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An approach to calculation of b-functions by using functional equations

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1 Introduction

It is well known that the b-function of a regular prehomogeneous vector space satisfies a certain functional equation. In this note, we shall explain the method of calculation of b-functions by using the functional equations. Starting with the case of one variable, we illustrate how we employ the functional equations to determine the explicit forms of b-functions.

Let (G,ρ,V) be an irreducible regular prehomogeneous vector space and f an irreducible relative invariant corresponding to a character ϕ. Denote by $(G,\rho^{\vee},V^{\vee})$ the dual prehomogeneous vector space and by $f^{\vee}$ an irreducible relative invariant on $V^{\vee}$ corresponding to the character ϕ^{-1}. Then the b-function $b_f(s)$ of f is defined as the polynomial of s satisfying

$$f^{\vee}(\mathrm{grad}_x)f(x)^{s+1}=b_f(s)f(x)^s. \tag{1.1}$$

M. Kashiwara [3] proved that the roots of $b_f(s)$ are negative rational numbers :

$$b_f(s)=b_0\prod_{j=1}^{d}(s+\alpha_j), \qquad (\alpha_j\in\mathbb{Q}_{>0}). \tag{1.2}$$

Moreover, by the regularity condition, $b_f(s)$ satisfies the following functional equation :

$$b_f(s)=(-1)^d b_f\left(-s-\frac{n}{d}-1\right), \tag{1.3}$$

where $d=\deg f$, $n=\dim V$. Then (1.2) and (1.3) imply a relation among α_j as

$$\{\alpha_1,\ldots,\alpha_d\}=\left\{\frac{n}{d}+1-\alpha_1,\ \ldots,\ \frac{n}{d}+1-\alpha_d\right\}.$$

Now let us suppose that $(s+\beta)$ is a factor of $b_f(s)$. We then obtain another factor $\left(s+\frac{n}{d}+1-\beta\right)$ of $b_f(s)$ by the above relation. Though these two factors may coincide, this simple observation is effective in the determination of $b_f(s)$.

In early days of the theory of prehomogeneous vector spaces, they used this observation to determine some b-functions, combining with the singular-orbit-method developed by M. Sato. However, if some factor $(s+\gamma)$ of $b_f(s)$ has the multiplicity $e \geq 2$, that is, $(s+\gamma)^e$ divides $b_f(s)$, we can not determine such e by this method. This difficulty was one of the motivations of microlocal calculus—so called SKKO algorithm [8], and in fact, all the b-functions of irreducible prehomogeneous vector spaces were settled by microlocal calculus [1].

For a prehomogeneous vector spaces with several relative invariants, we can define the b-functions of several variables. Also microlocal calculus is generalized to b-functions of several variables, and S. Kasai calculate microlocal structures of some non-irreducible prehomogeneous vector spaces. However, his results suggest that it is hard to apply the microlocal method for b-functions of several variables (see [11] and its references). Moreover, the author learned from A. Gyoja that K. Ukai could not determine some b-functions when he had used microlocal calculus.

On the other hand, K. Ukai [15, 16] approaches to explicit calculation of b-functions from quite a different view point. The method in [15] can be outlined as follows:

$$\boxed{\text{Contraction}} \quad + \quad \boxed{\text{Functional equation}} \quad + \quad \boxed{\text{Expansion formula}}$$

First we calculate the contraction of the prehomogeneous vector space in question. It is often easy to calculate the b-function after the contraction. Quoting the theorem of A. Gyoja which asserts that the exponential b-function is preserved under the contraction, we obtain the exponential b-function of the original space. Thus the roots of $b_f(s)$ are evaluated modulo $\mathbb{Z}$. Moreover, the expansion formula of the relative invariant involves some information (e.g. the product of the roots) about $b_f(s)$. Combining these data, we can recover the original b-function $b_f(s)$ from the exponential b-function. This is a framework of [15] [2].

In this note, we shall explain the method in [16], which is summarized as follows:

$$\boxed{\text{Functional equation}} \quad + \quad \boxed{\text{Localization}}$$

We recall the functional equation in § 2, and the localization of b-functions in § 3. In § 4, we actually calculate the b-functions along our method for $(GL_4 \times GL_2, \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1)$, and in § 5, we give a brief exposition on recent developments in explicit calculation of b-functions. Our method is classical and limited. However, once we find that we can apply this method for the prehomogeneous vector spaces in question, it works systematically and powerfully.

[1] There are two exceptions, namely, type (8) and (11) in [9]. In these cases, we need more advanced formulae in microlocal analysis.

[2] For the definitions of contractions and exponential b-functions, refer to [1, 13]. I learned the work of [15] in the excellent lecture of Professor Fumihiro Sato [10]. Although my talk in the conference was about the result on calculation based on the method in [15], I would like to explain more recent results. See [10, 11] for the subject on which I gave the talk.

2 a-Functions and b-functions

In this section, we give the definitions of a-functions and b-functions and some properties of them. For the detail, see [6, 7] and [1] in this volume.

Let G be a connected reductive algebraic group defined over $\mathbb{C}$, and $\rho : G \to GL(V)$ a rational representation of G on a finite dimensional vector space V. Assume that (G, ρ, V) is a prehomogeneous vector space and let $f_1, \dots, f_l$ be its fundamental relative invariants. Let $f_1^\vee, \dots, f_l^\vee$ be the irreducible relative invariants of the dual prehomogeneous vector space $(G, \rho^\vee, V^\vee)$ such that the characters of f_i and $f_i^\vee$ are the inverse of each other. We put $\underline{f} := (f_1, \dots, f_l)$, $\underline{f}^\vee := (f_1^\vee, \dots, f_l^\vee)$ and $V_{f_i} := \{v \in V;\ f_i(v) \neq 0\}$, $V_{\underline{f}} := \bigcap_{i=1}^l V_{f_i}$. For a multi-variable $\underline{s} = (s_1, \dots, s_l)$, we consider formally the powers $f_i^{s_i}$ and $f_i^{\vee s_i}$, their products $\underline{f}^{\underline{s}} := \prod_{i=1}^l f_i^{s_i}$ and $\underline{f}^{\vee \underline{s}} := \prod_{i=1}^l f_i^{\vee s_i}$.

Lemma 2.1. For any l-tuple $\underline{m} = (m_1, \dots, m_l) \in \mathbb{Z}_{\geq 0}^l$ of non-negative integers, we have

$$\underline{f}^{\underline{m}}(v)\underline{f}^{\vee \underline{m}}(\operatorname{grad} \log \underline{f}^{\underline{s}}(v)) = a_{\underline{m}}(\underline{s})$$

for all $v \in V_{\underline{f}}$ with some non-zero homogeneous polynomial $a_{\underline{m}}(\underline{s})$ which is independent of v.

We call $a_{\underline{m}}(\underline{s})$ the *a-function* of $\underline{f}$. When $\underline{m} = \varepsilon_i := (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears at ith place, we write $a_i(\underline{s})$ instead of $a_{\varepsilon_i}(\underline{s})$ for an abbreviation. We can easily see that $a_{\underline{m}}(\underline{s}) = \prod_{i=1}^l a_i(\underline{s})^{m_i}$ by definition. We have the following lemma about the structure of the a-function $a_{\underline{m}}(\underline{s})$.

Lemma 2.2. The a-function $a_{\underline{m}}(\underline{s})$ is expressed as the product of some linear forms :

$$a_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^N \left(\gamma_j(\underline{s})^{\gamma_j(\underline{m})}\right)^{\mu_j}.$$

Here $\underline{A}^{\underline{m}} = \prod_{i=1}^l A_i^{m_i}$ with $A_i \in \mathbb{C}^\times$, $N \in \mathbb{Z}_{>0}$, $\mu_j \in \mathbb{Z}_{>0}$, while each $\gamma_j(\underline{s})$ is a $\mathbb{Z}$-linear function $\sum_{i=1}^l \gamma_{ij} s_i$ with $\gamma_{ij} \in \mathbb{Z}_{\geq 0}$, $\mathrm{GCD}(\gamma_{1j}, \dots, \gamma_{lj}) = 1$.

Now we give the definition of the b-functions of several variables.

Lemma 2.3. For any l-tuple $\underline{m} = (m_1, \dots, m_l) \in \mathbb{Z}_{\geq 0}^l$ of non-negative integers, we have the functional equation

$$\underline{f}^{\vee \underline{m}}(\operatorname{grad})\underline{f}^{\underline{s}+\underline{m}} = b_{\underline{m}}(\underline{s})\underline{f}^{\underline{s}}$$

with some non-zero polynomial $b_{\underline{m}}(\underline{s})$ of $\underline{s}$.

We call the polynomial $b_{\underline{m}}(\underline{s})$ the *b-function* of $\underline{f}$. We write $b_i(\underline{s})$ instead of $b_{\varepsilon_i}(\underline{s})$ for an abbreviation. Let $a_{\underline{m}}(\underline{s})$ be the a-function as in Lemma 2.2. Then the following lemmas tell us the structures of $b_i(\underline{s})$ and $b_{\underline{m}}(\underline{s})$ to some extent.

Lemma 2.4. The b-function $b_i(\underline{s})$ is expressed as

$$b_i(\underline{s}) = A_i \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\varepsilon_i)-1} \prod_{r=1}^{\mu_j} (\gamma_j(\underline{s}) + \alpha_{j,r} + \nu).$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$.

Lemma 2.5. The b-function $b_{\underline{m}}(\underline{s})$ is expressed as

$$b_{\underline{m}}(\underline{s}) = \underline{A}^{\underline{m}} \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} (\gamma_j(\underline{s}) + \alpha_{j,r} + \nu).$$

with the same $\alpha_{j,r} \in \mathbb{Q}_{>0}$ as in Lemma 2.4.

Hence the calculation of $b_{\underline{m}}(\underline{s})$ is reduced to that of each $b_i(\underline{s})$ for $i = 1, \ldots, l$. If we know the a-function $a_{\underline{m}}(\underline{s})$, the remaining task is to determine the positive rational numbers $\alpha_{j,r}$ in Lemma 2.4. The following three tools are effective for the determination of $\alpha_{j,r}$.

(1) The results on the b-functions of irreducible prehomogeneous vector spaces.

(2) Functional equations satisfied by b-functions.

(3) Localization of b-functions.

Now we explain (1). By the definition of $b_i(\underline{s})$, we have that

$$f_i^\vee(\mathrm{grad})\underline{f}^{\underline{s}+\varepsilon_i} = b_i(\underline{s})\underline{f}^{\underline{s}}.$$

Putting $\underline{s} = s\varepsilon_i$ into the above, we have that

$$f_i^\vee(\mathrm{grad})f_i^{s+1} = b_i(s\varepsilon_i)f_i^s$$

and thus (if we ignore the scalar multiples)

$$b_i(s\varepsilon_i) = b_{f_i}(s) \tag{2.1}$$

where $b_{f_i}(s)$ is the b-function of f_i in the sense of (1.1). Hence the candidates for $\alpha_{j,r}$ are limited provided that the explicit form of $b_{f_i}(s)$ is known.

Next we shall state functional equations satisfied by b-functions. When (G, ρ, V) is a regular prehomogeneous vector space, a certain functional equation holds.

Lemma 2.6. If (G, ρ, V) is a regular prehomogeneous vector space, there exists a relative invariant whose character is $\det \rho(g)^2$. Here we denote by $\det \rho(g)$ the determinant of $\rho(g)$ in V. We define $2\underline{\kappa} \in \mathbb{Z}^l$ by the condition

$$\underline{f}^{2\underline{\kappa}}(\rho(g)v) = \det \rho(g)^2 \underline{f}^{2\underline{\kappa}}(v).$$

Theorem 2.7. Let the b-function $b_{\underline{m}}(\underline{s})$ be as in Lemma 2.5. We define a function $\beta_{\gamma_j}(u)$ of u by

$$\beta_{\gamma_j}(u) := \prod_{r=1}^{\mu_j} (u + \alpha_{j,r})$$

and let $\underline{\kappa}$ be in Lemma 2.6. Then for each $j = 1, \ldots, N$, the following functional equation holds:

$$\beta_{\gamma_j}(u) = (-1)^{\mu_j} \beta_{\gamma_j}(-u - \gamma_j(\underline{\kappa}) - 1).$$

3 Localization of b-functions

Now we consider the following situation.

Assumption 3.1. (1) Let (G, ρ, V) be a reductive prehomogeneous vector space.

(2) The representation $\rho : G \to GL(V)$ is of the form

$$\rho = \sigma \oplus \tau, \quad V = E \oplus F,$$

where E, F are some G-invariant subspaces of V and $\sigma : G \to GL(E)$, $\tau : G \to GL(F)$ are the subrepresentations of ρ. That is, we consider a non-irreducible prehomogeneous vector space.

(3) There exists a relative invariant polynomial f on V corresponding to a character ϕ : For all $g \in G$ and $(x, y) \in V = E \oplus F$, we have

$$f(\sigma(g)x, \tau(g)y) = \phi(g) f(x, y). \tag{3.1}$$

For simplicity, we assume that f contains both of the variables x of E and y of F.

Let $v_0 = (x_0, y_0) \in V$ be a generic point of (G, ρ, V). Then x_0 is a generic point of (G, σ, E). Furthermore, we put the following assumption.

Assumption 3.2. The generic isotropy subgroup G_{x_0} of (G, σ, E) at x_0 is reductive.

By the assumption above, (G_{x_0}, τ, F) is a reductive prehomogeneous vector space. We see that $f_F(y) = f(x_0, y)$ is a relative invariant of (G_{x_0}, τ, F). We thus obtain the b-function $b_{f_F}(s)$ of f_F in the sense of (1.1). Then the following theorem holds (cf. [12, 16]).

Theorem 3.3. Let $b_{f_F}(s)$ the b-function of f_F and $b_f(s)$ the b-function of f. Then $b_{f_F}(s)$ divides $b_f(s)$.

Although it seems that Assumptions 3.1, 3.2 can be replaced by some weaker condition, we can apply the above theorem for a sufficiently large class of prehomogeneous vector spaces. The author hopes to discuss the generalized theorem elsewhere.

4 An example of calculation

As an example, we shall calculate the b-functions of the following regular 2-simple prehomogeneous vector space (cf. [4]).

$$(G, \rho, V) = (GL_4 \times GL_2,\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1,\ \mathrm{Alt}_4^{\oplus 2} \oplus M_{4,2}).$$

Here $\mathrm{Alt}_4 = \{X \in M_4\,;\, {}^tX = -X\}$ and the representation ρ is defined by

$$\rho(g)x = \left((AX_1{}^tA, AX_2{}^tA)^tB\,;\, AY^tB\right)$$

for $g = (A, B) \in G$ and $x = (X_1, X_2\,; Y) \in V$. This prehomogeneous vector space has two fundamental relative invariants f_1, f_2 and their explicit constructions are given in [5]. Now we recall the construction of f_1. For $X, Y \in \mathrm{Alt}_4$, we put $\beta(X, Y) = \mathrm{Pf}(X+Y) - \mathrm{Pf}(X) - \mathrm{Pf}(Y)$ and define the matrix $\Phi(X_1, X_2)$ by

$$\Phi(X_1, X_2) = \begin{pmatrix} \beta(X_1, X_2) & \beta(X_1, X_2) \\ \beta(X_2, X_1) & \beta(X_2, X_2) \end{pmatrix} \in \mathrm{Sym}_2.$$

We can easily check that

$$\Phi\left((AX_1{}^tA, AX_2{}^tA)^tB\right) = (\det A) \cdot B\Phi(X_1, X_2)^tB.$$

So, if we define the polynomial function f_1 on V by

$$f_1(X_1, X_2, Y) := \det \Phi(X_1, X_2),$$

then f_1 is a relative invariant corresponding to the character $\phi_1 = (\det A)^2(\det B)^2$.

Since the construction of f_2 is much more complicated, we do not reproduce it here. However, we quote two useful pieces of information from [5].

(1) $\deg f_2 = 8$. (More precisely, $\deg_{(X_1,X_2)} f_2 = 4$, $\deg_Y f_2 = 4$.)

(2) The character ϕ_2 corresponding to f_2 is given by $\phi_2 = (\det A)^3 (\det B)^4$.

Actually, we do not need to know the explicit construction of f_2 if we know (1) and (2). Note that information about degrees and characters can be obtained from not only explicit construction of the relative invariant, but also the other methods such as calculation on isotropy subgroups.

In addition to, the coefficient $\underline{A}^{\underline{m}}$ of the b-function $b_{\underline{m}}(\underline{s})$ becomes meaningless, unless the relative invariants are normalized carefully. Henceforth, we shall ignore the scalar multiple $\underline{A}^{\underline{m}}$ in the calculation of a-functions and b-functions. Now let

$$X_{1,0} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad X_{2,0} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right), \quad Y_0 = \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right).$$

Then $v_0 = (X_{1,0}, X_{2,0}; Y_0)$ is a generic point of (G, ρ, V). For this v_0, we shall calculate the values $\operatorname{grad}\log f_1(v_0)$ and $\operatorname{grad}\log f_2(v_0)$. By the relative invariance of f_1, f_2, we have that

$$\begin{aligned} \langle \operatorname{grad}\log f_1(v_0), d\rho(A,B)v_0 \rangle &= d\phi_1(A,B)\,(= 2\operatorname{tr} A + 2\operatorname{tr} B). \\ \langle \operatorname{grad}\log f_2(v_0), d\rho(A,B)v_0 \rangle &= d\phi_2(A,B)\,(= 3\operatorname{tr} A + 4\operatorname{tr} B). \end{aligned}$$

for $(A, B) \in \mathrm{Lie}(G) = \mathfrak{gl}_4 \oplus \mathfrak{gl}_2$. However, since $\{d\rho(A,B)v_0;\ (A,B) \in \mathrm{Lie}(G)\} = V$ by the prehomogenuity, the above relations determine the values $\operatorname{grad}\log f_1(v_0)$ and $\operatorname{grad}\log f_2(v_0)$ uniquely, and the results are the following:

$$\operatorname{grad}\log f_1(v_0) = \left(\left(\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right), \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right) \right),$$

$$\operatorname{grad}\log f_2(v_0) = \left(\left(\begin{array}{cc|cc} 0 & 2 & 0 & -1 \\ -2 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{array}\right), \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right) \right).$$

We put $x_s = (X_{1,s}, X_{2,s}, ; Y_s) := \operatorname{grad}\log \underline{f}^{\underline{s}}(v_0) = s_1 \operatorname{grad}\log f_1(v_0) + s_2 \operatorname{grad}\log f_2(v_0)$, and calculate $f_1(x_s)$. It follows that

$$\begin{aligned} f_1(x_s) &= \det \begin{pmatrix} \beta(X_{1,s}, X_{1,s}) & \beta(X_{1,s}, X_{2,s}) \\ \beta(X_{2,s}, X_{1,s}) & \beta(X_{2,s}, X_{2,s}) \end{pmatrix} \\ &= -16\, s_1 (s_1 + s_2)^2 (s_1 + 2s_2) \end{aligned}$$

and thus the a-function $a_1(\underline{s})$ is given by

$$a_1(\underline{s}) = s_1 (s_1 + s_2)^2 (s_1 + 2s_2)$$

if we ignore the scalar multiple. By Lemma 2.2, we see that the only s_1, s_2, s_1+s_2 and s_1+2s_2 can appear as the factors of the a-function $a_{\underline{m}}(\underline{s})$. Moreover, the multiplicity μ_j in the lemma are determined except for μ_2 :

$$\begin{aligned}
\gamma_1(\underline{s}) &= s_1, & \mu_1 &= 1,\\
\gamma_2(\underline{s}) &= s_2, & \mu_2 &= ?,\\
\gamma_3(\underline{s}) &= s_1+s_2, & \mu_3 &= 2,\\
\gamma_4(\underline{s}) &= s_1+2s_2, & \mu_4 &= 1.
\end{aligned}$$

Again by Lemma 2.2, we have that

$$a_2(\underline{s}) = s_2^{\mu_2}(s_1+s_2)^2(s_1+2s_2)^2.$$

However, μ_2 must be equal to 4 because of $\deg a_2(\underline{s}) = \deg f_2 = 8$. Hence we obtain

$$\begin{aligned}
a_1(\underline{s}) &= s_1(s_1+s_2)^2(s_1+2s_2),\\
a_2(\underline{s}) &= s_2^4(s_1+s_2)^2(s_1+2s_2)^2.
\end{aligned}$$

Using the structure theorem of b-functions (Lemma 2.4), we see that

$$\begin{aligned}
b_1(\underline{s}) &= (s_1+\alpha_{1,1})(s_1+s_2+\alpha_{3,1})(s_1+s_2+\alpha_{3,2})(s_1+2s_2+\alpha_{4,1}),\\
b_2(\underline{s}) &= (s_2+\alpha_{2,1})(s_2+\alpha_{2,2})(s_2+\alpha_{2,3})(s_2+\alpha_{2,4})(s_1+s_2+\alpha_{3,1})\\
&\quad (s_1+s_2+\alpha_{3,2})(s_1+2s_2+\alpha_{4,1})(s_1+2s_2+\alpha_{4,1}+1)
\end{aligned}$$

with some $\alpha_{j,r} \in \mathbb{Q}_{>0}$.

Since f_1 is a relative invariant of the irreducible regular prehomogeneous vector space $(SL_4 \times GL_2, \Lambda_2 \otimes \Lambda_1) \cong (SO_6 \times GL_2, \Lambda_1 \otimes \Lambda_1)$, the b-function $b_{f_1}(s)$ of f_1 is given by

$$b_{f_1}(s) = (s+1)\left(s+\frac{3}{2}\right)(s+3)\left(s+\frac{5}{2}\right).$$

See [8, §9]. Combining this with (2.1), we have that

$$\{\alpha_{1,1}, \alpha_{3,1}, \alpha_{3,2}, \alpha_{4,1}\} = \left\{1,\ \frac{3}{2},\ 3,\ \frac{5}{2}\right\}. \tag{4.1}$$

Now we shall appeal to the functional equations. We easily see that $\underline{\kappa} = (1,2)$. So Theorem 2.7 implies the relations among $\{\alpha_{j,r}\}$ as

$$\begin{aligned}
\{\alpha_{1,1}\} &= \{2-\alpha_{1,1}\} \quad \therefore \alpha_{1,1} = 1.\\
\{\alpha_{2,1},\ \alpha_{2,2},\ \alpha_{2,3},\ \alpha_{2,4}\} &= \{3-\alpha_{2,1},\ 3-\alpha_{2,2},\ 3-\alpha_{2,3},\ 3-\alpha_{2,4}\}.\\
\{\alpha_{3,1},\ \alpha_{3,2}\} &= \{4-\alpha_{3,1},\ 4-\alpha_{3,2}\}.\\
\{\alpha_{4,1}\} &= \{6-\alpha_{4,1}\} \quad \therefore \alpha_{4,1} = 3.
\end{aligned}$$

Together with (4.1), we get

$$\{\alpha_{3,1}, \alpha_{3,2}\} = \left\{\frac{3}{2}, \frac{5}{2}\right\}.$$

So far, we have observed that

$$\begin{aligned}
b_1(\underline{s}) &= (s_1+1)\left(s_1+s_2+\frac{3}{2}\right)\left(s_1+s_2+\frac{5}{2}\right)(s_1+2s_2+3),\\
b_2(\underline{s}) &= (s_2+\alpha_{2,1})(s_2+\alpha_{2,2})(s_2+\alpha_{2,3})(s_2+\alpha_{2,4})\left(s_1+s_2+\frac{3}{2}\right)\\
&\quad \left(s_1+s_2+\frac{5}{2}\right)(s_1+2s_2+3)(s_1+2s_2+4).
\end{aligned}$$

Thus it remains to determine $\alpha_{2,r}$.

We shall make use of localization of b-functions here (see § 3) . In particular, we shall apply Theorem 3.3 to the case $E = \mathrm{Alt}_4^{\oplus 2}$, $F = M_{4,2}$. If we put $x_0 = (X_{1,0}, X_{2,0}) \in \mathrm{Alt}_4^{\oplus 2}$, then

$$\mathfrak{g}_{x_0} = \left\{ \left(\begin{array}{cc|cc} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ \hline & & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{array} \right), \begin{pmatrix} -a_{11}-a_{22} & 0 \\ 0 & -a_{33}-a_{44} \end{pmatrix} \right\},$$

and thus we have

$$(G_{x_0}, M_{4,2}) \cong (GL_2 \times GL_2, M_2 \oplus M_2).$$

More precisely, the latter prehomogeneous vector space is given as follows: For $(s,t) \in GL_2 \times GL_2$, the action is given by

$$M_2 \oplus M_2 \ni (u,v) \longmapsto \left(s\,u \begin{pmatrix} \det s^{-1} & \\ & \det t^{-1} \end{pmatrix}, t\,v \begin{pmatrix} \det s^{-1} & \\ & \det t^{-1} \end{pmatrix} \right).$$

This prehomogeneous vector space has two fundamental relative invariants $\det u$, $\det v$ and the relative invariant $f_2(X_{1,0}, X_{2,0}, Y)$ (that is, $f_F(y)$ in § 3) on $F = M_{4,2} \simeq M_2 \oplus M_2$ corresponds to $(\det u \, \det v)$ up to constant. To verify this fact, one does not need to do the actual calculation of the polynomial $f_2(X_{1,0}, X_{2,0}, Y)$. Instead, it is sufficient to compare their characters. As a consequence of Theorem 3.3, it follows that

the b-function $b_{f_2}(s)$ of f_2 is devided by $(s+1)^2(s+2)^2$,

and hence we obtain

$$\{\alpha_{2,1},\, \alpha_{2,2},\, \alpha_{2,3},\, \alpha_{2,4}\} = \{1,1,2,2\}.$$

Finally we observe that the b-function $b_{\underline{m}}(\underline{s})$ is given by

$$\begin{aligned} b_{\underline{m}}(\underline{s}) &= \left\{\prod_{\nu=0}^{m_1-1}(s_1+1+\nu)\right\}\left\{\prod_{\nu=0}^{m_2-1}(s_2+1+\nu)^2(s_2+2+\nu)^2\right\} \\ &\times \left\{\prod_{\nu=0}^{m_1+m_2-1}\left(s_1+s_2+\frac{3}{2}+\nu\right)\left(s_1+s_2+\frac{5}{2}+\nu\right)\right\} \\ &\times \left\{\prod_{\nu=0}^{m_1+2m_2-1}(s_1+2s_2+3+\nu)\right\}. \end{aligned}$$

5 Recent results on b-functions

Recently a large number of b-functions of prehomogeneous vector spaces was settled. Ukai [16] determines the b-functions of prehomogeneous vector spaces of Dynkin-Kostant type for exceptional groups, by using the method in § 4. A. Gyoja and Y. Kaneko determine such b-functions for classical groups, by using the castling transform (see [2]). See [1] in this volume for the prehomogeneous vector spaces of Dynkin-Kostant type.

Here we shall mention about the recent results on the b-functions of non-irreducible prehomogeneous vector spaces which are classified mainly by T. Kimura. A prehomogeneous vector space (G, ρ, V) is called *simple* if G is a simple algebraic group with scalar multiplications. Non-irreducible regular simple prehomogeneous vector spaces are classified by T. Kimura, and their b-functions are studied mainly by S. Kasai with use of microlocal analysis. However, the b-functions of the following two spaces had been open.

- $(GL(1)^2 \times Sp(3), \Lambda_3 \oplus \Lambda_1, V(14) \oplus V(6))$.
- $(GL(1)^4 \times SL(2n+1),\ \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1,\ V(n(2n+1)) \oplus V(2n+1) \oplus V(2n+1) \oplus V(2n+1))$.

In [11], these remaining b-functions were determined by using the method in § 4.

A prehomogeneous vector space (G, ρ, V) is called *2-simple* if G is the product of some two simple algebraic groups with scalar multiplications. Also 2-simple prehomogeneous vector spaces are classified (cf. [4]). Moreover, T. Kogiso et al. give the explicit construction of the relative invariants of 2-simple prehomogeneous vector spaces of type I (cf. [5]). Here the adjective "type I" means that it contains at least one non-trivial prehomogeneous vector space in the irreducible components. Making use of the results above, S. Wakatsuki [14] and the present author [12] have been trying to calculate the b-functions of regular 2-simple prehomogeneous vector spaces of type I. We note that some of them were settled already by Ukai [16]. Combining with his results, we have determined the b-functions of the prehomogeneous vector spaces which are listed in [4, pp.395–398] except for the following five cases :

- $(GL_1^3 \times SL_5 \times SL_2,\ \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1)$.
- $(GL_1^2 \times SL_5 \times SL_k,\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*) \qquad (k = 8, 9)$.
- $(GL_1^2 \times Spin_{10} \times SL_k,\ \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*) \qquad (k = 14, 15)$.

Here we denote by Λ' a half-spin representation of $Spin_{10}$.

We conclude this note by giving the table of the b-function of some regular 2-simple prehomogeneous vector spaces of type I. These are due to the present author [3]. In the table, l denotes the number of fundamental relative invariants and d_i denotes the degree of the relative invariant f_i. Here the numbering of the relative invariants follows [5]. The b-function of the relative invariant $\underline{f} = f_1^{m_1} \cdots f_l^{m_l}$ $(m_1, \ldots, m_l \in \mathbb{Z}_{\geq 0})$ is given by

$$b_{\underline{m}}(\underline{s}) = \prod_{j=1}^{N} \prod_{\nu=0}^{\gamma_j(\underline{m})-1} \prod_{r=1}^{\mu_j} (\gamma_j(\underline{s}) + \alpha_{j,r} + \nu).$$

The details of the results here are in the forthcoming paper [12].

(1) $(GL_1^2 \times SL_4 \times SL_2,\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1)$.

(2) $(GL_1^3 \times SL_4 \times SL_2,\ \Lambda_2 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1)$.

(3) $(GL_1^3 \times SL_4 \times SL_3,\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)$.

(4) $(GL_1^3 \times SL_4 \times SL_4,\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*)$.

(5) $(GL_1^3 \times SL_5 \times SL_2,\ \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + \Lambda_1 \otimes 1)$.

(6) $(GL_1^2 \times SL_5 \times SL_3,\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$.

(7) $(GL_1^2 \times Spin_{10} \times SL_3,\ \Lambda' \otimes 1 + 1 \otimes \Lambda_1)$.

(8) $(GL_1^2 \times Spin_{10} \times SL_3,\ \Lambda' \otimes 1 + 1 \otimes \Lambda_1^*)$.

(9) $(GL_1^2 \times Spin_{10} \times SL_3,\ \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$.

(10) $(GL_1^2 \times Spin_{10} \times SL_4,\ \chi \otimes \Lambda_1 + \Lambda' \otimes 1)$.

Here we denote by Λ' a half-spin representation of $Spin_{10}$ and by χ the vector representation.

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[3] I hope that these are new results. I would be grateful if you let me know something about the researches which I am missing.

	l	d_i	γ_j	$\alpha_{j,r}$
(1)	2		s_1	1
		4	s_2	$1^{\times 2}, 2^{\times 2}$
		8	s_1+s_2	$\frac{3}{2}, \frac{5}{2}$
			s_1+2s_2	3
(2)	2		s_1	1
		4	s_2	$1^{\times 2}, 2^{\times 2}$
		8	s_1+s_2	$\frac{3}{2}, \frac{5}{2}$
			s_1+2s_2	3
(3)	3		s_1	1
		6	s_2	$1, 2$
		5	s_3	$1, \frac{3}{2}$
		6	s_1+s_2	2
			s_1+s_3	$\frac{3}{2}, 2$
			$s_1+s_2+s_3$	$\frac{5}{2}, 3$
(4)	3		s_1	$1, \frac{3}{2}$
			s_2	1
		8	s_3	$1, 2$
		4	s_1+s_3	$2, \frac{5}{2}$
		8	s_2+s_3	2
			$2s_1+s_3$	3
			$s_1+s_2+s_3$	$\frac{5}{2}, 3$
(5)	3		s_1	1
		6	s_2	$1^{\times 2}, 2^{\times 2}$
		11	s_3	1
		2	s_1+s_2	$\frac{3}{2}, \frac{5}{2}$
			s_1+2s_2	3
			$2s_1+3s_2+s_3$	5
(6)	2		s_1	1
		15	s_2	$1, \frac{3}{2}$
		12	s_1+s_2	$\frac{3}{2}^{\times 2}, 2^{\times 2}$
			$2s_1+s_2$	$2, \frac{5}{2}$
			$3s_1+2s_2$	$3, 4$
(7)	2		s_1	$1, \frac{3}{2}, 3, \frac{7}{2}$
		12	s_2	$1, \frac{3}{2}$
		6	s_1+s_2	$2, 4$
			$3s_1+s_2$	$5, 8$

	l	d_i	γ_j	$\alpha_{j,r}$
(8)	2	12 10	s_1 s_2 s_1+s_2 $3s_1+2s_2$	$1,\ 3$ $1,\ \frac{3}{2}$ $\frac{3}{2},\ 2,\ \frac{7}{2},\ 4$ $5, 8$
(9)	2	6 10	s_1 s_2 s_1+s_2 s_1+2s_2	1 $1,\ \frac{3}{2},\ \frac{7}{2},\ 4$ $\frac{3}{2},\ 2,\ 4,\ \frac{9}{2}$ 5
(10)	2	8 12	s_1 s_2 s_1+s_2 s_1+2s_2	1 $1,\ 2,\ 3,\ 4$ $\frac{3}{2},\ 2,\ \frac{5}{2},\ \frac{7}{2},\ 4,\ \frac{9}{2}$ 5

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