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ISOTROPY REPRESENTATIONS ATTACHED TO THE ASSOCIATED CYCLES OF HARISH-CHANDRA MODULES

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1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a nontrivial involutive automorphism θ of $\mathfrak{g}$. We write $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ for the symmetric decomposition of $\mathfrak{g}$ given by θ, where $\mathfrak{k}$ and $\mathfrak{p}$ denote the $+1$ and -1 eigenspaces for θ, respectively. Let $K_{\mathbb{C}}$ be a connected complex algebraic group with Lie algebra $\mathfrak{k}$. We assume that the natural inclusion $\mathfrak{k}\hookrightarrow\mathfrak{g}$ gives rise to a group homomorphism from $K_{\mathbb{C}}$ to $G_{\mathbb{C}}^{\mathrm{ad}}$ through the exponential map. Here $G_{\mathbb{C}}^{\mathrm{ad}}$ denotes the adjoint group of $\mathfrak{g}$. Then, this homomorphism naturally induces the adjoint representation Ad of $K_{\mathbb{C}}$ on $\mathfrak{g}$.

We say that a finitely generated $\mathfrak{g}$-module $\boldsymbol{X}$ is a $(\mathfrak{g},K_{\mathbb{C}})$-module, or a Harish-Chandra module, if the action on $\boldsymbol{X}$ of the Lie subalgebra $\mathfrak{k}$ is locally finite and if it lifts up to a representation of $K_{\mathbb{C}}$ on $\boldsymbol{X}$ through the exponential map. It is a fundamental result of Harish-Chandra that the study of irreducible admissible representations of a real semisimple Lie group essentially reduces to that of irreducible $(\mathfrak{g},K_{\mathbb{C}})$-modules.

Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g},K_{\mathbb{C}})$-module. A $K_{\mathbb{C}}$-stable good filtration of $\boldsymbol{X}$ naturally gives rise to a graded, compatible $(S(\mathfrak{g}),K_{\mathbb{C}})$-module $\mathrm{gr}\,\boldsymbol{X}$ annihilated by $\mathfrak{k}$, where $S(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$. By Borho-Brylinski [1] and Vogan [17], [18], the *associated cycle* $\mathcal{C}(\boldsymbol{X})$ of $\boldsymbol{X}$ is defined to be the support $\mathcal{V}(\boldsymbol{X})$ of $\mathrm{gr}\,\boldsymbol{X}$ combined with the multiplicity at each irreducible component of $\mathcal{V}(\boldsymbol{X})$. The support $\mathcal{V}(\boldsymbol{X})$ is called the *associated variety* of $\boldsymbol{X}$. It is a $K_{\mathbb{C}}$-stable affine algebraic cone contained in the set of nilpotent elements in $\mathfrak{p}$, and each irreducible component of $\mathcal{V}(\boldsymbol{X})$ is the closure $\overline{\mathcal{O}}$ of a nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}$. As we have shown in [5] and [20], the variety $\mathcal{V}(\boldsymbol{X})$ controls some fundamental properties for $\boldsymbol{X}$.

The algebraic cycle $\mathcal{C}(\boldsymbol{X})$ describes a sort of asymptotic behavior of $\boldsymbol{X}$ (cf. [16]). Moreover, it is shown by Vogan [17, Theorem 2.13] that the multiplicity of $\boldsymbol{X}$ at an irreducible component $\overline{\mathcal{O}}$ of $\mathcal{V}(\boldsymbol{X})$ can be interpreted as the dimension of a certain finite-dimensional representation $(\varpi_{\mathcal{O}},\mathcal{W})$ of the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at an $X\in\mathcal{O}$. We call $\varpi_{\mathcal{O}}$ an *isotropy representation* attached to $\boldsymbol{X}$. In terms of $\varpi_{\mathcal{O}}$, the associated cycle $\mathcal{C}(\boldsymbol{X})$ of $\boldsymbol{X}$ is expressed as

$$\mathcal{C}(\boldsymbol{X})=\sum_{\mathcal{O}}\dim\varpi_{\mathcal{O}}\cdot[\overline{\mathcal{O}}]. \tag{1.1}$$

In this paper, we continue our study in [24] (also in [23]) concerning the associated cycle $\mathcal{C}(\boldsymbol{X})$ and the isotropy representation $\varpi_{\mathcal{O}}$ attached to a $(\mathfrak{g},K_{\mathbb{C}})$-module $\boldsymbol{X}$ with irreducible associated variety $\mathcal{V}(\boldsymbol{X})=\overline{\mathcal{O}}$.

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To be more precise, we first look at in Section 2 a relationship between the $(S(\mathfrak{g}), K_{\mathbf{C}})$-module $\operatorname{gr}\boldsymbol{X}$ and the induced representation $\Gamma(\mathcal{W}) = \operatorname{Ind}_{K_{\mathbf{C}}(X)}^{K_{\mathbf{C}}}(\varpi_{\mathcal{O}}, \mathcal{W})$ of $K_{\mathbf{C}}$ equipped with a natural $S(\mathfrak{g})$-action. A reciprocity law of Frobenius type for such an induced module (Proposition 2.4) plays an important role. In fact, it is effectively used to prove an irreducibility criterion for $\varpi_{\mathcal{O}}$ (Theorem 2.9). Section 2 does not contain new results, but it gives a survey of some part of [17, Sections 2–4] and [18, Lectures 6 and 7] in a slightly modified and simplified form (but for limited $\boldsymbol{X}$'s). Some remarks and examples in connection with our recent works ([23], [24]) are also included.

In order to identify the isotropy representation $\varpi_{\mathcal{O}}$, it is sometimes helpful to consider not only $\operatorname{gr}\boldsymbol{X}$ but also its $K_{\mathbf{C}}$-finite dual space consisting of certain (vector valued) polynomial functions on $\mathfrak{p}$. We present this technique in Section 3. A sufficient condition is given in Proposition 3.2 for $\operatorname{gr}\boldsymbol{X}$ being annihilated by the whole prime ideal I of $S(\mathfrak{g})$ defining $\overline{\mathcal{O}}$. In such a case, the isotropy representation $\varpi_{\mathcal{O}}$ can be described by means of the principal symbol of a differential operator on $\mathfrak{p}$ of gradient-type (see [24]).

In Section 4, we focus our attention on the irreducible Harish-Chandra modules $\boldsymbol{X}$ of discrete series. As is well known, such an $\boldsymbol{X}$ has irreducible associated variety (cf. [21], [22]). The multiplicities in the associated cycles for discrete series have been intensively studied by Chang [2], [3], by using the theory of D-modules on the flag variety for $\mathfrak{g}$. He succeeds to describe $\mathcal{C}(\boldsymbol{X})$ explicitly for the real rank one case. Taniguchi applies in [14] and [15] the results of Chang in order to specify Whittaker functions associated with discrete series for some noncompact unitary or orthogonal Lie groups. Here in this paper, we would like to propose another approach to identify $\mathcal{C}(\boldsymbol{X})$, by using a realization of $\boldsymbol{X}$ as the kernel of an invariant differential operator of gradient-type on the Riemannian symmetric space (cf. [7], [13]; [19], [25]). Through this approach, we can construct a certain $K_{\mathbf{C}}(X)$-submodule $\mathcal{U}_{\lambda}(Q_c)$ of the representation $(\varpi_{\mathcal{O}}^*, \mathcal{W}^*)$ contragredient to $\varpi_{\mathcal{O}}$, by improving our arguments in [22]. Moreover some evidences are given for this subrepresentation being large enough in the whole $\varpi_{\mathcal{O}}^*$, by means of our technique given in Section 3. The main results are given as Theorem 4.4 and Corollary 4.6.

An enlarged version of this article will appear elsewhere.

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2. Graded module $\operatorname{gr}\boldsymbol{X}$ and induced representation $\Gamma(\mathcal{W})$

As in Section 1, let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K_{\mathbf{C}})$-module with irreducible associated variety $\mathcal{V}(\boldsymbol{X}) = \overline{\mathcal{O}}$, where $\mathcal{O}$ is a nilpotent $K_{\mathbf{C}}$-orbit in $\mathfrak{p}$. This section introduces some elementary aspects of Vogan's theory on the associated cycle and the isotropy representation attached to $\boldsymbol{X}$. The results in this section may be read off from [17] and [18] with a little bit of effort.

2.1. Associated cycle and isotropy representation. To start with, let us introduce our key notion precisely. Take an irreducible $K_{\mathbf{C}}$-submodule (τ, V_τ) of $\boldsymbol{X}$, which yields a $K_{\mathbf{C}}$-stable good filtration of $\boldsymbol{X}$ in the following way:

$$(2.1) \qquad \boldsymbol{X}_0 \subset \boldsymbol{X}_1 \subset \cdots \subset \boldsymbol{X}_n \subset \cdots, \quad \boldsymbol{X}_n := U_n(\mathfrak{g})V_\tau \ (n = 0, 1, 2, \ldots).$$

Here $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, and we write $U_n(\mathfrak{g})$ $(n = 0, 1, \ldots)$ for the natural increasing filtration of $U(\mathfrak{g})$. This filtration gives rise to a graded

$(S(\mathfrak{g}), K_{\mathbb{C}})$-module $M = \operatorname{gr} \boldsymbol{X}$, annihilated by $S(\mathfrak{k})$, as follows:

$$M = \operatorname{gr} \boldsymbol{X} = \bigoplus_{n=0}^{\infty} M_n \quad \text{with} \quad M_n := \boldsymbol{X}_n / \boldsymbol{X}_{n-1} \ (\boldsymbol{X}_{-1} := \{0\}). \tag{2.2}$$

We note that

$$M_n = S^n(\mathfrak{g}) V_\tau \simeq S^n(\mathfrak{p}) V_\tau \quad \text{with } M_0 = V_\tau, \tag{2.3}$$

where $S^n(\mathfrak{v})$ is the homogeneous component of the symmetric algebra $S(\mathfrak{v})$ of degree n. By definition, the associated variety $\mathcal{V}(\boldsymbol{X})$ of $\boldsymbol{X}$ is identified with the affine algebraic variety of $\mathfrak{g}$ given by the annihilator ideal $\operatorname{Ann}_{S(\mathfrak{g})} M$ in $S(\mathfrak{g})$ of M:

$$\mathcal{V}(\boldsymbol{X}) = \{ Z \in \mathfrak{g} \mid f(Z) = 0 \text{ for all } f \in \operatorname{Ann}_{S(\mathfrak{g})} M \} \subset \mathfrak{p}, \tag{2.4}$$

where $S(\mathfrak{g})$ is viewed as the ring of polynomial functions on $\mathfrak{g}$ by identifying $\mathfrak{g}$ with its dual space through the Killing form B of $\mathfrak{g}$.

The Hilbert Nullstellensatz tells us that the radical of $\operatorname{Ann}_{S(\mathfrak{g})} M$ coincides with the prime ideal $I = I(\mathcal{V}(\boldsymbol{X}))$ defining the irreducible variety $\mathcal{V}(\boldsymbol{X})$: $I = \sqrt{\operatorname{Ann}_{S(\mathfrak{g})} M}$. So we see $I^n M = \{0\}$ for some positive integer n, and we write n_0 for the smallest n of this nature. Then, one gets a strictly decreasing filtration of the $(S(\mathfrak{g}), K_{\mathbb{C}})$-module M as

$$M = I^0 M \supsetneq I^1 M \supsetneq \cdots \supsetneq I^{n_0} M = \{0\}. \tag{2.5}$$

By the *multiplicity* $\operatorname{mult}_I(\boldsymbol{X})$ of $\boldsymbol{X}$ at I is meant the length as an $S(\mathfrak{g})_I$-module of the localization M_I of $M = \operatorname{gr} \boldsymbol{X}$ at the prime ideal I. Then, the associated cycle $\mathcal{C}(\boldsymbol{X})$ of $\boldsymbol{X}$ turns to be

$$\mathcal{C}(\boldsymbol{X}) = \operatorname{mult}_I(\boldsymbol{X}) \cdot [\overline{\mathcal{O}}] \quad \text{with} \quad \mathcal{V}(\boldsymbol{X}) = \overline{\mathcal{O}}. \tag{2.6}$$

Note that this cycle does not depend on the choice of a good filtration (2.1) of $\boldsymbol{X}$.

Now, let us explain how the multiplicity $\operatorname{mult}_I(\boldsymbol{X})$ can be interpreted as the dimension of an isotropy representation. For this, we take an element X in the open $K_{\mathbb{C}}$-orbit $\mathcal{O} \subset \mathcal{V}(\boldsymbol{X})$. Set $K_{\mathbb{C}}(X) := \{k \in K_{\mathbb{C}} \mid \operatorname{Ad}(k)X = X\}$, the isotropy subgroup of $K_{\mathbb{C}}$ at X. We write $\mathfrak{m}(X)$ for the maximal ideal of $S(\mathfrak{g})$ which defines the one point variety $\{X\}$ in $\mathfrak{g}$:

$$\mathfrak{m}(X) := \sum_{Y \in \mathfrak{g}} (Y - B(Y, X)) S(\mathfrak{g}) \quad \text{for } X \in \mathcal{O}. \tag{2.7}$$

For each $j = 0, \ldots, n_0 - 1$, we introduce a finite-dimensional $(S(\mathfrak{g}), K_{\mathbb{C}}(X))$-representation $\varpi_{\mathcal{O}}(j)$ acting on

$$\mathcal{W}(j) := I^j M / \mathfrak{m}(X) I^j M, \tag{2.8}$$

in the canonical way, and we set

$$(\varpi_{\mathcal{O}}, \mathcal{W}) := \bigoplus_{j=0}^{n_0 - 1} (\varpi_{\mathcal{O}}(j), \mathcal{W}(j)). \tag{2.9}$$

We call $\varpi_{\mathcal{O}}$ the *isotropy representation* attached to the data $(\boldsymbol{X}, V_\tau, \mathcal{O})$, where V_τ yields the filtration (2.1) of $\boldsymbol{X}$. The following lemma is essential for our succeeding discussion.

Lemma 2.1 (cf. [17, Corollary 2.7] and [24, Remark 2.2]). *Let N be a finitely generated $(S(\mathfrak{g}), K_{\mathbb{C}})$-module such that $IN = \{0\}$. Then, the length of $S(\mathfrak{g})_I$-module N_I is equal to the dimension of the vector space $N/\mathfrak{m}(X)N$ for every $X \in \mathcal{O}$.*

This lemma tells us that the length of the $S(\mathfrak{g})_I$-module $(I^jM/I^{j+1}M)_I$ equals $\dim \varpi_{\mathcal{O}}(j)$ by noting that the ideal I annihilates the subquotient $I^jM/I^{j+1}M$ of M. Together with the exactness of localization, we immediately get the following

Proposition 2.2. *One has* $\mathrm{mult}_I(\boldsymbol{X}) = \dim \varpi_{\mathcal{O}}$. *Moreover, the equality*

$$\mathrm{mult}_I(\boldsymbol{X}) = \dim \varpi_{\mathcal{O}}(0) = \dim M/\mathfrak{m}(X)M \tag{2.10}$$

holds if and only if the support of the $S(\mathfrak{g})$*-module* IM *is contained in the boundary* $\partial\mathcal{O} = \overline{\mathcal{O}}\backslash\mathcal{O}$.

Remark 2.3. The representation $\varpi_{\mathcal{O}}(0)$ in (2.10) never vanishes because the annihilator ideal $\mathrm{Ann}_{S(\mathfrak{g})}M/IM$ is equal to I (cf. [23, Lemma 3.4]). Moreover, the equality (2.10) holds for a number of unitarizable $(\mathfrak{g}, K_{\mathbf{C}})$-modules $\boldsymbol{X}$ with unique extreme $K_{\mathbf{C}}$-types V_τ. See Example 2.7 and Theorem 4.4 (1).

2.2. **Induced module** $\Gamma(\mathcal{Z})$. Let $(\varpi, \mathcal{Z})$ be a finite-dimensional $(S(\mathfrak{g}), K_{\mathbf{C}}(X))$-module with $X \in \mathcal{O}$. We write $\Gamma(\mathcal{Z})$ for the space of all left $K_{\mathbf{C}}$-finite, holomorphic functions $f : K_{\mathbf{C}} \to \mathcal{Z}$ satisfying

$$f(yh) = \varpi(h)^{-1}f(y) \quad (y \in K_{\mathbf{C}},\ h \in K_{\mathbf{C}}(X)).$$

Namely, $\Gamma(\mathcal{Z})$ consists of all $K_{\mathbf{C}}$-finite, holomorphic cross sections of the $K_{\mathbf{C}}$-homogeneous vector bundle $K_{\mathbf{C}} \times_{K_{\mathbf{C}}(X)} \mathcal{Z}$ on $K_{\mathbf{C}}/K_{\mathbf{C}}(X) \simeq \mathcal{O}$. Then, $\Gamma(\mathcal{Z})$ has a structure of $(S(\mathfrak{g}), K_{\mathbf{C}})$-module by the following actions:

$$(D \cdot f)(y) := \varpi(\mathrm{Ad}(y)^{-1}D)f(y), \quad (k \cdot f)(y) := f(k^{-1}y),$$

for $D \in S(\mathfrak{g})$, $k \in K_{\mathbf{C}}$ and $f \in \Gamma(\mathcal{Z})$. We call $\Gamma(\mathcal{Z})$ the $(S(\mathfrak{g}), K_{\mathbf{C}})$*-module induced from* ϖ. We note that, if $\mathcal{Z}$ is annihilated by the maximal ideal $\mathfrak{m}(X)$, the $S(\mathfrak{g})$-action on $\Gamma(\mathcal{Z})$ turns to be the multiplication of functions on the orbit $\mathcal{O}$:

$$(D \cdot f)(y) = D(\mathrm{Ad}(y)X)f(y). \tag{2.11}$$

In this case, the annihilator in $S(\mathfrak{g})$ of any nonzero function $f \in \Gamma(\mathcal{Z})$ coincides with the prime ideal I defining $\overline{\mathcal{O}}$.

Let M be any $(S(\mathfrak{g}), K_{\mathbf{C}})$-module. If ρ is an $(S(\mathfrak{g}), K_{\mathbf{C}}(X))$-homomorphism from M to $\mathcal{Z}$, we define a function $T_m : K_{\mathbf{C}} \to \mathcal{Z}$ for each $m \in M$ by putting

$$T_m(y) := \rho(y^{-1} \cdot m) \quad (y \in K_{\mathbf{C}}). \tag{2.12}$$

Then it is standard to verify that T_m lies in $\Gamma(\mathcal{Z})$ and that the map $T : m \mapsto T_m$ $(m \in M)$ gives an $(S(\mathfrak{g}), K_{\mathbf{C}})$-homomorphism from M to $\Gamma(\mathcal{Z})$. More precisely, one readily obtains the following reciprocity law of Frobenius type.

Proposition 2.4. *Under the above notation, the assignment* $\rho \mapsto T$ *sets up a linear isomorphism*

$$\mathrm{Hom}_{S(\mathfrak{g}),K_{\mathbf{C}}(X)}(M, \mathcal{Z}) \simeq \mathrm{Hom}_{S(\mathfrak{g}),K_{\mathbf{C}}}(M, \Gamma(\mathcal{Z})). \tag{2.13}$$

Here, for Ω*-modules* A *and* B*, we denote by* $\mathrm{Hom}_\Omega(A, B)$ *the space of* Ω*-homomorphisms from* A *to* B.

2.3. **Homomorphism $T = \oplus_j \tilde{T}(j)$.** We now return to our setting in Section 2.1, where $M = \operatorname{gr} \boldsymbol{X}$ for an irreducible $(\mathfrak{g}, K_{\mathbb{C}})$-module $\boldsymbol{X}$ with $\mathcal{V}(\boldsymbol{X}) = \overline{\mathcal{O}}$. Take an integer j such that $0 \leq j \leq n_0 - 1$. Let $\rho(j)$ denote the natural quotient map from $I^j M$ to $\mathcal{W}(j) = I^j M / \mathfrak{m}(X) I^j M$. Correspondingly, we get an $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism $T(j) : I^j M \to \Gamma(\mathcal{W}(j))$ by Proposition 2.4. It follows that

$$\operatorname{Ker} T(j) = \bigcap_{Y \in \mathcal{O}} \mathfrak{m}(Y) I^j M \supset I^{j+1} M, \tag{2.14}$$

by the definition of $T(j)$ together with $\mathfrak{m}(Y) \supset I$ $(Y \in \mathcal{O})$.

Proposition 2.5. *The kernel* $\operatorname{Ker} T(j)$ *of* $T(j)$ *is the largest* $(S(\mathfrak{g}), K_{\mathbb{C}})$*-submodule of* $I^j M$ *among those* N *having the following two properties*: (i) $N \supset I^{j+1} M$, *and,* (ii) *the support of* $N / I^{j+1} M$ *is contained in* $\partial \mathcal{O}$.

Proof. First, we show that $\operatorname{Ker} T(j)$ have two properties (i) and (ii). The inclusion (2.14) assures (i). As for (ii), we consider a short exact sequence of $(S(\mathfrak{g}), K_{\mathbb{C}})$-modules:

$$0 \longrightarrow \operatorname{Ker} T(j) / I^{j+1} M \longrightarrow I^j M / I^{j+1} M \longrightarrow I^j M / \operatorname{Ker} T(j) \longrightarrow 0. \tag{2.15}$$

Each module is annihilated by I. In view of Lemma 2.1, we find that the multiplicity of $I^j M / \operatorname{Ker} T(j)$ at I is equal to the dimension of vector space

$$(I^j M / \operatorname{Ker} T(j)) / \mathfrak{m}(X) (I^j M / \operatorname{Ker} T(j)) \simeq I^j M / (\mathfrak{m}(X) I^j M + \operatorname{Ker} T(j)) = \mathcal{W}(j).$$

Here, the last equality follows from $\operatorname{Ker} T(j) \subset \mathfrak{m}(X) I^j M$ (see (2.14)). This shows that the length of $S(\mathfrak{g})_I$-module $I^j M / I^{j+1} M$ and that of $I^j M / \operatorname{Ker} T(j)$ coincide with one another. Hence $(\operatorname{Ker} T(j) / I^{j+1} M)_I$ vanishes by (2.15). This means that the support of $\operatorname{Ker} T(j) / I^{j+1} M$ is contained in $\partial \mathcal{O}$.

Second, let N be any $(S(\mathfrak{g}), K_{\mathbb{C}})$-submodule of $I^j M$ with two properties (i) and (ii) in question. (2.14) tells us that $T(j)$ naturally induces an $(S(\mathfrak{g}), K_{\mathbb{C}})$-module map from $I^j M / I^{j+1} M$ to $\Gamma(\mathcal{W}(j))$ which we denote by $\tilde{T}(j)$. Then, $\tilde{T}(j)(N / I^{j+1} M)$ must vanish by virtue of (2.11) together with the property (ii) for N. This proves $N \subset \operatorname{Ker} T(j)$. □

As for the injectivity of $T(j)$, one gets the following consequence of Proposition 2.5.

Corollary 2.6. *The homomorphism* $T(j) : I^j M \to \Gamma(\mathcal{W}(j))$ *is injective if and only if* $\operatorname{Ann}_{S(\mathfrak{g})} m = I$ *for all* $m \in I^j M \backslash \{0\}$. *In this case, one has* $I^{j+1} M = \{0\}$, *i.e.,* $j = n_0 - 1$.

Example 2.7. We encounter the situation in the above corollary with $j = 0$, for example, if $\boldsymbol{X}$ is a unitarizable highest weight module of a simple hermitian Lie algebra $\mathfrak{g}$, and V_τ in (2.1) is the extreme $K_{\mathbb{C}}$-type of $\boldsymbol{X}$. Note that the associated variety of such an $\boldsymbol{X}$ is the closure of a "holomorphic" nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$. See [23, Section 3.2] for details.

Summing up $\tilde{T}(j)$'s on $I^j M / I^{j+1} M$ $(j = 0, \ldots, n_0 - 1)$, we obtain an $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism $T := \oplus_j \tilde{T}(j)$:

$$\hat{M}(I) := \bigoplus_j I^j M / I^{j+1} M \xrightarrow{T} \bigoplus_j \Gamma(\mathcal{W}(j)) \simeq \Gamma(\mathcal{W}), \tag{2.16}$$

where the support of the kernel $\operatorname{Ker} T$ is contained in $\partial \mathcal{O}$.

Remark 2.8. By using the "microlocalization technique", Vogan constructed a new $K_{\mathbb{C}}$-stable $\mathbb{Z}$-gradation on $\boldsymbol{X}$ such that the corresponding graded module embeds into $\Gamma(\mathcal{W})$ as a representation of $K_{\mathbb{C}}$ (see [17, Theorem 4.2]). Thanks to this result, one always has $\boldsymbol{X} \hookrightarrow \Gamma(\mathcal{W})$ as $K_{\mathbb{C}}$-modules. Noting that $\hat{M}(I) \simeq \boldsymbol{X}$ as $K_{\mathbb{C}}$-modules, we find that the above $T : \hat{M}(I) \to \Gamma(\mathcal{W})$ must be an isomorphism if T is surjective.

2.4. **Irreducibility of $\varpi_{\mathcal{O}}$.** The results in Sections 2.1–2.3 lead us to prove the following natural criterion for the irreducibility of isotropy representation $(\varpi_{\mathcal{O}}, \mathcal{W})$ of $K_{\mathbb{C}}(X)$.

Theorem 2.9 (cf. [18, Proposition 7.6]; see also [24, Section 5]). *The following two conditions on $\boldsymbol{X}$ are equivalent to each other.*

(a) *$(\varpi_{\mathcal{O}}, \mathcal{W})$ is irreducible as a $K_{\mathbb{C}}(X)$-module.*

(b) *If N is any $(S(\mathfrak{g}), K_{\mathbb{C}})$-submodule of $M = \operatorname{gr} \boldsymbol{X}$, either the support of N or that of the quotient M/N is contained in $\partial\mathcal{O}$.*

In this case, we have $\varpi_{\mathcal{O}} = \varpi_{\mathcal{O}}(0)$, or equivalently, the support of IM is contained in $\partial\mathcal{O}$ by Proposition 2.2.

Proof. The implication (a) $\Rightarrow$ (b) is an easy consequence of the exactness of localization. In what follows let us prove (b) $\Rightarrow$ (a). First, we note that the condition (b) together with Remark 2.3 implies that the support of IM is contained in $\partial\mathcal{O}$. Thus one gets $\varpi_{\mathcal{O}} = \varpi_{\mathcal{O}}(0)$, or,

$$\mathcal{W} = \mathcal{W}(0) = M/\mathfrak{m}(X)M.$$

Now, suppose by contraries that $\mathcal{W}$ is not irreducible. Then, there exists a $K_{\mathbb{C}}(X)$-stable subspace C of M such that $M \supsetneq C \supsetneq \mathfrak{m}(X)M$ and that $\mathcal{Z} := M/C$ is irreducible as a $K_{\mathbb{C}}(X)$-module. The condition $C \supset \mathfrak{m}(X)M$ assures that C is $S(\mathfrak{g})$-stable. Thus $\mathcal{Z}$ becomes an $(S(\mathfrak{g}), K_{\mathbb{C}}(X))$-module annihilated by $\mathfrak{m}(X)$.

Next, we consider two induced $(S(\mathfrak{g}), K_{\mathbb{C}})$-modules $\Gamma(\mathcal{W})$ and $\Gamma(\mathcal{Z})$. The quotient map $\mathcal{W} = M/\mathfrak{m}(X)M \longrightarrow \mathcal{Z} = M/C$ gives rise to an $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism, say γ, from $\Gamma(\mathcal{W})$ to $\Gamma(\mathcal{Z})$ in the canonical way. Set $T' := \gamma \circ T(0)$, where $T(0) : M \to \Gamma(\mathcal{W})$ is the $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism defined in Section 2.3. Then, as shown in the proof of [18, Proposition 7.9], the image $T'(M)$ of T' is a finitely generated $(S(\mathfrak{g}), K_{\mathbb{C}})$-submodule of $\Gamma(\mathcal{Z})$ whose isotropy representation is isomorphic to $\mathcal{Z}$. This combined with $T'(M) \simeq M/\operatorname{Ker} T'$ tells us that the multiplicity of $\operatorname{Ker} T'$ at the prime ideal I is equal to $\dim \mathcal{W} - \dim \mathcal{Z} > 0$. By the assumption (b), we find that the support of $M/\operatorname{Ker} T' \simeq T'(M)$ is contained in $\partial\mathcal{O}$. This necessarily implies $\operatorname{Ker} T' = M$, i.e., $T' = 0$, because the $S(\mathfrak{g})$-module $T'(M)$ $(\subset \Gamma(\mathcal{Z}))$ admits no embedded associated primes by (2.11). Finally, the resulting equality $T' = 0$ means that

$$y^{-1} \cdot m + \mathfrak{m}(X)M = T(0)_m(y) \in C/\mathfrak{m}(X)M \quad \text{for all } y \in K_{\mathbb{C}} \text{ and } m \in M.$$

This contradicts $C \neq M$. □

Example 2.10. Let $\boldsymbol{X}$ be an irreducible unitarizable highest weight $(\mathfrak{g}, K_{\mathbb{C}})$-module of a simple hermitian Lie algebra $\mathfrak{g}$, with extreme $K_{\mathbb{C}}$-type V_τ. In [23, Section 5], we have described the isotropy representation $\varpi_{\mathcal{O}} = \varpi_{\mathcal{O}}(0)$ explicitly, when $\boldsymbol{X}$ is the theta lift of an irreducible representation of the compact groups $G' = O(k)$, $U(k)$ and $Sp(k)$ with respect to the reductive dual pairs $(G, G') = (Sp(n, \mathbb{R}), O(k))$, $(SU(p, q), U(k))$ and $(SO^*(2n), Sp(k))$, respectively. In particular, one finds that the representation $\varpi_{\mathcal{O}}$ is irreducible if the dual pair (G, G') is in the stable range with smaller member G'. In this case, $\boldsymbol{X} \leftrightarrow \varpi_{\mathcal{O}}^*$ essentially gives the Howe duality correspondence. See also [11].

It should be an important problem to specify the isotropy representations $\varpi_{\mathcal{O}}$ attached to (singular) unitary highest weight modules $\boldsymbol{X}$, for the remaining simple hermitian Lie algebras $\mathfrak{g}$ with real forms $\mathfrak{o}(p,2)$, EIII and EVII. Toward this problem, an interesting investigation has been made by Kato and Ochiai [9, Section 5.2] for EVII case.

3. Utility of the dual $(S(\mathfrak{g}), K_{\mathbb{C}})$-module

In this section, we do *not* assume that the associated variety $\mathcal{V}(\boldsymbol{X})$ of $\boldsymbol{X}$ is irreducible. Let $M = \operatorname{gr} \boldsymbol{X}$ be the graded $(S(\mathfrak{g}), K_{\mathbb{C}})$-module attached to an irreducible $(\mathfrak{g}, K_{\mathbb{C}})$-module $\boldsymbol{X}$ by (2.1) and (2.2). For any nilpotent element $X \in \mathfrak{p}$, we can define $\mathfrak{m}(X)$, $\mathcal{W}(0) = M/\mathfrak{m}(X)M$ and $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism $T(0) : M \to \Gamma(\mathcal{W}(0))$, just as in Section 2. We are going to make a simple observation on the $K_{\mathbb{C}}$-finite dual M^* of M realized as a space of V_τ^*-valued polynomial functions on $\mathfrak{p}$. This will be helpful to describe in Section 4 the isotropy representation $\varpi_{\mathcal{O}}$ for discrete series $\boldsymbol{X}$.

First, the tensor product $S(\mathfrak{p}) \otimes V_\tau$ admits a natural structure of $(S(\mathfrak{g}), K_{\mathbb{C}})$-module so that $\mathfrak{k}$ annihilates the whole $S(\mathfrak{p}) \otimes V_\tau$. Since $M = S(\mathfrak{p})V_\tau$ with $V_\tau = M_0$, there exists a unique surjective $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism, say π, from $S(\mathfrak{p}) \otimes V_\tau$ to M such that $\pi(1 \otimes v) = v$ for $v \in V_\tau$. We write N for the kernel of π. This is a graded $(S(\mathfrak{g}), K_{\mathbb{C}})$-submodule of $S(\mathfrak{p}) \otimes V_\tau$. On the other hand, we identify $S(\mathfrak{p}^*) \otimes V_\tau^*$ with the space of polynomial functions on $\mathfrak{p}$ with values in $V_\tau^* = \operatorname{Hom}_{\mathbb{C}}(V_\tau, \mathbb{C})$. This space also becomes an $(S(\mathfrak{g}), K_{\mathbb{C}})$-module on which $\mathfrak{g}$ acts by directional differentiation through the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$. Note that the action of $S(\mathfrak{g})$ on $S(\mathfrak{p}^*) \otimes V_\tau^*$ is locally finite.

Then, it is standard to verify that

$$\begin{aligned}(3.1)\qquad & (S(\mathfrak{p}) \otimes V_\tau) \times (S(\mathfrak{p}^*) \otimes V_\tau^*) \ni (D \otimes v, f) \longmapsto \langle D \otimes v, f \rangle \in \mathbb{C}, \\ & \langle D \otimes v, f \rangle := (({}^T D \cdot f)(0), v)_{V_\tau^* \times V_\tau},\end{aligned}$$

gives a nondegenerate $(S(\mathfrak{g}), K_{\mathbb{C}})$-invariant pairing, where ^{T} denotes the principal automorphism of $S(\mathfrak{p})$ such that ${}^T Y = -Y$ for $Y \in \mathfrak{p}$, and $(\cdot\,,\,\cdot)_{V_\tau^* \times V_\tau}$ is the dual pairing on $V_\tau^* \times V_\tau$. Now, let M^* denote the $K_{\mathbb{C}}$-finite dual space of M, viewed as an $(S(\mathfrak{g}), K_{\mathbb{C}})$-module through the contragredient representation. We write $N^\perp$ for the orthogonal of N in $S(\mathfrak{p}^*) \otimes V_\tau^*$ with respect to $\langle \cdot\,,\,\cdot \rangle$. Then, (3.1) yields a nondegenerate invariant pairing

$$(3.2)\qquad \langle \cdot\,,\,\cdot \rangle_1 : M \times N^\perp \to \mathbb{C},$$

which gives an isomorphism of $(S(\mathfrak{g}), K_{\mathbb{C}})$-modules as

$$(3.3)\qquad M^* \simeq N^\perp \subset S(\mathfrak{p}^*) \otimes V_\tau^*.$$

For an integer $n \geq 0$, we denote by $(N^\perp)_n$ the homogeneous component of $N^\perp$ of degree n: $(N^\perp)_n := N^\perp \cap (S^n(\mathfrak{p}^*) \otimes V_\tau^*)$.

Noting that $M = V_\tau + \mathfrak{m}(X)M$, we have a natural $K_{\mathbb{C}}(X)$-homomorphism

$$V_\tau \longrightarrow \mathcal{W}(0) = M/\mathfrak{m}(X)M \longrightarrow 0.$$

This induces an embedding $\mathcal{W}(0)^* \hookrightarrow V_\tau^*$ by passing to the dual. In this way, we regard $\mathcal{W}(0)^*$ as a $K_{\mathbb{C}}(X)$-submodule of V_τ^*.

For each integer $n \geq 0$, let Ψ_n be the $K_{\mathbb{C}}$-submodule of $S^n(\mathfrak{p}^*) \otimes V_\tau^*$ generated by the vectors $X^n \otimes v^*$ $(v^* \in \mathcal{W}(0)^*)$:

$$\Psi_n := \langle X^n \otimes v^* \mid v^* \in \mathcal{W}(0)^* \rangle_{K_{\mathbb{C}}}. \tag{3.4}$$

Here the polynomial function $X^n \otimes v^* \in S^n(\mathfrak{p}^*) \otimes V_\tau^*$ is defined by

$$X^n \otimes v^* : \mathfrak{p} \ni Z \longmapsto B(X,Z)^n v^* \in \mathbb{C}$$

through the Killing form B of $\mathfrak{g}$. We set

$$\Psi := \bigoplus_{n=0}^{\infty} \Psi_n \subset S(\mathfrak{p}^*) \otimes V_\tau^*.$$

Then, it is standard to verify the following

Lemma 3.1. (1) *Ψ is an $(S(\mathfrak{g}), K_{\mathbb{C}})$-submodule of $S(\mathfrak{p}^*) \otimes V_\tau^*$ contained in $N^\perp$.*

(2) *We write ${}^\perp\Psi$ for the orthogonal of Ψ in M with respect to the pairing $\langle \cdot , \cdot \rangle_1$ on $M \times N^\perp$. Let $T(0) : M \to \Gamma(\mathcal{W}(0))$ be the $(S(\mathfrak{g}), K_{\mathbb{C}})$-homomorphism defined in Section 2.3. Then, one gets*

$$\operatorname{Ker} T(0) \cap M_n = {}^\perp\Psi \cap M_n \quad \textit{for every integer } n \geq 0, \tag{3.5}$$

In particular, ${}^\perp\Psi = \oplus_n {}^\perp\Psi \cap M_n$ is contained in $\operatorname{Ker} T(0)$.

Proof. (1) It is easy to see that Ψ is $(S(\mathfrak{g}), K_{\mathbb{C}})$-stable by noting that

$$Y \cdot ((\mathrm{Ad}(y)X)^n \otimes y \cdot v^*) = nB(\mathrm{Ad}(y)X, Y)(\mathrm{Ad}(y)X)^{n-1} \otimes y \cdot v^* \in \Psi_{n-1},$$

for $Y \in \mathfrak{g}$, $v^* \in \mathcal{W}(0)^*$ and $y \in K_{\mathbb{C}}$. To prove $\Psi \subset N^\perp$, let φ_{v^*} denote the linear form on $S(\mathfrak{p}) \otimes V_\tau$, which is the pull back of $v^* \in \mathcal{W}(0)^*$ through the quotient map

$$S(\mathfrak{p}) \otimes V_\tau \xrightarrow{\pi} M \longrightarrow \mathcal{W}(0) = M/\mathfrak{m}(X)M.$$

Then φ_{v^*} is zero on the subspace $\mathfrak{m}(X) \otimes V_\tau + N \subset S(\mathfrak{p}) \otimes V_\tau$.

If $\tilde{m} = \sum_j Y_j^n \otimes v_j$ $(Y_j \in \mathfrak{p}, v_j \in V_\tau)$ is a homogeneous element of N of degree n, it follows that

$$\langle \tilde{m}\, , X^n \otimes v^* \rangle = \sum_j n! B(X, Y_j)^n (v_j, v^*)_{V_\tau \times V_\tau^*} = n! \varphi_{v^*}(\tilde{m}) = 0,$$

by noting that $Y_j^n - B(X, Y_j)^n \in \mathfrak{m}(X)$. Hence one gets $(\mathrm{Ad}(y)X)^n \otimes y \cdot v^* \in y \cdot N^\perp = N^\perp$ for all $v^* \in \mathcal{W}(0)^*$, $y \in K_{\mathbb{C}}$ and $n \geq 0$. Thus we obtain (1).

(2) Let $m = \sum_j Y_j^n \cdot v_j$ be an element of M_n with $Y_j \in \mathfrak{p}$ and $v_j \in V_\tau$. Just in the proof of (1), we see that $m \in {}^\perp\Psi$ if and only if

$$\begin{aligned} 0 = \langle\, m\, , (\mathrm{Ad}(y)X)^n \otimes y \cdot v^* \rangle &= \sum_j n! B(Y_j, \mathrm{Ad}(y)X)^n\, (y^{-1} \cdot v_j,\, v^*)_{V_\tau \times V_\tau^*} \\ &= n! (T(0)_m(y),\, v^*)_{\mathcal{W}(0) \times \mathcal{W}(0)^*} \end{aligned}$$

for all $v^* \in \mathcal{W}(0)^*$ and $y \in K_{\mathbb{C}}$. This means $m \in \operatorname{Ker} T(0)$. □

Now, let $\mathcal{O} = \mathrm{Ad}(K_{\mathbb{C}})X$ be the nilpotent $K_{\mathbb{C}}$-orbit through X. We write I for the ideal defining the Zariski closure $\overline{\mathcal{O}}$ of $\mathcal{O}$, and let $\{D_i |\ i = 1, \ldots, r\}$ be a finite set of homogeneous elements of $S(\mathfrak{p})$ which generates the ideal I. We set $n(i) := \deg D_i$.

Proposition 3.2. *Assume that $\Psi_{n(i)} = (N^\perp)_{n(i)}$ for $i = 1, \ldots, r$. Then we get $I \subset$ $\mathrm{Ann}_{S(\mathfrak{g})} M$. In this case, the multiplicity of M at I equals* $\dim \mathcal{W}(0)$ *by Lemma* 2.1.

Proof. We see for every $v \in V_\tau$ that

$$T(0)_{D_i v}(y) = (D_i \cdot T(0)_v)(y) = D_i(\mathrm{Ad}(y)X)\,T(0)_v(y) = 0 \quad (y \in K_{\mathbb{C}}),$$

since $D_i \in I$ and $\mathrm{Ad}(y)X \in \mathcal{O}$. It then follows that $D_i v = 0$ by Lemma 3.1 (2) together with the assumption $\Psi_{n(i)} = (N^\perp)_{n(i)}$, which is equivalent to ${}^\perp\Psi \cap M_{n(i)} = \{0\}$. Hence, D_i annihilates V_τ and so the whole $M = S(\mathfrak{p})V_\tau$. This shows the assertion. □

In particular, if the associated variety $\mathcal{V}(X)$ is irreducible, i.e., $\mathcal{V}(X) = \overline{\mathcal{O}}$, as in previous sections, the conclusion of the above proposition turns to be

$$I = \mathrm{Ann}_{S(\mathfrak{g})} M, \quad \text{and} \quad \mathcal{W} = \mathcal{W}(0). \tag{3.6}$$

In this case, the dual $K_{\mathbb{C}}(X)$-module $\mathcal{W}^* = \mathcal{W}(0)^*$ can be specified by means of the principal symbol of a differential operator on $\mathfrak{p}$ of gradient-type (see [24, Theorem 4.1]). Moreover, the $(S(\mathfrak{g}), K_{\mathbb{C}})$-module Ψ is almost equal to $N^\perp \simeq M^*$, because the support of ${}^\perp\Psi \subset \mathrm{Ker}\, T(0)$ is contained in $\partial\mathcal{O}$ by Proposition 2.5.

4. Isotropy representation attached to discrete series

In this section, we assume that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an equi-rank algebra (cf. [12]), i.e., $\mathrm{rank}\,\mathfrak{g} = \mathrm{rank}\,\mathfrak{k}$, and we study the isotropy representations attached to irreducible $(\mathfrak{g}, K_{\mathbb{C}})$-modules of discrete series, by refining our discussion in [22].

4.1. **Discrete series.** We begin with a quick review on the discrete series representations, and let us fix our notation. As is well known, the complex Lie algebra $\mathfrak{g}$ has a θ-stable real form $\mathfrak{g}_0$ such that

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \quad \text{with} \quad \mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0,\ \mathfrak{p}_0 := \mathfrak{p} \cap \mathfrak{g}_0,$$

gives a Cartan decomposition of $\mathfrak{g}_0$. Such a real form $\mathfrak{g}_0$ is unique up to $K_{\mathbb{C}}$-conjugacy. Take a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$, and we write $\mathfrak{t}$ for the complexification of $\mathfrak{t}_0$ in $\mathfrak{k}$. Since $\mathfrak{g}$ is an equi-rank algebra, $\mathfrak{t}$ turns to be a Cartan subalgebra of $\mathfrak{g}$. We write Δ for the root system of $(\mathfrak{g}, \mathfrak{t})$. The subset of compact (resp. noncompact) roots will be denoted by Δ_c (resp. Δ_n).

Let G be a connected Lie group with Lie algebra of $\mathfrak{g}_0$ such that $K_{\mathbb{C}}$ is the complexification of a maximal compact subgroup K of G. An irreducible unitary representation σ of G is called a member of discrete series if the matrix coefficients of σ are square-integrable on G. We are concerned with the irreducible $(\mathfrak{g}, K_{\mathbb{C}})$-modules $\boldsymbol{X}$ of discrete series, consisting of K-finite vectors for such σ's. For example, we refer to [7], [13], and also [19, I, Section 1] for the parametrization and realization of discrete series representations.

Now, let $\boldsymbol{X} = \boldsymbol{X}_\Lambda$ be the $(\mathfrak{g}, K_{\mathbb{C}})$-module of discrete series with Harish-Chandra parameter $\Lambda \in \mathfrak{t}^*$. Since the parameter Λ is regular and real on $\sqrt{-1}\mathfrak{t}_0$, there exists a unique positive system Δ^+ of Δ for which Λ is dominant:

$$\Delta^+ := \{\alpha \in \Delta \mid (\Lambda, \alpha) > 0\}. \tag{4.1}$$

We denote by (τ, V_τ) the unique lowest $K_{\mathbb{C}}$-type of $\boldsymbol{X}$ which occurs in $\boldsymbol{X}$ with multiplicity one. Set $\Delta_c^+ := \Delta^+ \cap \Delta_c$ (resp. $\Delta_n^+ := \Delta^+ \cap \Delta_n$). The Δ_c^+-dominant highest weight λ (say) for τ is called the Blattner parameter of $\boldsymbol{X}$. Then, λ is expressed as $\lambda = \Lambda - \rho_c + \rho_n$ with $\rho_c := (1/2) \cdot \sum_{\alpha \in \Delta_c^+} \alpha$ and $\rho_n := (1/2) \cdot \sum_{\beta \in \Delta_n^+} \beta$.

4.2. **Results of Hotta-Parthasarathy.** In what follows, we assume that the Blattner parameter λ of $\boldsymbol{X}$ is *far from the walls* (defined by compact roots) in the sense of [19, I, Definition 1.7]. Let $M = \operatorname{gr}\boldsymbol{X} = \oplus_{n\geq 0} M_n$ be the graded $(S(\mathfrak{g}), K_{\mathbb{C}})$-module defined through the lowest $K_{\mathbb{C}}$-type V_τ. As in Section 3, we have a natural quotient map $\pi : S(\mathfrak{p}) \otimes V_\tau \to M$ with $N = \operatorname{Ker}\pi$. This subsection explains the structure of graded modules M, N, and $M^* \simeq N^\perp$ by interpreting the results of Hotta-Parthasarathy [7].

For this, we first decompose the tensor product $\mathfrak{p} \otimes V_\tau$ as

$$\mathfrak{p} \otimes V_\tau = V_\tau^+ \oplus V_\tau^- \quad \text{as } K_{\mathbb{C}}\text{-modules},$$

where $V_\tau^\pm$ denotes the sum of irreducible $K_{\mathbb{C}}$-submodules of $\mathfrak{p} \otimes V_\tau$ with highest weights $\lambda \pm \beta$ ($\beta \in \Delta_n^+$), respectively. The inclusion $V_\tau^- \hookrightarrow \mathfrak{p} \otimes V_\tau$ naturally induces a quotient map of $K_{\mathbb{C}}$-modules:

$$P : \mathfrak{p}^* \otimes V_\tau^* = (\mathfrak{p} \otimes V_\tau)^* \longrightarrow (V_\tau^-)^*. \tag{4.2}$$

Hereafter, we replace $\mathfrak{p}^*$ by $\mathfrak{p}$ through the identification $\mathfrak{p} = \mathfrak{p}^*$ by the Killing form $B_{|\mathfrak{p}\times\mathfrak{p}}$.

Let B_c be the Borel subgroup of $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{b}_c = \mathfrak{t} \oplus \sum_{\alpha\in\Delta_c^-} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root subspace of $\mathfrak{g}$ corresponding to a root α. We set

$$\mathfrak{p}_\pm := \bigoplus_{\beta\in\Delta_n^+} \mathfrak{g}_{\pm\beta}. \tag{4.3}$$

Then, we have $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as vector spaces, and $\mathfrak{p}_-$ is stable under the action of B_c.

If U is a holomorphic B_c-module, the i-th cohomology space $H^i(K_{\mathbb{C}}/B_c; U)$ of $K_{\mathbb{C}}/B_c$ with coefficients in the sheaf of holomorphic sections of the vector bundle $K_{\mathbb{C}} \times_{B_c} U$ has a structure of $K_{\mathbb{C}}$-module.

The following theorem can be read off from the proof of [7, Theorem 1], by taking into account the Blattner multiplicity formula [8] for discrete series. (See also [13]; [22].)

Theorem 4.1 (Hotta-Parthasarathy). (1) *One has* $N = S(\mathfrak{p})V_\tau^-$.

(2) *The orthogonal $N^\perp$ of N in $S(\mathfrak{p}) \otimes V_\tau^*$ coincides with the kernel of the differential operator $\mathcal{D}$ on $\mathfrak{p}$ of gradient-type defined as follows:*

$$(\mathcal{D}f)(Z) := P\Big(\sum_\ell X_\ell \otimes (X_\ell \cdot f)(Z)\Big) \quad (f \in S(\mathfrak{p}) \otimes V_\tau^*,\ Z \in \mathfrak{p}). \tag{4.4}$$

Here $\{X_\ell\}_\ell$ is an orthonormal basis of $\mathfrak{p}_0$ with respect to the Killing form.

(3) *For every integer $n \geq 0$, the dual M_n^* of M_n is isomorphic to the cohomology space:*

$$H^s(K_{\mathbb{C}}/B_c;\ S^n(\mathfrak{p}_-) \otimes \mathbb{C}_{-\lambda-2\rho_c}) \quad \textit{with} \quad s := \dim K_{\mathbb{C}}/B_c,$$

as a $K_{\mathbb{C}}$-module. Here $\mathbb{C}_{-\lambda-2\rho_c}$ denotes the one dimensional B_c-module corresponding to $-\lambda - 2\rho_c \in \mathfrak{t}^*$.

Remark 4.2. The operator $\mathcal{D}$ in the above theorem gives the "polynomialization" of an invariant differential operator of gradient-type (on the Riemannian symmetric space for $(\mathfrak{g}_0, \mathfrak{k}_0)$) whose kernel realizes the maximal globalization of dual $(\mathfrak{g}, K_{\mathbb{C}})$-module $\boldsymbol{X}^*$ of discrete series.

4.3. **Description of associated cycle.** For a positive number c, we say that a linear form μ on $\mathfrak{t}$ satisfies the condition (FFW(c)) if

$$\text{(FFW}(c)) \qquad (\mu, \alpha) \geq c \quad \text{for all} \quad \alpha \in \Delta_c^+.$$

Theorem 4.1 coupled with the Borel-Weil Bott theorem for the group $K_{\mathbb{C}}$ leads us to the following proposition, which is crucial to describe the associated cycle of $\boldsymbol{X}$.

Proposition 4.3 (cf. [22, Section 6.1]). (1) *Let v_λ^* be a nonzero lowest weight vector of V_τ^* of weight $-\lambda$. Then, $N^\perp = \operatorname{Ker} \mathcal{D}$ contains the $K_{\mathbb{C}}$-submodule $\langle S(\mathfrak{p}_-) \otimes v_\lambda^* \rangle_{K_{\mathbb{C}}}$ generated by $S(\mathfrak{p}_-) \otimes v_\lambda^*$.*

(2) *For any integer $n \geq 0$, there exists a positive constant c_n such that*

$$(N^\perp)_n = \langle S^n(\mathfrak{p}_-) \otimes v_\lambda^* \rangle_{K_{\mathbb{C}}} \tag{4.5}$$

holds if the Blattner parameter λ satisfies the condition (FFW(c_n)).

Now, let $\mathcal{O}$ be the unique nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$ which intersects $\mathfrak{p}_-$ densely. Then one sees that $\overline{\mathcal{O}} = \operatorname{Ad}(K_{\mathbb{C}})\mathfrak{p}_-$. As before, we write I for the prime ideal of $S(\mathfrak{g})$ defining $\overline{\mathcal{O}}$. It follows from the the claim (1) in Proposition 4.3 that $\operatorname{Ann}_{S(\mathfrak{g})} M \subset I$, i.e., $\mathcal{V}(\boldsymbol{X}) \supset \overline{\mathcal{O}}$. Also, the same claim shows $\mathfrak{p}_- \otimes v_\lambda^* \in \operatorname{Ker} P$, which can be easily verified by noting that $-\lambda - \beta$ ($\beta \in \Delta_n^+$) cannot be a weight of $(V_\tau^-)^*$.

Take an element $X \in \mathcal{O} \cap \mathfrak{p}_-$. By virtue of our discussion in [24, Section 3], we find that the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(0)^* = (M/\mathfrak{m}(X)M)^* \subset V_\tau^*$ consists exactly of all the vectors $v^* \in V_\tau^*$ satisfying $P(X \otimes v^*) = 0$. Let $N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-)$ be the totality of elements $k \in K_{\mathbb{C}}$ such that $\operatorname{Ad}(k)X \in \mathfrak{p}_-$ (cf. [3]). For any subset R of $N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-)$, we denote by $\mathcal{U}_\lambda(R)$ the $K_{\mathbb{C}}(X)$-submodule of V_τ^* generated by $R^{-1} \cdot v_\lambda^*$:

$$\mathcal{U}_\lambda(R) := \langle R^{-1} \cdot v_\lambda^* \rangle_{K_{\mathbb{C}}(X)}.$$

Then, we readily find from $\mathfrak{p}_- \otimes v_\lambda^* \in \operatorname{Ker} P$ that

$$\mathcal{U}_\lambda(R) \subset \mathcal{W}(0)^*, \quad \text{and so} \quad \langle X^n \otimes \mathcal{U}_\lambda(R) \rangle_{K_{\mathbb{C}}} \subset \Psi_n \subset (N^\perp)_n, \tag{4.6}$$

for every $n \geq 0$. Moreover one gets the equality

$$\langle X^n \otimes \mathcal{U}_\lambda(R) \rangle_{K_{\mathbb{C}}} = \langle S^n(\mathfrak{p}_-) \otimes v_\lambda^* \rangle_{K_{\mathbb{C}}}, \tag{4.7}$$

if $\operatorname{Ad}(R)X \subset \mathfrak{p}_-$ is Zariski dense in $\mathfrak{p}_-$. This is true when R equals the whole $N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-)$, because $\operatorname{Ad}(N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-))X = \mathcal{O} \cap \mathfrak{p}_-$ is dense in $\mathfrak{p}_-$.

As in Section 3, we take homogeneous generators D_i ($i = 1, \dots, r$) of the ideal I such that $\deg D_i = n(i)$. We set $c(I) := \max_i(c_{n(i)})$. By virtue of Proposition 3.2 together with (4.5), (4.6) and (4.7), we come to the following conclusion.

Theorem 4.4. *Assume that the Blattner parameter λ of discrete series $\boldsymbol{X}$ is far from the walls and that it satisfies the condition* (FFW($c(I)$)).

(1) *One gets $I = \operatorname{Ann}_{S(\mathfrak{g})} M$ and so $\mathcal{V}(\boldsymbol{X}) = \operatorname{Ad}(K_{\mathbb{C}})\mathfrak{p}_- = \overline{\mathcal{O}}$. Moreover, the $K_{\mathbb{C}}(X)$-module $\mathcal{W}^*$ contragredient to the isotropy representation $(\varpi_{\mathcal{O}}, \mathcal{W})$ is described as*

$$\mathcal{W}^* = \mathcal{W}(0)^* = \{v^* \in V_\tau^* \mid P(X \otimes v^*) = 0\}, \tag{4.8}$$

where $X \in \mathcal{O} \cap \mathfrak{p}_-$, and $P : \mathfrak{p} \otimes V_\tau^* \to (V_\tau^-)^*$ is the $K_{\mathbb{C}}$-homomorphism in (4.2).

(2) *Let R be a subset of $N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-)$ such that* $\mathrm{Ad}(R)X$ *is Zariski dense in $\mathfrak{p}_-$. Then, the $K_{\mathbb{C}}(X)$-submodule $\mathcal{U}_\lambda(R) = \langle R^{-1} \cdot v_\lambda^* \rangle_{K_{\mathbb{C}}(X)} \subset \mathcal{W}^*$ is exhaustive in the following sense: for every integer $n \geq 0$, one has*

$$\langle X^n \otimes \mathcal{W}^* \rangle_{K_{\mathbb{C}}} = \langle X^n \otimes \mathcal{U}_\lambda(R) \rangle_{K_{\mathbb{C}}} \tag{4.9}$$

if λ satisfies FFW(c_n).

Remark 4.5. (1) The assertions $I = \mathrm{Ann}_{S(\mathfrak{g})} M$ and $\mathcal{V}(\boldsymbol{X}) = \mathrm{Ad}(K_{\mathbb{C}})\mathfrak{p}_- = \overline{\mathcal{O}}$ have been obtained in [22]. But, in that paper, we did not discuss the possibility of applying the results to describe the isotropy representation.

(2) One should get a result similar to Theorem 4.4, more generally for the derived functor modules $\mathcal{A}_{\mathfrak{q}}(\lambda)$.

(3) Compare Theorem 4.4 (1) with Chang's result [3, Proposition 1.4] established by means of the localization theory of Harish-Chandra modules.

4.4. **Submodule $\mathcal{U}_\lambda(Q_c)$.** In this subsection, we give a natural choice of $R \subset N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-)$ for which we expect to have the property (4.9). Let Π be the set of simple roots in Δ^+. We write $S = \Pi \cap \Delta_c$ for the totality of compact simple roots. Then, there exists a unique element $H_S \in \mathfrak{t}$ such that

$$\alpha(H_S) = \begin{cases} 0 & \text{if } \alpha \in S, \\ 1 & \text{if } \alpha \in \Pi \backslash S. \end{cases}$$

The adjoint action of H_S yields a gradation on the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{g} = \bigoplus_j \mathfrak{g}(j) \quad \text{with} \quad \mathfrak{g}(j) := \{Z \in \mathfrak{g} |\ (\mathrm{ad}\, H_S)Z = jZ\}.$$

Here j runs through the integers such that $|j| \leq \delta(H_S)$ with the highest root δ. Note that

$$\mathfrak{k} = \bigoplus_{j:even} \mathfrak{g}(j), \qquad \mathfrak{p} = \bigoplus_{j:odd} \mathfrak{g}(j) \quad \text{with} \quad \mathfrak{p}_\pm = \bigoplus_{j>0,odd} \mathfrak{g}(\pm j).$$

Now, we set

$$\mathfrak{q} := \bigoplus_{j \leq 0} \mathfrak{g}(j), \quad \mathfrak{l} := \mathfrak{g}(0) \subset \mathfrak{k} \quad \text{and} \quad \mathfrak{u} := \bigoplus_{j<0} \mathfrak{g}(j).$$

Then, $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ gives the Levi decomposition of the standard parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ associated with the subset S of Π. We write Q (resp. Q_c) for the parabolic subgroup of $G_{\mathbb{C}} := G_{\mathbb{C}}^{\mathrm{ad}}$ (resp. of $K_{\mathbb{C}}$) with Lie algebra $\mathfrak{q}$ (resp. $\mathfrak{q} \cap \mathfrak{k}$). The group Q (resp. Q_c) admits the Levi decomposition $Q = LU$ (resp. $Q_c = L_c U_c$), where L and U (resp. L_c and U_c) are the connected subgroups of Q (resp. Q_c) with Lie algebras $\mathfrak{l}$ and $\mathfrak{u}$ (resp. $\mathfrak{l}$ and $\mathfrak{u} \cap \mathfrak{k}$) respectively. Note that $\mathrm{Ad}(L_c) = L$. The parabolic subgroup Q acts on its nilradical $\mathfrak{u}$, and so Q_c acts on $\mathfrak{p}_- = \mathfrak{p} \cap \mathfrak{u}$ by the adjoint action. Thus, Q_c is contained in $N_{K_{\mathbb{C}}}(X, \mathfrak{p}_-)$ for all $X \in \mathfrak{p}_-$, and the corresponding $K_{\mathbb{C}}(X)$-submodule $\mathcal{U}_\lambda(Q_c)$ of V_τ^* turns to be

$$\mathcal{U}_\lambda(Q_c) = \langle (V_\lambda^{L_c})^* \rangle_{K_{\mathbb{C}}(X)}. \tag{4.10}$$

Here, $(V_\lambda^{L_c})^* = U(\mathfrak{l}) v_\lambda^*$ denotes the irreducible L_c-submodule of V_τ^* generated by the lowest weight vector v_λ^*.

We can now apply Theorem 4.4 to deduce

Corollary 4.6. *Under the assumption in Theorem* 4.4, *the* $K_{\mathbb{C}}(X)$*-submodule* $\mathcal{U}_\lambda(Q_c)$ *of* $\mathcal{W}^*$ *is exhaustive in the sense of* (4.9), *if* $\mathfrak{p}_-$ *is a prehomogeneous vector space under the adjoint action of the group* Q_c, *and if* $X \in \mathcal{O} \cap \mathfrak{p}_-$ *lies in the open* Q_c*-orbit in* $\mathfrak{p}_-$.

4.5. **Relation to the Richardson orbit.** We end this article by looking at the condition for $\mathfrak{p}_-$ in Corollary 4.6, and also some related conditions, in relation to the Richardson $G_{\mathbb{C}}$-orbit associated with the parabolic subalgebra $\mathfrak{q}$.

First, let us recall some basic facts on the Richardson orbit (cf. [6, Chapter 5]). The $G_{\mathbb{C}}$-stable subset $G_{\mathbb{C}} \cdot \mathfrak{u}$ ($\subset \mathfrak{g}$) forms an irreducible affine variety of $\mathfrak{g}$ whose dimension is equal to $2 \dim \mathfrak{u}$. Noting that $G_{\mathbb{C}} \cdot \mathfrak{u}$ consists of nilpotent elements only, there exists a unique $G_{\mathbb{C}}$-orbit $\tilde{\mathcal{O}}$ such that

$$\overline{\tilde{\mathcal{O}}} = G_{\mathbb{C}} \cdot \mathfrak{u},$$

by the finiteness of the number of nilpotent $G_{\mathbb{C}}$-orbits in $\mathfrak{g}$. $\tilde{\mathcal{O}}$ is called the Richardson $G_{\mathbb{C}}$-orbit associated with $\mathfrak{q}$. The parabolic subgroup Q acts on $\mathfrak{u}$ prehomogenenously, and $\tilde{\mathcal{O}} \cap \mathfrak{u}$ turns to be a single Q-orbit in $\mathfrak{u}$. Moreover, the centralizer in $\mathfrak{g}$ of any element $X \in \tilde{\mathcal{O}} \cap \mathfrak{u}$ is contained in $\mathfrak{q}$.

Now, we have two nilpotent $G_{\mathbb{C}}$-orbits $G_{\mathbb{C}} \cdot \mathcal{O}$ and $\tilde{\mathcal{O}}$ with the closure relation $G_{\mathbb{C}} \cdot \mathcal{O} \subset \overline{\tilde{\mathcal{O}}}$. By virtue of a result of Kostant-Rallis [10, Proposition 5], this relation implies that

$$\dim \mathcal{O} = \frac{1}{2} \dim G_{\mathbb{C}} \cdot \mathcal{O} \leq \frac{1}{2} \dim \tilde{\mathcal{O}} = \dim \mathfrak{u}. \tag{4.11}$$

In particular, we find that the Gelfand-Kirillov dimension $\dim \mathcal{V}(\boldsymbol{X}) = \dim \mathcal{O}$ of discrete series $\boldsymbol{X}$ cannot exceed $\dim \mathfrak{u}$. The following proposition tells us when these two orbits turn to be equal.

Proposition 4.7. *The following three conditions* (a), (b) *and* (c) *on the positive system* $\Delta^+ = \{\alpha | \ (\Lambda, \alpha) > 0\}$ *are equivalent with each other*:

$$\text{(a) } G_{\mathbb{C}} \cdot \mathcal{O} = \tilde{\mathcal{O}}, \quad \text{(b) } \dim \mathcal{O} = \dim \mathfrak{u}, \quad \text{(c) } \tilde{\mathcal{O}} \cap \mathfrak{p}_- \neq \emptyset.$$

In this case, $\mathcal{O} \cap \mathfrak{p}_-$ *is a single open* Q_c*-orbit in* $\mathfrak{p}_-$, *and so one gets the conclusion of Corollary* 4.6.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is a direct consequence of (4.11). The condition (a) immediately implies (c), since $\mathcal{O}$ ($\subset G_{\mathbb{C}} \cdot \mathcal{O} = \tilde{\mathcal{O}}$) contains an element of $\mathfrak{p}_-$. Conversely, if $\tilde{\mathcal{O}} \cap \mathfrak{p}_- \neq \emptyset$, this is a nonempty open subset of $\mathfrak{p}_-$, since $\tilde{\mathcal{O}} \cap \mathfrak{p}_- = (\tilde{\mathcal{O}} \cap \mathfrak{u}) \cap \mathfrak{p}_-$ with $\tilde{\mathcal{O}} \cap \mathfrak{u}$ open in $\mathfrak{u}$. Hence, $\tilde{\mathcal{O}} \cap \mathfrak{p}_-$ intersects $\mathcal{O}$. We thus get (c) $\Rightarrow$ (a). This proves the equivalence of three conditions in question.

Next,we assume the condition (b) ($\Leftrightarrow$ (a) $\Leftrightarrow$ (c)), and let X be any element of $\mathcal{O} \cap \mathfrak{p}_-$. We write $\mathfrak{z}_{\mathfrak{s}}(X)$ for the centralizer of X in a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. By noting that $\mathfrak{z}_{\mathfrak{g}}(X) \subset \mathfrak{q}$, the dimension of the Q_c-orbit $\mathrm{Ad}(Q_c)X$ is calculated as

$$\begin{aligned} \dim \mathrm{Ad}(Q_c)X &= \dim \mathfrak{q} \cap \mathfrak{k} - \dim \mathfrak{z}_{\mathfrak{q} \cap \mathfrak{k}}(X) = (\dim \mathfrak{k} - \dim \mathfrak{u} \cap \mathfrak{k}) - \dim \mathfrak{z}_{\mathfrak{k}}(X) \\ &= \dim \mathcal{O} - \dim \mathfrak{u} \cap \mathfrak{k} = \dim \mathfrak{u} - \dim \mathfrak{u} \cap \mathfrak{k} = \dim \mathfrak{p}_-, \end{aligned}$$

where we used the condition (b) for the forth equality. This shows that the orbit $\mathrm{Ad}(Q_c)X$ is open in $\mathfrak{p}_-$ for every $X \in \mathcal{O} \cap \mathfrak{p}_-$. We thus find that $\mathcal{O} \cap \mathfrak{p}_-$ forms a single Q_c-orbit, because of the uniqueness of the open Q_c-orbit in $\mathfrak{p}_-$. $\square$

Remark 4.8. Each of the conditions (a), (b) and (c) in Proposition 4.7 is equivalent to Assumption 2.5 in [2] concerning the generically finiteness of the moment map defined on the conormal bundle $T^*_{Z_1}(G_{\mathbb{C}}/Q)$, where Z_1 is a closed $K_{\mathbb{C}}$-orbit in $G_{\mathbb{C}}/Q$ through the origin eQ.

Suggested by Corollary 4.6 and Proposition 4.7, let us consider the following three conditions on the positive system Δ^+:

(C1) $\tilde{\mathcal{O}} \cap \mathfrak{p}_- \neq \emptyset$ ($\Leftrightarrow$ $\dim \mathcal{O} = \dim \mathfrak{u}$ $\Leftrightarrow$ $G_{\mathbb{C}} \cdot \mathcal{O} = \tilde{\mathcal{O}}$, by Proposition 4.7),

(C2) $\mathcal{O} \cap \mathfrak{p}_-$ is a single Q_c-orbit,

(C3) $\mathfrak{p}_-$ is a prehomogeneous vector space under the adjoint action of Q_c.

Proposition 4.7 says (C1) $\Rightarrow$ (C2), and the implication (C2) $\Rightarrow$ (C3) is obvious.
As for the conditions (C2) and (C3), we can show the following

Proposition 4.9. *One gets* (C3) *if* $\mathcal{O} \cap \mathfrak{g}(-1) \neq \emptyset$. *Moreover, the equality* $\mathrm{Ad}(Q_c)(\mathcal{O} \cap \mathfrak{g}(-1)) = \mathcal{O} \cap \mathfrak{p}_-$ *assures* (C2).

Proof. Let $X \in \mathcal{O} \cap \mathfrak{g}(-1)$. Since $\mathcal{O} = \mathrm{Ad}(K_{\mathbb{C}})X$ contains a nonempty open subset of $\mathfrak{p}_-$, we find that $[\mathfrak{k}, X] \supset \mathfrak{p}_-$. We set $\mathfrak{k}_+ := \oplus_{j>0}\mathfrak{g}(2j)$. Then $\mathfrak{k} = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{k}_+$ is a direct sum of vector spaces. Then it follows from the assumption $X \in \mathfrak{g}(-1)$ that $[\mathfrak{k}_+, X] \subset \mathfrak{p}_+$ and $[\mathfrak{k} \cap \mathfrak{q}, X] \subset \mathfrak{p}_-$. We thus obtain

$$\mathfrak{p}_- = [\mathfrak{k}, X] \cap \mathfrak{p}_- = [\mathfrak{k} \cap \mathfrak{q}, X].$$

Hence $\mathrm{Ad}(Q_c)X$ is open in $\mathfrak{p}_-$, and one gets (C3).
The above argument shows that any element $X \in \mathcal{O} \cap \mathfrak{p}_-$ lies in the unique open Q_c-orbit in $\mathfrak{p}_-$. This proves the latter claim, too. □

Following Gross-Wallach [4], we say that a discrete series $(\mathfrak{g}, K_{\mathbb{C}})$-module $\boldsymbol{X}$ is *small* if $\delta(H_S) \leq 2$, or equivalently, $\mathfrak{g}(j) = \{0\}$ if $|j| \geq 3$. Here δ is the highest root of Δ^+ as before. In this case, one has $\mathfrak{p}_- = \mathfrak{g}(-1)$, and so the above proposition implies

Corollary 4.10. *The positive system* Δ^+ *corresponding to a small discrete series admits the property* (C2).

Remark 4.11. By case-by-case analysis, Chang [3] proved the property (C2) for any discrete series representations of simple Lie groups of $\mathbb{R}$-rank one.

We can give an explicit, combinatorial algorithm to find out whether or not a given Δ^+ satisfies the condition (C1), for the case of arbitrary discrete series of $SU(p,q)$. We will discuss it elsewhere.

References

[1] W. Borho and J.-L. Brylinski, Differential operators on homogeneous spaces. III: Characteristic varieties of Harish-Chandra modules and of primitive ideals, Invent. Math., **80** (1986), 1–68.

[2] J.-T. Chang, Characteristic cycles of holomorphic discrete series, Trans. Amer. Math. Soc., **334** (1992), 213–227.

[3] J.-T. Chang, Characteristic cycles of discrete series for $\mathbb{R}$-rank one groups, Trans. Amer. Math. Soc., **341** (1994), 603–622.

[4] B. Gross and N.R. Wallach, Restriction of small discrete series representations to symmetric subgroups, *in*: The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), 255–272, Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., Providence, RI, 2000.

[5] A. Gyoja and H. Yamashita, Associated variety, Kostant-Sekiguchi correspondence, and locally free $U(\mathfrak{n})$-action on Harish-Chandra modules, J. Math. Soc. Japan, **51** (1999), 129–149.

[6] J.-E. Humphreys, Conjugacy classes in semisimple algebraic groups, Mathematical Surveys and Monographs, **43**, American Mathematical Society, Providence, RI, 1995.

[7] R. Hotta and R. Parthasarathy, Multiplicity formulae for discrete series, Invent. Math. **26** (1974), 133–178.

[8] H. Hecht and W. Schmid, A proof of Blattner's conjecture, Invent. Math. **31** (1975), 129–154.

[9] S. Kato and H. Ochiai, The degrees of orbits of the multiplicity-free actions, *in*: "Nilpotent orbits, associated cycles and Whittaker models for highest weight representations", Astérisque, **273** (2001), 139–158.

[10] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces, Amer. J. Math., **83** (1971), 753–809.

[11] K. Nishiyama, H. Ochiai and K. Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules – Hermitian symmetric case, *in*: "Nilpotent orbits, associated cycles and Whittaker models for highest weight representations", Astérisque, **273** (2001), 13–80.

[12] H. Rubenthaler, A new proof of the Borel-de Siebenthal theorem, the classification of equi-rank groups, and some compact and semi-compact dual pairs, Prépublications IRMA, l'Université Louis Pasteur, no. **45** (1998), 21 pages.

[13] W. Schmid, Some properties of square-integrable representations of semisimple Lie groups, Ann. of Math., **102** (1975), 535–564.

[14] K. Taniguchi, Discrete series Whittaker functions of $\mathrm{SU}(n,1)$ and $\mathrm{Spin}(2n,1)$, J. Math. Sci. Univ. Tokyo, **3** (1996), no. 2, 331–377.

[15] K. Taniguchi, On discrete series Whittaker functions $\mathrm{SO}_0(2n,2)$ (Japanese), *in*: "Automorphic forms on $\mathrm{Sp}(2;R)$ and $\mathrm{SU}(2,2)$, II (Kyoto, 1998)", RIMS Kōkyūroku, **1094** (1999), 11–27.

[16] D.A. Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules, Invent. Math. **48** (1978), 75–98.

[17] D.A. Vogan, Associated varieties and unipotent representations, *in*: "Harmonic Analysis on Reductive Groups (W.Barker and P.Sally eds.)", Progress in Math., Vol. **101**, Birkhäuser, 1991, 315–388.

[18] D. A. Vogan, The method of coadjoint orbits for real reductive groups, *in*: "Representation Theory of Lie Groups (eds. J. Adams and D. Vogan)", 177–238, IAS/Park City Mathematics Series **8**, AMS, 2000.

[19] H. Yamashita, Embeddings of discrete series into induced representations of semisimple Lie groups, I: General theory and the case of $SU(2,2)$, Japan. J. Math.,**16** (1990), 31–95; II: generalized Whittaker models for $SU(2,2)$, J. Math. Kyoto Univ., **31** (1991), 543–571.

[20] H. Yamashita, Criteria for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras and applications to Harish-Chandra modules: a study in relation to the associated varieties, J. Funct. Anal., **121** (1994), 296–329.

[21] H. Yamashita, Associated varieties and Gelfand-Kirillov dimensions for the discrete series of a semisimple Lie group, Proc. Japan Acad., **70A** (1994), 50–55.

[22] H. Yamashita, Description of the associated varieties for the discrete series representations of a semisimple Lie group: An elementary proof by means of differential operators of gradient type, Comment. Math. Univ. St. Paul., **47** (1998), 35–52.

[23] H. Yamashita, Cayley transform and generalized Whittaker models for irreducible highest weight modules, *in*: "Nilpotent orbits, associated cycles and Whittaker models for highest weight representations", Astérisque, **273** (2001), 81–137.

[24] H. Yamashita, Associated cycles of Harish-Chandra modules and differential operators of gradient type, RIMS Kōkyūroku, **1183** (2001), 157–167.

[25] T. Yoshinaga and H. Yamashita, The embeddings of discrete series into principal series for an exceptional real simple Lie group of type G_2, J. Math. Kyoto Univ., **36** (1996), 557–595.

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