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A NOTE ON THE ODDS-THEOREM

愛知大学経営学部　玉置光司 (Mitsushi Tamaki)
Department of Business Administration, Aichi University

1 Introduction and summary

This paper is motivated by the odds-Theorem in Bruss[2], which is directly applicable to some optimal stopping problems involving independent indicator functions. We first review the Bruss odds-theorem. Let, for given n, $I_1, I_2, \cdots, I_n$ be indicators of independent events $A_1, A_2, \cdots, A_n$ defined on some probability space $(\Omega, \mathcal{A}, P)$. We observe $I_1, I_2, \cdots$ sequentially and may stop at any of these, but may not recall on preceding I_k's. If $I_k = 1$, we say that k is a "success time". Let $\mathcal{T}$ denote the class of all rules t such that $\{t = k\} \in \sigma(I_1, I_2, \cdots, I_k)$, the sigma field generated by $I_1, I_2, \cdots, I_k$. We wish to stop on the last success and so seek a stopping rule $\tau_n \in \mathcal{T}$ maximizing $P(I_t = 1, I_{t+1} = 0, \cdots, I_n = 0)$ over all $t \in \mathcal{T}$ and its value. Let $p_j = P(I_j = 1) = 1 - q_j$ and $r_j = p_j/q_j,\ 1 \leq j \leq n$. Then Bruss[2] gives the following theorem.

Theorem 1 (Bruss odds-theorem)

An optimal rule τ_n for stopping on the last success exists and is to stop on the first index (if any) k with $I_k = 1$ and $k \geq s$, where

$$s = \sup\left\{1, \sup\left\{1 \leq k \leq n : \sum_{j=k}^{n} r_j \geq 1\right\}\right\},$$

with $\sup\{\phi\} = -\infty$ (This is assumed throughout this paper).

The optimal reward (win probability) is given by $V(n) = \left(\prod_{j=s}^{n} q_j\right)\left(\sum_{j=s}^{n} r_j\right)$.

A typical application of the odds-theorem is the celebrated classical secretary problem where we want to maximize the probability of stopping on rank 1 in a random permutations of n candidates (all $n!$ permutations being equally likely and the overall best having rank 1), that is, on the last record. Success is sometimes referred to as record. It is easy to check that I_k's are independent with $p_k = 1/k$. Hence $r_k = 1/(k-1)$ and so the odds-theorem immediately yields the well known results

$$\begin{aligned} s &= \sup\left\{1, \sup\left\{1 \leq k \leq n : \sum_{j=k}^{n} \frac{1}{j-1} \geq 1\right\}\right\}, \\ V(n) &= \frac{s-1}{n} \sum_{j=s}^{n} \frac{1}{j-1}. \end{aligned}$$

In this note, we attempt to generalize the odds-theorem to include the case of uncertain selection. This is motivated by Smith[3] who considered a version of the secretary problem where each candidate has the right to refuse an offer of selection, that is, he/she accepts an offer only with a known fixed probability β $(0 < \beta < 1)$, independent of his/her rank and the arrival time. The objective is still to maximize the probability of stopping on the last record, i.e., selecting the very best candidate. For ease of description, we call a stopping rule *threshold* or more specifically r-threshold if it passes over the first $r-1$ candidates and then makes an offer to records successively until an offer is accepted or the final stage is reached. The following theorem is the main result of this note.

Theorem 2 (generalized odds-theorem)

An optimal rule σ_n for stopping on the last record (success) exists and is described as the s-threshold rule, where

$$s = \sup\left\{1, \sup\left\{1 \le k \le n : \sum_{j=1}^{n-k+1} (1-\beta)^{j-1}\beta R_j^{(k)} \ge 1\right\}\right\},$$

where

$$R_j^{(k)} = \sum_{k \le i_1 < i_2 < \cdots < i_j \le n} r_{i_1} r_{i_2} \cdots r_{i_j}.$$

The optimal reward (win probability) is given by

$$V(n) = \left(\prod_{j=s}^{n} q_j\right)\left[\sum_{j=1}^{n-s+1} (1-\beta)^{j-1}\beta R_j^{(s)}\right].$$

Remark : If we call $R_j^{(k)}$ the sum of the remaining odds when the k-threshold needs to make j offers until the offer of selection is eventically accepted, the $\sum_{j=1}^{n-k+1}(1-\beta)^{j-1}\beta R_j^{(k)}$ can be interpreted as the expected sum of the remaining odds. Thus, in this generalized odds-theorem also optimal stopping rule can be given by so called "stop-at-1 algorithm" (see Bruss, section 2).

2 Derivation of results

Let v_i denote the reward attainable when we pass over the first $(i-1)$ candidates and then proceed optimally. Then, from the principle of optimality, we have

$$v_{i-1} = p_i \max\{\beta a_i + (1-\beta)v_i, v_i\} + q_i v_i, \tag{2.1}$$

with $v_n = 0$ and $a_i \equiv \prod_{j=i+1}^{n} q_j$, because the optimal rule makes no offer to non-records and the reward is a_i if the i-th candidate is a record and accepts the offer.

Lemma 2.1

The optimal stopping rule is a s-threshold with

$$s = \sup\{1, \sup\{0 \le k \le n-1 : a_k \le v_k\}. \tag{2.2}$$

Proof. We show this only when $p_j < 1, 1 \le j \le n$. Note that Equation (2.1) can be written as

$$v_{i-1} = v_i + \beta p_i \max\{a_i - v_i, 0\}. \tag{2.3}$$

Hence, to show that the optimal rule is threshold, it suffices to show that a_i is increasing in i, while v_i is non-increasing in i. The former is evident from its definition and the latter is also immediate from (2.3). Thus the proof is complete.

We now turn to derivation of v_k for $k \ge s-1$.

Lemma 2.2

Let X denote the total number of records, that is, $X = \sum_{j=1}^{n} I_j$. Then the distribution of X is given by, for $0 \le k \le n$,

$$P(X=k) = \sum_{j=k}^{n}(-1)^{j-k}\binom{j}{k}S_j,$$

where

$$S_j \equiv \sum_{1\le i_1<i_2<\cdots<i_j\le n} p_{i_1}p_{i_2}\cdots p_{i_j}.$$

Proof. See, for example, Blom et al[1].

Lemma 2.3

Let Q_n be the win probability under the 1-threshold. Then

$$Q_n = \left(\frac{\beta}{1-\beta}\right)\sum_{j=1}^{n}(-1)^j(\beta^j - 1)S_j.$$

Proof. It is easy to see, from Lemma 2.2,

$$\begin{aligned} Q_n &= \sum_{k=1}^{n}(1-\beta)^{k-1}\beta P(X=k) \\ &= \left(\frac{\beta}{1-\beta}\right)\sum_{j=1}^{n}\left[\sum_{k=1}^{j}\binom{j}{k}(1-\beta)^k(-1)^{j-k}\right]S_j \\ &= \left(\frac{\beta}{1-\beta}\right)\sum_{j=1}^{n}\left[(1-\beta-1)^j - (-1)^j\right]S_j, \end{aligned}$$

which is the desired result.

To derive another expression for Q_n, the following lemma is helpful.

Lemma 2.4

$$S_j = \left(\prod_{j=1}^{n} q_j\right) \sum_{k=j}^{n} \binom{k}{j} R_k,$$

where $R_k \equiv R_k^{(1)}$.

Proof.

$$\begin{aligned}
S_j \Big/ \left(\prod_{j=1}^{n} q_j\right) &= \sum_{i_1<i_2<\cdots<i_j} \frac{p_{i_1}p_{i_2}\cdots p_{i_j}}{q_1 q_2 \cdots q_n} \\
&= \sum_{i_1<i_2<\cdots<i_j} (r_{i_1}r_{i_2}\cdots r_{i_j}) \prod_{t\neq i_1,\cdots,i_j} (1+r_t) \\
&= \sum_{i_1<i_2<\cdots<i_j} (r_{i_1}r_{i_2}\cdots r_{i_j}) \left(1 + \sum_{i\neq i_1,\cdots,i_j} r_i + \sum_{i,j\neq i_1,\cdots,i_j} r_i r_j + \cdots\right) \\
&= \sum_{k=j}^{n} \binom{k}{j} R_k,
\end{aligned}$$

which completes the proof.

Lemma 2.5

$$Q_n = \left(\prod_{j=1}^{n} q_j\right) \left[\sum_{j=1}^{n} (1-\beta)^{j-1} \beta R_j\right].$$

Proof. From Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
Q_n \Big/ \left(\prod_{j=1}^{n} q_j\right) &= \left(\frac{\beta}{1-\beta}\right) \sum_{j=1}^{n} (-1)^j (\beta^j - 1) \left[S_j \Big/ \prod_{i=1}^{n} q_i\right] \\
&= \left(\frac{\beta}{1-\beta}\right) \sum_{j=1}^{n} (-1)^j (\beta^j - 1) \left[\sum_{k=j}^{n} \binom{k}{j} R_k\right] \\
&= \left(\frac{\beta}{1-\beta}\right) \sum_{k=1}^{n} \left[\sum_{j=1}^{k} \binom{k}{j} \{(-\beta)^j - (-1)^j\}\right] R_k \\
&= \beta \sum_{k=1}^{n} (1-\beta)^{k-1} R_k,
\end{aligned}$$

which is the desired result.

The following result is an immediate consequence from Lemma 2.5.

Lemma 2.6

Let $Q_n^{(k)}$ be the win probability under the k-threshold. Then

$$Q_n^{(k)} = \left(\prod_{j=k}^{n} q_j\right)\left[\sum_{j=1}^{n-k+1}(1-\beta)^{j-1}\beta R_j^{(k)}\right].$$

Proof. Appling Lemma 2.6 to Lemma 2.1 yields Theorem 2, because, for $k \geq s-1$, $v_k = Q_n^{(k+1)}$.

Remark : Smith's results is immediate from Theorem 2 because

$$\sum_{j=1}^{n-k+1}(1-\beta)^{j-1}\beta R_j^{(k)} = \left(\frac{\beta}{1-\beta}\right)\left[\prod_{j=k}^{n}\{1+(1-\beta)r_j\}-1\right],$$

from the identity

$$\prod_{j=k}^{n}(1+r_j x) = 1 + \sum_{j=1}^{n-k+1} R_j^{(k)} x^j.$$

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[3] M.T.Smith (1975), "A secretary problem with uncertain employment", *Jornal of Applied Probability* **12**, 620-624.