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A MATHEMATICAL APPROACH TO INTERMITTENCY (1),(2)

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1 Piecewise $C^0$-invertible Systems

Let $(T, X, Q = \{X_i\}_{i \in I})$ be a piecewise $C^0$-invertible system i.e., $X$ is a compact metric space with metric $d$, $T : X \to X$ is a noninvertible map which is not necessarily continuous, and $Q = \{X_i\}_{i \in I}$ is a countable disjoint partition $Q = \{X_i\}_{i \in I}$ of $X$ such that $\bigcup_{i \in I} \text{int} X_i$ is dense in $X$ and satisfy the following properties.

(01) For each $i \in I$ with $\text{int} X_i \neq \emptyset$, $T|_{\text{int} X_i} : \text{int} X_i \to T(\text{int} X_i)$ is a homeomorphism and $(T|_{\text{int} X_i})$ extends to a homeomorphism $v_i$ on $\overline{d(T(\text{int} X_i))}$.

(02) $T(\bigcup_{\text{int} X_i = \emptyset} X_i) \subset \bigcup_{\text{int} X_i = \emptyset} X_i$.

(03) $\{X_i\}_{i \in I}$ generates $\mathcal{F}$, the sigma algebra of Borel subsets of $X$.

Let $i = (i_1 \ldots i_n) \in I^n$ satisfy $\text{int}(X_{i_1} \cap T^{-1} X_{i_2} \cap \ldots T^{-(n-1)} X_{i_n}) \neq \emptyset$. Then we define $X_i := X_{i_1} \cap T^{-1} X_{i_2} \cap \ldots T^{-(n-1)} X_{i_n}$ which is called a cylinder of rank $n$ and write $|i| = n$. By (01), $T^n|_{\text{int} X_{i_1 \ldots i_n}} : \text{int} X_{i_1 \ldots i_n} \to T^n(\text{int} X_{i_1 \ldots i_n})$ is a homeomorphism and $(T^n|_{\text{int} X_{i_1 \ldots i_n}})^{-1}$ extends to a homeomorphism $v_{i_1} \circ v_{i_2} \circ \ldots \circ v_{i_n} = v_{i_1 \ldots i_n} : \overline{d(T^n(\text{int} X_i))} \to \overline{d(\text{int} X_i)}$.

We impose on $(T, X, Q)$ the next condition which gives a nice countable states symbolic dynamics similar to sofic shifts (cf. [11]):

Finite Range Structure $\mathcal{U} = \{\text{int}(T^n \cdot X_{i_1 \ldots i_n}) : \forall X_{i_1 \ldots i_n}, \forall n > 0\}$ consists of finitely many open subsets $U_1 \ldots U_N$ of $X$.

In particular, we say that $(T, X, Q)$ satisfies Bernoulli property if $\overline{d(T(\text{int} X_i))} = X (\forall i \in I)$ so that $\mathcal{U} = \{\text{int} X\}$ and that $(T, X, Q)$ satisfies Markov property if $\text{int}(\overline{d(T(\text{int} X_i))} \cap \text{int}(T(\text{int} X_j))) \neq \emptyset$ implies $\overline{d(T(\text{int} X_i))} \supset \overline{d(\text{int} X_j)}$. $(T, X, Q)$ satisfying Bernoulli (Markov) property is called a piecewise $C^0$-invertible Bernoulli (Markov) system respectively. We say that $X_i \in Q$ is a full cylinder if $\overline{d(T(\text{int} X_i))} = X$. We assume further the next condition:
(Transitivity) $\text{int} X = \bigcup_{k=1}^{N} U_k$ and $\forall l \in \{1, 2, \ldots N\}$, $\exists s_l < \infty$ such that for each $k \in \{1, 2, \ldots N\}$, $U_k$ contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank $s_l$ such that $T^{s_l}(\text{int} X^{(k,l)}(s_l)) = U_l$.

2 Topological pressure for potentials of weak bounded variation

Definition We say that $\phi$ is a potential of weak bounded variation (WBV) if there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n \to \infty} (1/n) \log C_n = 0$ and $\forall n \geq 1, \forall X_{i_1 \ldots i_n} \in \mathcal{V}_{j=0}^{n-1} T^{-j} Q$,

$$\sup_{x \in X_{i_1 \ldots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x)) / \inf_{x \in X_{i_1 \ldots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x)) \leq C_n.$$  

(C.f.[11,13,15-19])

We define a partition function for each $n > 0$ and for each $U_k \in \mathcal{U}$ as follows:

$$Z_n(U_k, \phi) := \sum_{i : |i| = n, \text{int} (TX_i) = U_k \supset \text{int} X_i} \sum_{v_{i_1} x = x \in \text{cl}(\text{int} X_i)} \exp(\sum_{h=0}^{n-1} \phi T^h(x)).$$

We further define:

$$Z_n(U_k, \phi) = \sum_{i : |i| = n, \text{int} (TX_i) = U_k \supset \text{int} X_i} \sup_{x \in \text{int} X_i} \exp(\sum_{h=0}^{n-1} \phi T^h(x))$$

and

$$Z_n(U_k, \phi) = \sum_{i : |i| = n, \text{int} (TX_i) = U_k \supset \text{int} X_i} \inf_{x \in \text{int} X_i} \exp(\sum_{h=0}^{n-1} \phi T^h(x)).$$

Lemma 2.1 ([17]) Let $(T, X, Q)$ be a piecewise $C^0$-invertible Markov system with finite range structure satisfying the transitivity. Let $\phi$ be a potential of WBV. For each $U_k \in \mathcal{U}$, $\lim_{n \to \infty} \frac{1}{n} \log Z_n(U_k, \phi)$, $\lim_{n \to \infty} \frac{1}{n} \log Z_n(U_k, \phi)$ exist and the limits does not depend on $k$. Furthermore,

$$P_{\text{top}}(T, \phi) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(X, \phi)$$

$$= \lim_{n \to \infty} \frac{1}{n} \log Z_n(U_k, \phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(U_k, \phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(U_k, \phi),$$

where

$$\log Z_n(X, \phi) := \sum_{i : |i| = n, \text{int} (TX_i) = \text{int} X_i} \sum_{v_{i_1} x = x \in \text{cl}(\text{int} X_i)} \exp(\sum_{h=0}^{n-1} \phi T^h(x)).$$

We define

$$\mathcal{W}_0(T) := \{ \phi : X \to \mathbb{R} | \phi \text{satisfies WBV and } P_{\text{top}}(T, \phi) < \infty \}.$$

Then we can easily see that the pressure function $P_{\text{top}}(T, .) : \mathcal{W}_0(T) \to \mathbb{R}$ satisfies continuity for bounded functions and convexity.
3 Weak Gibbs measures associated to potentials of WBV

Definition ([7],[11],[13],[15-19]) A Borel probability measure $\nu$ is called a weak Gibbs measure for a function $\phi$ with a constant $P$ if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that $\nu$-a.e.,

$$K_n^{-1} \leq \frac{\nu(X_{i_1\ldots i_n}(x))}{\exp(\sum_{i=0}^{n-1} \phi T^i(x) + nP)} \leq K_n,$$

where $X_{i_1\ldots i_n}(x)$ denotes the cylinder containing $x$.

Definition A Borel probability measure $\nu$ on $X$ is called an $f$-conformal measure if

$$d(\nu T)|_{X_i} = f|_{X_i}(\forall i \in I).$$

Lemma 3.1 ([17]) Let $(T, X, Q)$ be a piecewise $C^0$-invertible Markov system with FRS satisfying the transitivity and $\operatorname{int}X \in U$. Let $\phi \in \mathcal{W}_0(T)$ and $\nu$ be an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure. Then $\nu$ is a weak Gibbs measure for $\phi$ with $-P_{\text{top}}(T, \phi)$.

For $\phi : X \to \mathbb{R}$ we define the Ruelle-Perron-Frobenius operator $\mathcal{L}_\phi$ by

$$\mathcal{L}_\phi g(x) = \sum_{i \in I} \exp[\phi(v_i(x))]g(v_i(x)) (\forall g \in C(X), \forall x \in X).$$

Lemma 3.2 ([11],[13]) If there exist $p > 0$ and a Borel probability measure $\nu$ on $X$ satisfying $\mathcal{L}_\phi \nu = p\nu$, then $\nu$ is an $\exp[\log p - \phi]$-conformal measure and $p = \exp[P_{\text{top}}(T, \phi)]$.

4 Indifferent periodic points associated to potentials of WBV

Lemma 4.1 $P_{\text{top}}(T, \phi) \geq \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0)(\forall x_0 \in X, T^q x_0 = x_0)$.

Definition $x_0$ is called an indifferent periodic point with period $q$ with respect to $\phi$ if $P_{\text{top}}(T, \phi) = \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0)$. If there exists an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure $\nu$, then $x_0$ satisfies

$$\frac{d(\nu T^q)}{d\nu}|_{X_{i_1\ldots i_q}(x_0)} = \exp[qP_{\text{top}}(T, \phi) - \sum_{h=0}^{q-1} \phi T^h(x_0)] = 1.$$

If $x_0$ is not indifferent, then we call $x_0$ a repelling periodic point.

Proposition 4.1 ([16-17]) Let $x_0$ be an indifferent periodic point with period $q$ with respect to $\phi \in \mathcal{W}_0(T)$. Let $\nu$ be an $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure. Then

(i) $\forall s \geq 1, P_{\text{top}}(T, s\phi) = sP_{\text{top}}(T, \phi)$ and $\forall s < 1, P_{\text{top}}(T, s\phi) \geq sP_{\text{top}}(T, \phi)$.

(ii) $\nu(X_{i_1\ldots i_n}(x_0))$ decays subexponentially fast.
5 Jump transformations

Let $J$ be a subset of the index set $I$ and let $B_{1} = \bigcup_{i \in J} X_{i}$. Define $B_{1} := \{X_{i} \in Q : X_{i} \subseteq B_{1}\}$ and for each $n > 1$ $B_{n} := \{X_{i_{1} \ldots i_{n}} \in \sqrt{\cap_{i=1}^{n-1} T^{-i} Q : X_{i_{k}} \subseteq B_{1}^{*}(k = 1, \ldots, n-1), X_{i_{n}} \subseteq B_{1}\}$. Define a function $R : X \to \mathbb{N} \cup \{\infty\}$ by $R(x) = \inf\{n \geq 0 : T^{n}x \in B_{1}\} + 1$. Then we see that $B_{n} := \{x \in X | R(x) = n\} = \cup_{X_{i_{1} \ldots i_{n}} \in B_{n}} X_{i_{1} \ldots i_{n}}$ and $D_{n} := \{x \in X | R(x) > n\} = \cap_{i=0}^{n} T^{-i}B_{1}^{*}$. Now we define the jump transformation $T^{*} : \cap_{n=1}^{\infty} B_{n} \to X$ by $T^{*}x = T^{R(x)}x$. We denote $X^{*} := X \backslash (\cap_{i=0}^{\infty} T^{*-i}(\cap_{n=0} D_{n}))$ and $I^{*} := \cup_{n \geq 1}\{(i_{1} \ldots i_{n}) \in I^{n} : X_{i_{1} \ldots i_{n}} \subseteq B_{n}\}$. Then it is easy to see that $(T^{*}, X^{*}, Q^{*} = \{X_{i_{1} \ldots i_{n}}\}_{i_{1} \ldots i_{n} \in I^{*}})$ is a piecewise $C^{0}$-invertible Markov system with FRS and the property (1) $B_{n+1} = D_{n} \cap T^{-n}B_{1}$ is valid for $n \geq 1$. Let $\phi : X \to \mathbb{R}$ be a potential of WBV with $P_{\text{top}}(T, \phi) < \infty$. We assume further the next condition:

(Local Bounded Distortion) $\exists \theta > 0$ and $\forall X_{i_{1} \ldots i_{n}} \in B_{n}, \exists 0 < L_{\phi}(i_{1} \ldots i_{n}) < \infty$ such that
\[
|\phi v_{i_{1} \ldots i_{n}}(x) - \phi v_{i_{1} \ldots i_{n}}(y)| \leq L_{\phi}(i_{1} \ldots i_{n})d(x, y)^{\theta}
\]
and
\[
\sup_{n \geq 1} \sup_{X_{i_{1} \ldots i_{n}} \in B_{n}} \sum_{j=0}^{n-1} L_{\phi}(i_{j+1} \ldots i_{n}) < \infty.
\]

Define $\phi^{*} : \cup_{n=1}^{\infty} B_{n} \to \mathbb{R}$ by $\phi^{*}(x) = \sum_{i=0}^{R(x)-1} \phi T^{i}(x)$ and denote the local inverses to $T^{*}|_{X_{i}}$ by $v_{i}$. Then $\{\phi^{*}v_{i}\}$ is a family of equi-Hölder continuous functions and if $T^{*}$ satisfies the next property then $\phi^{*}$ satisfies summability of variation.

(Exponential Instability) $\sigma^{*}(n) := \sup_{i \in I^{*}, |i| = n} \text{diam} X_{i}$ decays exponentially fast as $n \to \infty$.

The summable variation allows one to show the existence of an unique equilibrium Gibbs state $\mu^{*}$ for $\phi^{*}$ under the existence of an exp$[P_{\text{top}}(T, \phi) - \phi]$ -conformal measure $\nu$ on $X$ with $\nu(\cap_{n=0} D_{n}) = 0$ and $\mu^{*} \sim \nu|_{X^{*}}$. The following formula gives a $T$-invariant $\sigma$-finite measure $\mu \sim \nu$.

(2) : $\mu(E) = \sum_{n=0}^{\infty} \mu^{*}(D_{n} \cap T^{-n}E)$.

If $\sum_{n=0}^{\infty} \nu(D_{n}) < \infty$, then $\mu$ is finite. In particular, $\mu(B_{1}) = \mu^{*}(X^{*}) > 0$, since $\nu(X^{*}) = 1$. If the reference measure $\nu$ is ergodic, then both $\mu, \mu^{*}$ are ergodic, too.

Theorem 5.1 (A construction of conformal measures) ([17]) Let $(T, X, Q)$ be a piecewise $C^{0}$-invertible Markov system with FRS satisfying transitivity. Let $T^{*}$ be the jump transformation associated to a union of full cylinders of rank 1 which satisfies exponential instability. Let $\phi : X \to \mathbb{R}$ be a potential of WBV satisfying (LBD), $P_{\text{top}}(T, \phi) < \infty$ and
\[ \|L_{\phi}^{-1}\| < \infty. \text{ Suppose either } P_{\text{top}}(T^*, \phi^*) \geq 0 \text{ or } \|L_{(\phi - P_{\text{top}}(T^*, \phi^*))^{-1}}\| < \infty. \text{ Then there exists a Borel probability measure } \nu \text{ on } X \text{ supported on } X^* \text{ satisfying} \]

\[ \frac{d\nu T}{d\nu} |_{x_i} = \exp[P_{\text{top}}(T, \phi) - \phi](\forall i \in I) \]

and \( \nu(\cup_{i \in I} \partial X_i) = 0. \)

We can associate the indifferent periodic points \( x_0 \) with respect to \( \phi \) to the Marginal sets \( \cap_{n \geq 0} D_n \).

**Proposition 5.1 ([17])**

(i) (Failure of bounded distortion)

\[ C_{nq}(x_0) := \sup_{x,y \in X_{1\cdots nq(x_0)}} \sup_{x,y \in X_{1\cdots nq(x_0)}} \frac{\exp[\sum_{i=0}^{nq-1} \phi T^i(x)]}{\exp[\sum_{i=0}^{nq-1} \phi T^i(y)]} \rightarrow \infty \]

monotonically as \( n \rightarrow \infty. \)

(ii) (Singularity of the invariant density) \( x_0 \in \cap_{n \geq 0} D_n \) and \( \frac{d\nu}{dx}(x_0) = \infty. \)

For a \( T \)-invariant probability measure \( m \) on \( (X, \mathcal{F}) \), \( I_m \) denotes the conditional information of \( Q \) with respect to \( T^{-1} \mathcal{F} \).

**Theorem 5.2 (Variational principle) ([17])** Let \( \nu \) be the \( \exp[P_{\text{top}}(T, \phi) - \phi] \)-conformal measure obtained under assumptions in Theorem 5.1. We assume further that \( \Gamma := \cap_{n \geq 0} D_n \) consists of periodic points. If \( \int_X R d\nu < \infty \) and \( H_{\nu}(Q^*) < \infty \), then there exists a \( T \)-invariant ergodic probability measure \( \mu \) equivalent to \( \nu \) which satisfies the following variational principle.

\[ P_{\text{top}}(T, \phi) = h_\mu(T) + \int_X \phi d\mu \geq h_m(T) + \int_X \phi dm \]

for all \( T \)-invariant ergodic probability measure \( m \) on \( X \) with \( I_m + \phi \in L^1(m) \) satisfying \( h_m(T) < \infty \) or \( \int_X \phi dm > -\infty. \)

**Corollary 5.1 (Phase transition)** We assume all conditions in Theorem 5.2. If \( \Gamma \) consists of indifferent periodic points with respect to \( \phi \), then the set of equilibrium states for \( \phi \) is the convex hull of \( \mu \) and the set of invariant Borel probability measures supported on \( \Gamma. \)
6 Slow decay of correlations

We denote $v'_{i_{1}...i_{n}}(x) = \frac{d(\mu v_{i_{1}.i_{n}})}{d\mu}(x)$ and let $P_{\mu} : L^{1}(\mu) \to L^{1}(\mu)$ be the normalized transfer operator with respect to $\mu$, i.e.,

$$P_{\mu}f(x) = \sum_{i \in I} v'_i(x)f(\psi_i(x))1_{TX_i}(x)(\forall f \in L^{1}(\mu)).$$

In this section, we shall establish bounds on the $L^1$-convergence of iterated transfer operators $\{P_{\mu}^{n}\}_{n \geq 1}$ and bounds on the decay of correlations relative to bounded functions $f$ satisfying a weak Lipschitz-type condition defined by:

(6-1) $\exists 0 < L_f < \infty$ such that

$$\sup_{X_{i(m)} \subset D_{i(m)}} \sup_{x,y \in X_{i(m)}} |f(x) - f(y)| \leq L_f \sigma(m) \ (\forall m > 0)$$

under the following conditions.

(6-2) $\Delta_1(k) := \sup_{n \geq 1} \sup_{i(n) \in A_n} \sup_{X_{j(k)} \subset D_{k}} \sup_{x,y \in X_{j(k)}} |1 - \frac{\psi_{i(n)}'(x)}{\psi_{i(n)}'(y)}| \to 0$ as $k \to \infty$.

(6-3) $\Delta_2(k) := \sup_{X_{j(k)} \subset D_{k}} \sup_{x,y \in X_{j(k)}} |1 - \frac{(d\mu/d\nu')(x)}{(d\mu/d\nu')(y)}| \to 0$ as $k \to \infty$.

Here $\sigma(m) := \sup_{i(m) \in A_m} diam X_{i(m)}$, $i(m)$ denotes a sequence $i_1 \ldots i_m$ of length $m$ and $D_{m}$ denotes $X \backslash D_{m}$.

Remark (1) If $d\mu^*/d\nu$ is H"{o}lder continuous (with exponent $\theta$), $\Delta_2(m)$ can be bounded from above by $O(\Delta_1(m)) + O(\sigma(m)^{\theta})$. For all examples, we can easily estimate both $\Delta_1(m)$ and $\sigma(m)$.

Remark (2) The condition (6-1) is milder than the usual Lipschitz condition. For example, for $S_{\beta}(x) = x + x^{1+\beta} \bmod 1$ $f(x) = x^{-\delta}$ for any $0 < \delta < \beta$ is a non-Lipschitz unbounded function satisfying (6-1).

We denote $\Delta(k) := \max_{i=1,2} \Delta_i(k)$.

Theorem 6.1 (Polynomial bounds) Let $(T, X, Q)$ be a piecewise $C^{0}$-invertible Bernoulli system and let $\nu$ and $\mu$ be the probability measures obtained in Theorems 5.1 and 5.2 respectively. Suppose that (6-2) and (6-3) are satisfied. Assume further that all $\mu(D_n)$, $\Delta(n)$ and $\sigma(n)$ decay polynomially fast. Then $\forall f \in L^{\infty}(\mu)$ satisfying (6-1) we have the following results.

1. (Rates of $L^1$-convergence of $\{P_{\mu}^{n}f\}_{n \geq 1}$) $\forall n \geq 1$ and $\forall 0 < \epsilon < 1$

$$||P_{\mu}^{n}f - \int_{X} f d\mu||_{1} \leq \max\{O(\mu(D_{[m]})), O(\Delta([n^\epsilon])), O(\sigma(2[n^\epsilon]))\}.$$
2. (Decay of correlations) \( \forall g \in L^\infty(\mu) \) and \( \forall 0 < \epsilon < 1 \)

\[ | \int_X f(gT^n)d\mu - \int_X f d\mu \int_X gd\mu | \leq \max\{O(\mu(D[n^{\epsilon}])), O(\Delta([n^{\epsilon}])), O(\sigma(2[n^{\epsilon}]))\}. \]

The next result gives sufficient conditions for (6-2).

**Lemma 6.1** Suppose that \( \{\phi\psi_i\}_{i \in I}, \{\phi^*\psi^*_i\}_{\underline{i} \in I^*} \) are equi-Hölder continuous with exponents \( \theta_1, \theta_2 \) respectively. Then \( \forall X_{i_1 \ldots i_m} \subset D^c_m \) and \( \forall (j_1 \ldots j_n) \in A_n \) such that \( X_{j_k} \subset B_1 \) and \( X_{j_{k+1} \ldots j_n} \subset D_{n-k} \) and \( \forall x, y \in X \) we have

\[
|1 - \frac{\psi_{j_1 \ldots j_n}(\psi_{i_1 \ldots i_m} x)}{\psi_{j_1 \ldots j_n}(\psi_{i_1 \ldots i_m} y)}| \\
\leq \max\{O(\sigma(m + n - k)^{\theta_1}), O(\sum_{i = [m/2]}^{\infty} \sup_{x \in X_{i_1 \ldots i_t} \subset B_i} \{diam X_{i_1 \ldots i_t}\}^{\theta_1}), O(\sigma(\lfloor m/2 \rfloor)^{\theta_1})\}. 
\]

**References**


