Efficiency of Set Optimization with Weighted Criteria

Nonlinear Analysis and Convex Analysis

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1 Introduction

In this paper, we consider efficiency of set-valued optimization problems with weighted criteria. Let \((E, \leq)\) be an ordered topological vector space, \(C\) the ordering cone in \((E, \leq)\), and assume that \(C\) is a closed set. Also \(C^+ = \{x^* \in E^* \mid \langle x^*, x \rangle \geq 0, \forall x \in C\}\) and we choose a weight set \(W\), a subset of \(C^+\). Let \(A\) be the family of all nonempty compact convex sets in \(E\), and \(B\) a nonempty subfamily of \(A\). Our purpose is to consider about minimal elements of \(B\) with weighted criteria.

In this paper, we introduce some concepts concerned with set-limit and cone-completeness, to characterize existence of such minimal elements. Also we consider completeness of some metric space including the whole space \(A\).

Definition 1.1 \(\emptyset \neq A, B \subset E\),

\[
A \leq_W B \iff \langle z^*, A + C \rangle \supset \langle z^*, B \rangle, \forall z^* \in W
\]

\[
A \leq_W^u B \iff \langle z^*, A \rangle \subset \overline{\langle z^*, B - C \rangle}, \forall z^* \in W
\]

Definition 1.2 (Minimal for a Family with Weight)

\(B_0\) is \((l,W)\)-minimal in \(B\) if \(B_0 \in B\) and condition \(B \leq_W B_0\) implies \(B_0 \leq_W B\).

\(B_0\) is \((u,W)\)-minimal in \(B\) if \(B_0 \in B\) and condition \(B \leq_W^u B_0\) implies \(B_0 \leq_W^u B\).

Similarly we can define \((l,W)\)-maximal and \((u,W)\)-maximal. In this paper we treat only the \((l,W)\)-minimal notion.
2 Characterization of Efficiency

Definition 2.1 ((l, W)-Decreasing, (l, W)-Complete, (l, W)-Section)
A net of sets \( \{A_\lambda\} \) in \( \mathcal{A} \) is said to be \((l, W)\)-decreasing if
\[
\lambda < \lambda' \implies A_{\lambda'} \leq^l_W A_{\lambda}
\]
A subfamily \( \mathcal{D} \subset \mathcal{A} \) is said to be \((l, W)\)-complete if there is no \((l, W)\)-decreasing net \( \{D_\lambda\} \) in \( \mathcal{D} \) such that
\[
\mathcal{D} \subset \{ A \in \mathcal{A} \mid \exists \lambda \text{ such that } A \not\leq^l_W D_\lambda \}
\]
Let \( A \in \mathcal{A} \) and \( \mathcal{D} \subset \mathcal{A} \). Then the family
\[
\mathcal{D}(A) = \{ D \in \mathcal{D} \mid D \leq^l_W A \}
\]
is called an \((l, W)\)-section in \( \mathcal{D} \)

Theorem 2.1 (Existence of \((l, W)\)-minimal sets)
\( \mathcal{B} \) has an \((l, W)\)-minimal set if and only if \( \mathcal{B} \) has a nonempty \((l, W)\)-complete section

Definition 2.2 (W-limit, W-set limit)
Let \( \{a_\lambda\}_{\Lambda} \) be a net of \( E \), \( x \in E \), then
\[
\lim_{\lambda}^W a_\lambda \ni x \iff \forall y^* \in W, \langle y^*, a_\lambda \rangle \to \langle y^*, x \rangle.
\]
the set \( \lim_{\lambda}^W a_\lambda \) is called \( W \)-limit of \( \{a_\lambda\} \) Also let \( \{A_\lambda\}_{\lambda \in \Lambda} \) be a net of \( \mathcal{A} \), \( x \in E \), then
\[
\liminf_{\lambda \in \Lambda}^W A_\lambda \ni x \iff \exists \{a_\lambda\} \text{ such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \text{ and } \lim_{\lambda}^W a_\lambda \ni x
\]
\[
\limsup_{\lambda \in \Lambda}^W A_\lambda \ni x \iff \exists \{a_\lambda'\} \subset \{a_\lambda\}: \text{a subnet such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \text{ and } \lim_{\lambda'}^W a'_\lambda \ni x
\]
these are called \( W \)-lower and \( W \)-upper limits, resp.

Definition 2.3 ((l, W) and (u, W)-Set limits)
\[
\liminf_{\lambda \in \Lambda}^l A_\lambda = \liminf_{\lambda \in \Lambda}^W (A_\lambda + C)
\]
\[
\liminf_{\lambda \in \Lambda}^u A_\lambda = \liminf_{\lambda \in \Lambda}^W (A_\lambda - C)
\]
\[
\limsup_{\lambda \in \Lambda}^l A_\lambda = \limsup_{\lambda \in \Lambda}^W (A_\lambda + C)
\]
\[
\limsup_{\lambda \in \Lambda}^u A_\lambda = \limsup_{\lambda \in \Lambda}^W (A_\lambda - C)
\]
Proposition 2.1 If $A_{\lambda}$ is $(l, W)$-decreasing then

$$A \leq_{W}^{l} A_{\lambda} \iff A \leq_{W}^{\iota\iota} \liminf_{\lambda \in \Lambda} A_{\lambda}$$

Theorem 2.2 The following are equivalent:

- $B$ has an $(l, W)$-minimal set
- $B$ has a nonempty $(l, W)$-complete section
- There exists $A_{0} \in A$ such that $B(A_{0}) = \{B \in B \mid B \leq_{W}^{l} A_{0}\}$ is $(l, W)$-complete
- For any $(l, W)$-decreasing net $\{B_{\lambda}\}$ in $B$, there exists $A_{0} \in A$ such that $A_{0} \leq_{W}^{l} \liminf_{\lambda \in \Lambda}^{l} B_{\lambda}$

Corollary 2.1 Let $F$ be a set-valued map from a subset $X$ of a topological space into $E$. If $X$ is compact and

$$x_{\lambda} \to x_{0}, \{F(x_{\lambda})\} : (l, W)\text{-decreasing} \implies F(x_{0}) \leq_{W}^{l} \liminf_{\lambda \in \Lambda}^{l} F(x_{\lambda})$$

then there is an $(l, W)$-minimal set in $\{F(x) \mid x \in X\}$.

3 Completeness

In this section, we consider about completeness of metric space $(A/\equiv_{W}^{l}, d)$. At first we define a quotient space $A/\equiv_{W}^{l}$ as follows:

$$A/\equiv_{W}^{l} = \{[A] \mid A \in A\},$$

where $[A] = \{B \in A \mid A \equiv_{W}^{l} B\}$ for each $A \in A$. In this space, we define an order relation. For $[A], [B] \in A/\equiv_{W}^{l}$,

$$[A] \leq_{W}^{l} [B] \overset{\text{def}}{=} A \leq_{W}^{l} B$$

Then $\leq_{W}^{l}$ is an order relation on $A/\equiv_{W}^{l}$. Next, we define a metric on the space. For $[A], [B] \in A/\equiv_{W}^{l}$,

$$d([A], [B]) = \sup_{y^{*} \in W} |\min\langle y^{*}, A\rangle - \min\langle y^{*}, B\rangle|$$

Then $d$ is a metric on $A/\equiv_{W}^{l}$.

Now we have a question. Is $d$ complete?

Counterexample 3.1 $E = \mathbb{R}^{2}, C = \mathbb{R}_{+}^{2}, W = [(1, 0), (0, 1)], A_{n} = \{(x_{1}, x_{2}) \in E \mid 0 \leq x_{1}, x_{2} \leq n, 1 \leq x_{1}x_{2}\}$. Then $\{[A_{n}]\}$ is a Cauchy sequence in $A/\equiv_{W}^{l}$, but $\{[A_{n}]\}$ does not converges to any elements of $A/\equiv_{W}^{l}$. (For example, $A_{0} = \{(x_{1}, x_{2}) \in E \mid 0 \leq x_{1}, x_{2}, 1 \leq x_{1}x_{2}\}, d(A_{n}, A_{0}) \to 0$ as $n \to \infty$)
How conditions assure the completeness? Concerning the question, we have the following two theorems.

**Theorem 3.1** \{\{A_n\}\} is a Cauchy sequence in \(\mathcal{A}/\equiv_{W}\), and there exists a compact subset \(K\) of \(E\) such that \(A_n \subset K\) for each \(n\).

**Proof.** Let \(\mu_{A_n} : W \to \mathbb{R}\) defined by

\[
\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \quad y^* \in W
\]

then there exists a continuous function \(\mu_0 : W \to \mathbb{R}\) such that \(\mu_{A_n}\) converges to \(\mu_0\) uniformly on \(W\). For \(y^* \in W\), there exists \(a_{y^*} \in K\) such that \(\mu_0(y^*) = \langle y^*, a_{y^*} \rangle\). Let \(A_0 := \{a_{y^*} \mid y^* \in W\}\), then \(\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \text{co} A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\text{co}} A_0} \langle y^*, a \rangle\).

Also we have \(\text{co} A_0 \in \mathcal{A}\), and then we conclude the proof. \(\square\)

**Theorem 3.2** \{\{A_n\}\} is a Cauchy sequence in \(\mathcal{A}/\equiv_{W}\), and there exists a compact subset \(K\) of \(E\) and a sequence \(\{x_n\} \subset E\) such that \(x_n + A_n \subset K\) for each \(n\). Assume that \(C^+ - C^+ = E^*\) and \(E\) is reflexive, then \(\{\{A_n\}\}\) converges some element of \(\mathcal{A}\).

**Proof.** Let \(\mu_{A_n} : W \to \mathbb{R}\) defined by

\[
\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \quad y^* \in W
\]

then there exists a continuous function \(\mu_0 : W \to \mathbb{R}\) such that \(\mu_{A_n}\) converges to \(\mu_0\) uniformly on \(W\). From condition \(x_n + A_n \subset K\), there exists \(M\) such that \(|\langle y^*, x_n \rangle| \leq M\) for each \(y^* \in W\) and \(n\), and by assumption \(C^+ - C^+ = E^*\), we have \(|\langle y^*, x_n \rangle| \leq M\) for each \(y^* \in E^*\) and \(n\). Using uniform boundedness theorem, we have \(\|x_n\| \leq M\) for each \(n\). Then we can choose a subsequence \(\{x_{n'}\}\) and \(x_0 \in E\) such that \(\{x_{n'}\}\) converges to \(x_0\) weakly.

For \(y^* \in W\), there exists \(a_{y^*} \in K\) such that \(\langle y^*, x_0 \rangle + \mu_0(y^*) = \langle y^*, a_{y^*} \rangle\). Let \(A_0 := \{a_{y^*} - x_0 \mid y^* \in W\}\), then \(\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \text{co} A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\text{co}} A_0} \langle y^*, a \rangle\) for each \(y^* \in W\). Also we have \(\overline{\text{co}} A_0 \in \mathcal{A}\), then we complete the proof. \(\square\)

**References**

