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A note on interval games and their saddle points

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Abstract
In this note, we consider the interval matrix game which is an interval generation of the traditional matrix game. The saddle-points of the interval matrix game are defined and characterized as equilibrium points of corresponding non-zero sum parametric games. Numerical examples are given. Also, these results are extended to the fuzzy matrix games.

Key words: Interval matrix game, Saddle point, Interval arithmetics, (β, β')-equilibrium.

1 Introduction and notations

In usual matrix game theory(cf. [15, 16]), all the elements of the payoff matrix are assumed to be exactly given. But in a real application, we often encounter the case where the information on the required data includes imprecision or ambiguity because of uncertain environment.

In order to deal with such case, it is more reasonable to estimate the elements of the payoff matrix by intervals. As for interval approaches to linear programming problem and decision processes, refer, for example, to [2, 12] and [4] respectively.

In this note, we consider the interval matrix game which is an interval generation of the traditional matrix game. The saddle points of the interval matrix game are defined and characterized as equilibrium points of corresponding non-zero sum parametric games. Also, these results are extended to the fuzzy matrix games.

In the reminder of this section, we shall give some notation on interval arithmetics(cf. [8]) and some preliminaries related to preference relation on intervals.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{C}$ the set of all bounded and closed intervals in $\mathbb{R}$.

Note that $\mathbb{R} \subset \mathbb{C}$ by identifying $r \in \mathbb{R}$ with $[r, r] \in \mathbb{C}$. We will give a partial order $\succcurlyeq, \succ$ on $\mathbb{C}$ by the following definition.

For $[a, b], [c, d] \in \mathbb{C}$, $[a, b] \succcurlyeq [c, d]$ if $a \geqq c$ and $b \geqq d$, and $[a, b] \succ [c, d]$ if $[a, b] \succcurlyeq [c, d]$ and $[a, b] \neq [c, d]$.

The Hausdorff metric(cf. [5]) on $\mathbb{C}$ is defined by δ, i.e.,

$$\delta([a,b],[c,d]) := |a-c| \vee |b-d| \quad \text{for} \quad [a,b],[c,d] \in \mathbb{C},$$

where $x \vee y = \max\{x,y\}$.

Let $\mathbb{C}_+ := \{\boldsymbol{a} \in \mathbb{C} \mid \boldsymbol{a} = [a,b] \succcurlyeq [0,0]\}$ be the set of nonnegative intervals. Let $\mathbb{C}^m$ and $\mathbb{C}^{m\times n}$ be the set of all m-dimensional column vectors and $m \times n$ matrices, called interval vectors and interval matrices respectively, whose elements are in $\mathbb{C}$, i.e.,

$$\mathbb{C}^m := \{\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_m)^t \mid \boldsymbol{a}_i \in \mathbb{C}\ (1 \leqq i \leqq m)\} \quad \text{and}$$
$$\mathbb{C}^{m\times n} := \{\boldsymbol{A} = (\boldsymbol{a}_{ij}) \mid (\boldsymbol{a}_{ij}) \in \mathbb{C}\ (1 \leqq i \leqq m, 1 \leqq j \leqq n)\}.$$

We shall identify $m \times 1$ interval matrices with interval vectors and 1×1 interval matrices with intervals, so that $\mathbb{C} = \mathbb{C}^{1\times 1}$ and $\mathbb{C}^m = \mathbb{C}^{m\times 1}$. Also, we denote by $\mathbb{C}^m_+$ and $\mathbb{C}^{m\times n}_+$ the subsets of componentwise non-negative elements in $\mathbb{C}^m$ and $\mathbb{C}^{m\times n}$. We equip $\mathbb{C}^{m\times n}$ with componentwise relations $\preccurlyeq$, $\prec$, $\succcurlyeq$, $\succ$.

Similarly, we can define $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$ as the set of real m-dimensional column vectors and real $m \times n$ matrices. Note that $\mathbb{R}^{m\times n} \subset \mathbb{C}^{m\times n}$.

For any $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{C}^{m\times n}$ with $\boldsymbol{a}_{ij} = [a^-_{ij}, a^+_{ij}]$, $\boldsymbol{A}$ will be denoted by $\boldsymbol{A} = [A^-, A^+]$, where $A^- = (a^-_{ij}) \in \mathbb{R}^{m\times n}, A^+ = (a^+_{ij}) \in \mathbb{R}^{m\times n}$ and $[A^-, A^+] = \{A \in \mathbb{R}^{m\times n} \mid A^- \preccurlyeq A \preccurlyeq A^+\}$.

The following arithmetics are used in Section 2.

For $\boldsymbol{A} = (\boldsymbol{a}_{ij}), \boldsymbol{B} = (\boldsymbol{b}_{ij}) \in \mathbb{C}^{m\times n}$ and $\lambda \in R_+$,

$$\boldsymbol{A} + \boldsymbol{B} = \{A + B \mid A \in \boldsymbol{A} \text{ and } B \in \boldsymbol{B}\} \tag{1.1}$$
$$\lambda \boldsymbol{A} = \{\lambda A \mid A \in \boldsymbol{A}\}, \tag{1.2}$$

where for $C = (c_{ij})$ and $D = (d_{ij}) \in \mathbb{R}^{m\times n}, C + D = (c_{ij} + d_{ij})$.

Observing $\boldsymbol{A} + \boldsymbol{B} = [A^- + B^-, A^+ + B^+] \in \mathbb{C}^{m\times n}$. For any given $\mathbb{D} \subset \mathbb{C}, \boldsymbol{c} \in \mathbb{D}$ is called a minimal(maximal) point of $\mathbb{D}$ if

$$\{\boldsymbol{d} \in \mathbb{D} \mid \boldsymbol{d} \prec (\succ) \boldsymbol{c}\} = \emptyset. \tag{1.3}$$

The set of all minimal(maximal) point of $\mathbb{D}$ will be defined by $\operatorname{Min} \mathbb{D}(\operatorname{Max} \mathbb{D})$(cf. [6, 11, 14]).

Since the partial order $\preccurlyeq$ on $\mathbb{C}$ is equivalent to the vector ordering on $\mathbb{R}^2$ with $\mathbb{R}^2_+$ as the corresponding order cone, the following fact follows easily(cf. [1, 11]).

Lemma 1.1. *Let $\mathbb{D}$ be a compact and convex subset of $\mathbb{C}$. Then $[a,b] \in \operatorname{Min} \mathbb{D}(\operatorname{Max} \mathbb{D})$ if and only if there exists $\beta \in (0,1)$ such that $\beta a + (1-\beta)b \leqq (\geqq) \beta c + (1-\beta)d$ for all $[c,d] \in \mathbb{D}$.*

In Section 2, an interval matrix game is specified and their saddle points are characterized as equilibrium points of the corresponding non-zero sum parametric game. In Section 3, a fuzzy matrix game is investigated.

2 Interval matrix games

The two person interval matrix game is defined by the $m\times n$ interval matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{C}^{m\times n}$, where player 1(maximizer) and player 2(minimizer) have m pure strategies $\{i \mid i = 1, 2, \dots, m\}$ and n pure strategies $\{j \mid j = 1, 2, \dots, n\}$ and if player 1 and 2 select $i(1 \leqq i \leqq n)$ and $j(1 \leqq j \leqq m)$ respectively, the payoff of player 1 is estimated to be in $\boldsymbol{a}_{ij} \in \mathbb{C}$.

Let X and Y be the set of all mixed strategies for player 1 and 2 respectively, i.e.,

$$X = \{x = (x_1, x_2, \dots, x_m)^t \in \mathbb{R}^m_+ \mid \sum_{j=1}^{m} x_i = 1\} \quad \text{and}$$

$$Y = \{y = (y_1, y_2, \dots, y_n)^t \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} y_i = 1\}.$$

Then, for any selected pair $(x, y) \in X \times Y$ the expected payoff for player 1 is estimated by

$$f(x, y) := x^t \boldsymbol{A} y, \tag{2.1}$$

where

$$x^t \boldsymbol{A} y = \sum x_i y_j \boldsymbol{a}_{ij}.$$

By arithmetics in (1.1) and (1.2), for any $[a, b], [c, d] \in \mathbb{C}$, $[a, b] + [c, d] = [a + c, b + d]$ and $\lambda[a, b] = [\lambda a, \lambda b](\lambda \geqq 0)$. So, the following holds obviously.

Lemma 2.1. *For any $x \in X$ and $y \in Y$, it holds that*

$$f(x, y) = [x^t A^- y, x^t A^+ y] \in \mathbb{C}. \tag{2.2}$$

Definition 1. (cf. [[6], [14]]) Let $(x^*, y^*) \in X \times Y$ and $\boldsymbol{A} \in \mathbb{C}^{m\times n}$. Then, (x^*, y^*) is said to be a *saddle point of the interval matrix game* $\boldsymbol{A}$ if the following holds:

$$f(x^*, y^*) \in \operatorname{Max} f(X, y^*) \cap \operatorname{Min} f(x^*, Y), \tag{2.3}$$

where for any $(x, y) \in X \times Y$, $f(X, y) = \{f(x', y) \mid x' \in X\}$ and $f(x, Y) = \{f(x, y') \mid y' \in Y\}$.

We note that $f(X, y)$ and $f(x, Y)$ are compact and convex subset of $\mathbb{C}$.

In order to characterize the saddle point of $\boldsymbol{A}$, we introduce a parametric equilibrium points.

For each $\beta \in (0, 1)$ and $\boldsymbol{A} = [A^-, A^+] \in \mathbb{C}^{m\times n}$, let $A(\beta) = \beta A^+ + (1 - \beta)A^-$.

Definition 2. For any $\beta, \beta' \in [0, 1]$, the point $(x^*, y^*) \in X \times Y$ is said to be a (β, β')-*equilibrium point* if the following (i)–(ii) holds:

(i) $x^{*t} A(\beta) y \geqq x^{*t} A(\beta) y^*$ for all $y \in Y$ and

(ii) $x^t A(\beta') y^* \leqq x^{*t} A(\beta') y^*$ for all $x \in X$.

We note that the (β,β)-equilibrium point (x^*,y^*) means that (x^*,y^*) is an optimal pair for the conventional matrix game $A(\beta)$, i.e.,

$$(2.4)\qquad x^tA(\beta)y^* \leqq x^{*t}A(\beta)y^* \leqq x^{*t}A(\beta)y \ \text{ for all } \ x\in X \ \text{ and } \ y\in Y.$$

Also, every non-zero sum finite game has an equilibrium point(cf. [7, 16]), so that for any $\beta,\beta'\in[0,1]$, a (β,β')-equilibrium point exists.

Applying Lemma 1.1 and 2.1, we have the following.

Theorem 2.1. *A point $(x^*,y^*)\in X\times Y$ is a saddle point for the interval matrix game $\boldsymbol{A}$ if and only if there exist $\beta,\beta'\in(0,1)$ such that (x^*,y^*) is a (β,β')-equilibrium point.*

Proof. By Lemma 1.1 and 2.1, that $f(x^*,y^*)\in \operatorname{Min} f(x^*,Y)$ means that there exists $\beta\in(0,1)$ satisfying

$$(2.5)\qquad \beta x^{*t}A^+y^* + (1-\beta)x^{*t}A^-y^* \leqq \beta x^{*t}A^+y + (1-\beta)x^{*t}A^-y \ \text{ for all } \ y\in Y.$$

Obviously, (2.5) is rewritten as follows.

$$(2.6)\qquad x^{*t}A(\beta)y^* \leqq x^{*t}A(\beta)y \ \text{ for all } \ y\in Y.$$

which is corresponding with (i) of Definition 2.

Similarly, $f(x^*,y^*)\in \operatorname{Max} f(X,y^*)$ means that there exists $\beta'\in(0,1)$ such that

$$(2.7)\qquad x^{*t}A(\beta')y^* \geqq x^tA(\beta)y^* \ \text{ for all } \ x\in X.$$

Thus, the proof is complete. ∎

The following results easily follow from Theorem 2.1.

Corollary 2.1. *If the point $(x^*,y^*)\in X\times Y$ is an optimal pair for the matrix game $A(\beta)$ $(\beta\in(0,1))$, (x^*,y^*) is a saddle point of the interval matrix game $\boldsymbol{A}$.*

Corollary 2.2. *For any $\boldsymbol{A}=([a_{ij}^-,a_{ij}^+])\in\mathbb{C}^{m\times n}$ with $a_{ij}^+-a_{ij}^-=c$ independent of i and j $(1\leqq i\leqq m, 1\leqq j\leqq n)$, the saddle point (x^*,y^*) of $\boldsymbol{A}$ is uniquely determined as a optimal pair for the matrix game $A^-=(a_{ij}^-)$.*

Proof. We note that $A(\beta)$ is rewritten as $A(\beta)=A^-+\beta(A^+-A^-)$. So that if $A^+-A^-=cE$, $A(\beta)$ and $A(\beta')$ is essentially equivalent for any $\beta,\beta'\in(0,1)$, where all the elements of $E\in\mathbb{R}^{m\times n}$ are 1. Thus, the statement of Corollary 2.2 follows obviously. ∎

The following is useful in finding the saddle point of the interval matrix game $\boldsymbol{A}$.

Corollary 2.3 (cf. [13]). *The point $(x^*,y^*)\in X\times Y$ is a saddle point for the interval matrix game $\boldsymbol{A}\in\mathbb{C}^{m\times n}$ if and only if there exist $\beta,\beta'\in(0,1)$ such that (x^*,y^*) is point of a solution to*

$$(2.8)\qquad \left\{\begin{array}{c} A(\beta)y+\lambda = x^tA(\beta)y1_m \\ x^tA(\beta')-\mu^t = x^tA(\beta')y1_n^t \\ \lambda^tx=0, \qquad \beta^ty=0 \\ x^t1_m=1, \qquad y^t1_n=1 \\ x,\lambda\in\mathbb{R}_+^m, \quad y,\mu\in\mathbb{R}_+^n, \end{array}\right.$$

where $1_m=(1,\dots,1)'\in\mathbb{R}_+^m$ and $1_n=(1,\dots,1)'\in\mathbb{R}_+^n$.

Remark. A $(1,0)$-equilibrium point $(x^*, y^*) \in X \times Y$ means that

(i) $x^{*t}A^+y \geqq x^{*t}A^+y^*$ for all $A \in \boldsymbol{A}$ and $y \in Y$ and

(ii) $x^tA^-y^* \leqq x^{*t}A^-y^*$ for all $A \in \boldsymbol{A}$ and $x \in X$.

This shows that (x^*, y^*) guarantees the best in the worst case. Thus, $(1,0)$-equilibrium point (x^*, y^*) will be called a pessimistic pair. By the same discussion as the above, the $(0,1)$-equilibrium point (x^*, y^*) will be called a utopian or optimistic pair.

3 Extensions to Fuzzy Games

In this section, the results in the preceding section will be extended to the multi-dimensional fuzzy payoff games.

We write a fuzzy set on $\mathbb{R}^p$ by its membership function $\widetilde{s} : \mathbb{R}^p \to [0,1]$ (see Novák [9] and Zadeh [17]). The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\widetilde{s}$ on $\mathbb{R}^p$ is defined as

$$\widetilde{s}_\alpha := \{x \in \mathbb{R}^p \mid \widetilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \widetilde{s}_0 := \mathrm{cl}\{x \in \mathbb{R}^p \mid \widetilde{s}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\widetilde{s}$ is called convex if

$$\widetilde{s}(\lambda x + (1-\lambda)y) \geq \widetilde{s}(x) \wedge \widetilde{s}(y) \quad x, y \in \mathbb{R}^p, \ \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$. Note that $\widetilde{s}$ is convex if and only if the α-cut $\widetilde{s}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^p)$ be the set of all convex fuzzy sets whose membership functions $\widetilde{s} : \mathbb{R}^p \to [0,1]$ are upper-semicontinuous and normal ($\sup_{x \in \mathbb{R}^p} \widetilde{s}(x) = 1$) and have a compact support. In the one-dimensional case $n = 1$, $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. Let $\mathbb{C}(\mathbb{R}^p)$ be the set of all compact convex subsets of $\mathbb{R}^p$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^p)$ are as follows: For $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^p)$ and $\lambda \geq 0$,

$$(\widetilde{s} + \widetilde{r})(x) := \sup_{\substack{x_1, x_2 \in \mathbb{R}^p \\ x_1 + x_2 = x}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\}, \tag{3.1}$$

$$(\lambda \widetilde{s})(x) := \begin{cases} \widetilde{s}(x/\lambda) & \text{if } \lambda > 0 \\ \mathbf{1}_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}^p), \tag{3.2}$$

where $\mathbf{1}_{\{\cdot\}}(\cdot)$ is an indicator.

By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any non-empty sets $A, B \subset \mathbb{R}^p$, the following holds immediately.

$$(\widetilde{s} + \widetilde{r})_\alpha = \widetilde{s}_\alpha + \widetilde{r}_\alpha \quad \text{and} \quad (\lambda \widetilde{s})_\alpha = \lambda \widetilde{s}_\alpha \quad (\alpha \in [0,1]). \tag{3.3}$$

Let K be a non-empty cone of $\mathbb{R}^p$. Using this K, we can define a pseudo order relation $\preccurlyeq_K$ on $\mathbb{R}^p$ by $x \preccurlyeq_K y$ if and only if $y - x \in K$. We introduce a pseudo order $\preccurlyeq_K$ on $\mathcal{F}(\mathbb{R}^p)$ (cf. [3]). Let $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^p)$. The relation $\widetilde{s} \preccurlyeq_K \widetilde{r}$ means the following (i)

(i) For any $x \in \mathbb{R}^p$, there exists $y \in \mathbb{R}^p$ such that $x \preccurlyeq_K y$ and $\widetilde{s} \leq \widetilde{r}$.

(ii) For any $y \in \mathbb{R}^p$, ther exists $x \in \mathbb{R}^p$ such that $x \preccurlyeq_K y$ and $\widetilde{s} \geq \widetilde{r}$.

For any $a \in \mathbb{R}^p$ and $\boldsymbol{d} \in \mathbb{C}(\mathbb{R}^p)$, the product of a and $\boldsymbol{d}$ is defined as

(3.4) $$a\boldsymbol{d} = \{a^t d \mid d \in \boldsymbol{d}\}.$$

We note that $a\boldsymbol{d} \in \mathbb{C}$.

Lemma 3.1. *[3] For any $\widetilde{s}, \widetilde{r} \in \mathcal{F}(\mathbb{R}^p)$, $\widetilde{s} \preccurlyeq_K \widetilde{r}$ if and only if $a\widetilde{s}_\alpha \leq a\widetilde{r}_\alpha$ for all $a \in K^+$ and $\alpha \in [0,1]$.*

Here, we consider the two person fuzzy matrix game defined by the $m \times n$ fuzzy matrix $\widetilde{\boldsymbol{A}} = (\widetilde{\boldsymbol{a}}_{ij}) \in \mathcal{F}(\mathbb{R}^p)^{m\times n}$. For any $x = (x_1, x_2, \dots, x_m)^t \in X$ and $y = (y_1, y_2, \dots, y_n)^t \in Y$, the expected payoff for player 1 is estimated (cf. [10]) by

(3.5) $$f(x,y) := x^t \widetilde{\boldsymbol{A}} y = \sum x_i y_j \widetilde{\boldsymbol{a}}_{ij}.$$

We note that $f(x,y) \in \mathcal{F}(\mathbb{R}^p)$ and its α-cut is given by

(3.6) $$f(x,y)_\alpha = \sum x_i y_j \widetilde{\boldsymbol{a}}_{ij,\alpha} \in \mathbb{C}(\mathbb{R}^p),$$

where $\widetilde{\boldsymbol{a}}_{ij,\alpha}$ is the α-cut of $\widetilde{\boldsymbol{a}}_{ij}$.

The saddle point of the fuzzy matrix game $\widetilde{\boldsymbol{A}}$ is defined similarly as that of the interval matrix game (see Definition 1 in Section 2).

For any $a \in \mathbb{R}^p$, noting $a\widetilde{\boldsymbol{a}}_{ij,\alpha} \in \mathbb{C}$, we denote $a\widetilde{\boldsymbol{a}}_{ij,\alpha}$ by $[\widetilde{a}^-_{ij,\alpha}(a), \widetilde{a}^+_{ij,\alpha}(a)]$ and set $A^-_\alpha(a) := (\widetilde{a}^-_{ij,\alpha}(a)) \in \mathbb{R}^{m\times n}$ and $A^+_\alpha(a) := (\widetilde{a}^+_{ij,\alpha}(a)) \in \mathbb{R}^{m\times n}$. Here, for $\alpha \in [0,1], \beta \in (0,1)$ and $a \in \mathbb{R}^p$, we put

(3.7) $$A_{\alpha,a}(\beta) = \beta A^+_\alpha(a) + (1-\beta) A^-_\alpha(a).$$

Then, the saddle points of the fuzzy matrix game $\widetilde{\boldsymbol{A}}$ will be characterized in the following theorem, whose proof is done by applying Lemma 3.1 and the ideas used in Section 2.

Theorem 3.1. *A point $(x^*, y^*) \in X \times Y$ is a saddle point of the fuzzy matrix game $\widetilde{\boldsymbol{A}}$ if and only if there exist two functions $\beta, \beta' : [0,1] \times K^+ \to (0,1)$ such that*

(3.8) $$x^{*t} A_{\alpha,a}(\beta(\alpha,a)) y \geqq x^{*t} A_{\alpha,a}(\beta(\alpha,a)) y^* \quad \textit{and}$$

(3.8) $$x^t A_{\alpha,a}(\beta'(\alpha,a)) y^* \leqq x^{*t} A_{\alpha,a}(\beta'(\alpha,a)) y^*$$

for all $\alpha \in [0,1]$ and $a \in K^+$.

4 Numerical Example

Here, we give numerical examples.

Example 1. Let $\boldsymbol{A} = \begin{pmatrix} [2,4], & [-2,0] \\ [0,2], & [1,3] \end{pmatrix} \in \mathbb{C}^{2\times 2}$.

Noting that $A^- = \begin{pmatrix} 2, & -2 \\ 0, & 1 \end{pmatrix}$ and $A^+ = \begin{pmatrix} 4, & 0 \\ 2, & 3 \end{pmatrix}$ and $A^+ - A^- = \begin{pmatrix} 2, & 2 \\ 2, & 2 \end{pmatrix}$. Thus, by Corollary 2.2, a saddle point (x^*, y^*) of $\boldsymbol{A}$ is unique and given by a optimal pair for A^-. After a simple calculation, we find that $x^* = \left(\frac{1}{5}, \frac{4}{5}\right)$, $y^* = \left(\frac{3}{5}, \frac{2}{5}\right)$ and $f(x^*, y^*) = \left[\frac{2}{5}, \frac{12}{5}\right]$.

Example 2. Let $\boldsymbol{A} = \begin{pmatrix} [3,4], & \left[-\frac{3}{2}, \frac{1}{2}\right] \\ \left[\frac{1}{2}, \frac{3}{2}\right], & [1,2] \end{pmatrix}$ with $A^- = \begin{pmatrix} 3, & -\frac{3}{2} \\ \frac{1}{2}, & 1 \end{pmatrix}$ and $A^+ = \begin{pmatrix} 4, & \frac{1}{2} \\ \frac{3}{2}, & 2 \end{pmatrix}$. Noting $\boldsymbol{A} = \begin{pmatrix} \beta + 3, & 2\beta - \frac{3}{2} \\ \beta + \frac{1}{2}, & \beta + 1 \end{pmatrix}$, for each $\beta \in (0,1)$, we solve the parametric equation (2.8) and find that the (β, β')-equilibrium point (x^*, y^*) is given by

$$x^* = \left(\frac{1}{10 - 2\beta'}, \frac{9 - 2\beta'}{10 - 2\beta'}\right), y^* = \left(\frac{5 - 2\beta}{1 - 2\beta}, \frac{5}{1 - 2\beta}\right) \text{ with}$$

$$f(x^*, y^*) = \left[\frac{2\beta\beta' - 15\beta' - 15\beta + 75}{(10 - 2\beta)(10 - 2\beta')}, \frac{6\beta\beta' - 35\beta - 35\beta' + 75}{(10 - 2\beta)(10 - 2\beta')}\right].$$

By Theorem 2.1, the set of all saddle points is specified by the set of all (β, β')-equilibrium points. Some saddle points and their values are given in Table 1.

	x^*	y^*	$f(x^*, y^*)$
$\beta = \frac{1}{3}\ \beta' = \frac{1}{3}$	$(\frac{3}{28}, \frac{25}{28})$	$(\frac{13}{28}, \frac{15}{28})$	$[\frac{587}{784}, \frac{177}{98}]$
$\beta = \frac{1}{3}\ \beta' = \frac{2}{3}$	$(\frac{3}{26}, \frac{23}{26})$	$(\frac{13}{28}, \frac{15}{28})$	$[\frac{68}{91}, \frac{1317}{728}]$
$\beta = \frac{2}{3}\ \beta' = \frac{1}{3}$	$(\frac{3}{28}, \frac{25}{28})$	$(\frac{11}{26}, \frac{15}{26})$	$[\frac{68}{91}, \frac{1317}{728}]$
$\beta = \frac{2}{3}\ \beta' = \frac{2}{3}$	$(\frac{3}{26}, \frac{23}{26})$	$(\frac{11}{26}, \frac{15}{26})$	$[\frac{503}{676}, \frac{306}{169}]$

Table 1. Saddle points and their values.

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