \\
\times \mathcal{L}_{(-1)^{n/2}}(f; k-n/2, s-k+(n+1)/2), \]

where $\alpha_{n,k}$ is a constant depending only on $n$ and $k$.

As for the proof, see [I-K2]. By the above theorem combined with the general theory of $L([f]_1^n, s)$ obtained by [M], we obtain

**Corollary.** Assume that $n \equiv 2 \mod 4$. Put

\[
\mathcal{L}_{-1}(f; \lambda, s) = \pi^{(2\lambda-2k)(s+\lambda-1/2)} \zeta(2s+2\lambda-2k) \Gamma(s+\lambda-1/2) \Gamma(s+\lambda-1) \mathcal{L}_{-1}(f; \lambda, s).
\]

Then $\mathcal{L}_{-1}(f; k-n/2, s)$ can be continued analytically to a meromorphic function of $s$ in the whole complex plane, and has the following functional equation:

\[
\mathcal{L}_{-1}(f; k-n/2, n+1-s-k) = \mathcal{L}_{-1}(f; k-n/2, s).
\]

**Remark.** If $n = 2$, the two terms in the above formula coincide with each other, and unify in one term. This is nothing but Böcherer's result [B,
3 Koecher-Maaß Dirichlet series for the Ikeda lifting

Let $f(z)$ be a normalized cuspidal Hecke eigenform of weight $2k - n$ with respect to $\Gamma_1$. Assume that $n$ and $k - n/2$ are even positive integers. Then Duke and Imamoglu conjectured that there exists a cuspidal Hecke eigenform $I(f)^n(Z)$ of weight $k$ with respect to $\Gamma_n$ such that

$$\zeta^*(I(f)^n, s) = \zeta(s) \prod_{i=1}^{n} L(f, s + k - i).$$

In [I], Ikeda constructed such a Hecke eigenform explicitly. Thus we call $I(f)^n(Z)$ the Ikeda lifting of $f$ to $\Gamma_n$. Let $\tilde{f}$ be the modular form of weight $k - n/2 + 1/2$ belonging to the Kohnen plus-space corresponding to $f$, and $E_{n/2+1/2}$ be the Cohen Eisenstein series of weight $n/2 + 1/2$. Let $L(\tilde{f}, s)$ and $L(E_{n/2+1/2}, s)$ be the Mellin transforms of $\tilde{f}$ and $E_{n/2+1/2}$, respectively, and $L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s)$ be the convolution product. Let

$$\tilde{f}(z) = \sum_{d_0} c(d_0) \exp(2\pi i |d_0| z),$$

where $d_0$ runs over all integers such that $(-1)^{k-n/2}d_0 \equiv 0, 1 \mod 4$. Then we note that $L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s)$ can be expressed as

$$L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s) = L(f, 2s)L(f, 2s-n+1)$$

$$\times \sum_{d_0} c(d_0)d_0^{s+(n-1)/2} \prod_p \{(1 + p^{-2s+k-1})(1 + \chi_p((-1)^{n/2}d_0)^2p^{-2s+k-2})$$

$$-\chi_p((-1)^{n/2}d_0)p^{-2s+k-2/3}\alpha_p(1 + p^{1/2-n/2}\alpha_p^{-1})(1 + p^{-1/2+n/2}\alpha_p^{-1})\},$$

where $\alpha_p$ denotes the Satake $p$-parameter determined by $f$.

**Theorem 2.** Under the above notation and assumption, we have

$$L(I(f)^n, s)$$

$$= 2^{n^2}\beta_{n,k}[L(\tilde{f}, s) \otimes L(E_{n/2+1/2}, s)] \prod_{i=1}^{n/2-1} L(f, 2s - 2i)$$
\[ +((-1)^{n/2} + 1)(-1)^{(n-2)/8} \prod_{i=1}^{n/2} L(f, 2s - 2i + 1), \]

where \( \beta_{n,k} \) is a constant depending only on \( n \) and \( k \).

As for the proof, see [I-K3].

References


[I-K3], An explicit formula for the Koecher-Maaß Dirichlet series for the Ikeda lifting, preprint.


Muroran Institute of Technology, 27-1 Mizumoto, Muroran, 050-8585 Japan

e-mail: hidenori@mmm. muroran-it.ac.jp