Efficiency of Set Optimization with Weighted Criteria

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1 Introduction

In this paper, we consider efficiency of set-valued optimization problems with weighted criteria. Let $(E, \leq)$ be an ordered topological vector space, $C$ the ordering cone in $(E, \leq)$, and assume that $C$ is a closed set. Also $C^+ = \{ x^* \in E^* \mid \langle x^*, x \rangle \geq 0, \forall x \in C \}$ and we choose a weight set $W$, a subset of $C^+$. Let $\mathcal{A}$ be the family of all nonempty compact convex sets in $E$, and $\mathcal{B}$ a nonempty subfamily of $\mathcal{A}$. Our purpose is to consider about minimal elements of $\mathcal{B}$ with weighted criteria.

In this paper, we introduce some concepts concerned with set-limit and cone-completeness, to characterize existence of such minimal elements. Also we consider completeness of some metric space including the whole space $\mathcal{A}$.

**Definition 1.1** $\emptyset \neq A, B \subset E,$

$$A \leq_B W B \iff \langle z^*, A + C \rangle \supset \langle z^*, B \rangle, \forall z^* \in W$$

$$A \leq_U W B \iff \langle z^*, A \rangle \subset \langle z^*, B - C \rangle, \forall z^* \in W$$

**Definition 1.2** (Minimal for a Family with Weight)

$B_0$ is $(l, W)$-minimal in $\mathcal{B}$ if $B_0 \in \mathcal{B}$ and condition $B \leq_B W B_0$ implies $B_0 \leq_B W B$.

$B_0$ is $(u, W)$-minimal in $\mathcal{B}$ if $B_0 \in \mathcal{B}$ and condition $B \leq_U W B_0$ implies $B_0 \leq_U W B$.

Similarly we can define $(l, W)$-maximal and $(u, W)$-maximal. In this paper we treat only the $(l, W)$-minimal notion.
2 Characterization of Efficiency

Definition 2.1 \((l, W)\)-Decreasing, \((l, W)\)-Complete, \((l, W)\)-Section
A net of sets \(\{A_\lambda\}\) in \(\mathcal{A}\) is said to be \((l, W)\)-decreasing if
\[ \lambda < \lambda' \implies A_{\lambda'} \leq_W A_\lambda \]
A subfamily \(D \subset \mathcal{A}\) is said to be \((l, W)\)-complete if there is no \((l, W)\)-decreasing net \(\{D_\lambda\}\) in \(D\) such that
\[ D \subset \{A \in \mathcal{A} | \exists \lambda \text{ such that } A \not\leq_W D_\lambda\} \]
Let \(A \in \mathcal{A}\) and \(D \subset \mathcal{A}\). Then the family
\[ \mathcal{D}(A) = \{D \in D | D \leq_W A\} \]
is called an \((l, W)\)-section in \(\mathcal{D}\)

Theorem 2.1 (Existence of \((l, W)\)-minimal sets)
\(B\) has an \((l, W)\)-minimal set if and only if \(B\) has a nonempty \((l, W)\)-complete section

Definition 2.2 \((W\)-limit, \(W\)-set limit\)
Let \(\{a_\lambda\}_{\Lambda}\) be a net of \(E, x \in E\), then
\[ \lim_W a_\lambda \ni x \iff \forall y^* \in W, \langle y^*, a_\lambda \rangle \to \langle y^*, x \rangle. \]
the set \(\lim_W a_\lambda\) is called \(W\)-limit of \(\{a_\lambda\}\) Also let \(\{A_\lambda\}_{\lambda \in \Lambda}\) be a net of \(\mathcal{A}, x \in E\), then
\[ \operatorname{Llim}_W A_\lambda \ni x \iff \exists \{a_\lambda\}\text{ such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \text{ and } \lim_W a_\lambda \ni x \]
\[ \operatorname{Llim}^u_W A_\lambda \ni x \iff \exists \{a'_\lambda\} \subset \{a_\lambda\}: \text{a subnet such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \]
and \(\lim_W a'_\lambda \ni x\)
these are called \(W\)-lower and \(W\)-upper limits, resp.

Definition 2.3 \((l, W)\) and \((u, W)\)-Set limits
\[ \operatorname{Llim}_W^l A_\lambda = \operatorname{Llim}_W (A_\lambda + C) \]
\[ \operatorname{Llim}_W^u A_\lambda = \operatorname{Llim}_W (A_\lambda - C) \]
\[ \operatorname{Llim}^l_W A_\lambda = \operatorname{Llim}_W (A_\lambda + C) \]
\[ \operatorname{Llim}^u_W A_\lambda = \operatorname{Llim}_W (A_\lambda - C) \]
Proposition 2.1 If $A_\lambda$ is $(l,W)$-decreasing then
\[ A \leq_W A_\lambda \iff A \leq_{W}^{l} \text{Lim inf}_{\lambda \in \Lambda} A_{\lambda} \]

Theorem 2.2 The following are equivalent:

- $B$ has an $(l,W)$-minimal set
- $B$ has a nonempty $(l,W)$-complete section
- There exists $A_0 \in A$ such that $B(A_0) = \{B \in B \mid B \leq_W A_0\}$ is $(l,W)$-complete
- For any $(l,W)$-decreasing net $\{B_\lambda\}$ in $B$, there exists $A_0 \in A$ such that $A_0 \leq_W \text{Lim inf}_{W\lambda\in\Lambda} B_\lambda$

Corollary 2.1 Let $F$ be a set-valued map from a subset $X$ of a topological space into $E$. If $X$ is compact and
\[ x_\lambda \to x_0, \{F(x_\lambda)\} : (l,W)\text{-decreasing} \]
\[ \implies F(x_0) \leq_W \text{Lim inf}_{W\lambda\in\Lambda} F(x_\lambda) \]
then there is an $(l,W)$-minimal set in $\{F(x) \mid x \in X\}$.

3 Completeness

In this section, we consider about completeness of metric space $(\mathcal{A}/\equiv_W, d)$. At first we define a quotient space $\mathcal{A}/\equiv_W$ as follows:
\[ \mathcal{A}/\equiv_W = \{[A] \mid A \in \mathcal{A}\}, \]
where $[A] = \{B \in \mathcal{A} \mid A \equiv_W B\}$ for each $A \in \mathcal{A}$. In this space, we define an order relation. For $[A], [B] \in \mathcal{A}/\equiv_W$,
\[ [A] \leq_W [B] \iff A \leq_W B \]
Then $\leq_W$ is an order relation on $\mathcal{A}/\equiv_W$. Next, we define a metric on the space. For $[A], [B] \in \mathcal{A}/\equiv_W$,
\[ d([A], [B]) = \sup_{y^* \in W} |\min\langle y^*, A\rangle - \min\langle y^*, B\rangle| \]
Then $d$ is a metric on $\mathcal{A}/\equiv_W$.

Now we have a question. Is $d$ complete?

Counterexample 3.1 $E = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $W = [(1,0), (0,1)]$, $A_n = \{(x_1, x_2) \in E \mid 0 \leq x_1, x_2 \leq n, 1 \leq x_1x_2\}$. Then $\{[A_n]\}$ is a Cauchy sequence in $\mathcal{A}/\equiv_W$, but $\{[A_n]\}$ does not converges to any elements of $\mathcal{A}/\equiv_W$. (For example, $A_0 = \{(x_1, x_2) \in E \mid 0 \leq x_1, x_2, 1 \leq x_1x_2\}$, $d(A_n, A_0) \to 0$ as $n \to \infty$)
How conditions assure the completeness? Concerning the question, we have the following two theorems.

**Theorem 3.1** \{[A_n]\} is a Cauchy sequence in $A/\equiv_W$, and there exists a compact subset $K$ of $E$ such that $A_n \subset K$ for each $n$.

**Proof.** Let $\mu_{A_n} : W \to \mathbb{R}$ defined by

$$\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \quad y^* \in W$$

then there exists a continuous function $\mu_0 : W \to \mathbb{R}$ such that $\mu_{A_n}$ converges to $\mu_0$ uniformly on $W$. For $y^* \in W$, there exists $a_{y^*} \in K$ such that $\mu_0(y^*) = \langle y^*, a_{y^*} \rangle$. Let $A_0 := \{a_{y^*} \mid y^* \in W\}$, then $\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \text{co}A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\text{co}}A_0} \langle y^*, a \rangle$. Also we have $\overline{\text{co}}A_0 \subset A$, and then we conclude the proof.

**Theorem 3.2** \{[A_n]\} is a Cauchy sequence in $A/\equiv_W$, and there exists a compact subset $K$ of $E$ and a sequence $\{x_n\} \subset E$ such that $x_n + A_n \subset K$ for each $n$. Assume that $C^+ - C^+ = E^*$ and $E$ is reflexive, then $\{[A_n]\}$ converges some element of $A$.

**Proof.** Let $\mu_{A_n} : W \to \mathbb{R}$ defined by

$$\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \quad y^* \in W$$

then there exists a continuous function $\mu_0 : W \to \mathbb{R}$ such that $\mu_{A_n}$ converges to $\mu_0$ uniformly on $W$. From condition $x_n + A_n \subset K$, there exists $M$ such that $|\langle y^*, x_n \rangle| \leq M$ for each $y^* \in W$ and $n$, and by assumption $C^+ - C^+ = E^*$, we have $|\langle y^*, x_n \rangle| \leq M$ for each $y^* \in E^*$ and $n$. Using uniform boundedness theorem, we have $\|x_n\| \leq M$ for each $n$. Then we can choose a subsequence $\{x_{n'}\}$ and $x_0 \in E$ such that $\{x_{n'}\}$ converges to $x_0$ weakly.

For $y^* \in W$, there exists $a_{y^*} \in K$ such that $\langle y^*, x_0 \rangle + \mu_0(y^*) = \langle y^*, a_{y^*} \rangle$. Let $A_0 := \{a_{y^*} - x_0 \mid y^* \in W\}$, then $\mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \text{co}A_0} \langle y^*, a \rangle = \inf_{a \in \overline{\text{co}}A_0} \langle y^*, a \rangle$ for each $y^* \in W$. Also we have $\overline{\text{co}}A_0 \subset A$, then we complete the proof.

**References**

