<table>
<thead>
<tr>
<th>Title</th>
<th>Inverse Problems from Economics and Game Theory (Nonlinear Analysis and Convex Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Komiya, Hidetoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2002), 1246: 45-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41711">http://hdl.handle.net/2433/41711</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Inverse Problems from Economics and Game Theory

Hidetoshi Komiya (小宮 英敏)
Faculty of Business and Commerce
Keio University (慶応大学商学部)

1 Introduction

We discuss open problems concerning inverses of theorems appearing in economics and game theory. We often finds the following Berge maximum theorem under convexity as a mathematical tools for optimal control problems in economics and game theory:

**Theorem 1** [Berge] Let $X$ be a subset of $l$-dimensional Euclidean space $R^l$ and let $Y$ be a subset of $m$-dimensional Euclidean space $R^m$. Let $u : X \times Y \to R$ be continuous and quasi-concave in its second variable, let $S : X \to Y$ be continuous and nonempty compact and convex-valued. Then, the correspondence $K : X \to Y$ defined by

$$K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}, \quad x \in X$$

is upper semicontinuous and compact and convex-valued.

It is known that inverses of Theorem 1 hold (cf. [3], [5]) and we shall treat a related open inverse problem in Section 2.

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $u : \Omega \times R^l_+ \to R_+$ a function with appropriate properties and $e \in L_1(\Omega, R^l_+)$. Then, for each $S \in \mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) \, d\mu(w) : x \in L_1(S, R^l_+), \int_S x \, d\mu = \int_S e \, d\mu \right\}.$$  

The map $v$ on $\mathcal{F}$ is called a market game. It is known that a market game is totally balanced and inner continuous at any $S \in \mathcal{F}$. (cf. [4]) We shall treat an open inverse problems concerning market games in Section 3.
2 Berge maximum theorem

In [3], the following inverse problem of Theorem 1 is considered:

Let $X$ be a subset of $\mathbb{R}^l$ and let $Y$ be a convex subset $\mathbb{R}^m$. Let $K : X \to Y$ be a nonempty compact convex-valued upper semi-continuous correspondence and let $S : X \to Y$ be a compact convex-valued continuous correspondence such that $K(x) \subset S(x)$ for $x \in X$. Then does there exist a continuous function $u : X \times Y \to \mathbb{R}$ such that

(i) $K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}$ for $x \in X$;

(ii) $u(x, y)$ is quasi-concave in $y$ for $x \in X$?

and is obtained the following result:

**Theorem 2** Let $X$ be a subset of $\mathbb{R}^l$. Let $K : X \to \mathbb{R}^m$ be a nonempty compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $v : X \times \mathbb{R}^m \to [0, 1]$ such that

(i) $K(x) = \{y \in \mathbb{R}^m : v(x, y) = \max_{z \in \mathbb{R}^m} v(x, z)\}$ for any $x \in X$;

(ii) $v(x, y)$ is quasi-concave in $y$ for any $x \in X$.

It is tried to generalize Theorem 2 to infinite dimensional case and a result is obtained in [5].

For a topological space $X$ and a subset $Y$ of a topological vector space, a correspondence $K : X \to Y$ is said to be $\sigma$-selectionable if there exists a sequence $\{K_n\}$ of continuous correspondences $K_n : X \to Y$ with compact convex values such that

(a) $K_{n+1}(x) \subset K_n(x)$ for any $x \in X$ and any $n \in \mathbb{N}$; and

(b) $K(x) = \bigcap_n K_n(x)$ for any $x \in X$.

It is known that an upper semicontinuous correspondence $K : X \to \mathbb{R}^m$, $X \subset \mathbb{R}^l$, with compact convex values is $\sigma$-selectionable and hence the following theorem obtained in [5] is a generalization of Theorem 2.
Theorem 3 Let $X$ be a topological space, and $Y$ a metric t.v.s. whose balls are convex, and $K : X \rightarrow Y$ a $\sigma$-selectionable map. Then there exists a continuous function $u : X \times Y \rightarrow [0, 1]$ such that

(i) $K(x) = \{ y \in Y : u(x, y) = \max_{z \in Y} f(x, z) \}$ for any $x \in X$; and

(ii) $u(x, y)$ is quasi-concave in $y$ for any $x \in X$.

It is not known that the assumption of $\sigma$-selectionability of the correspondence $K$ can be removed or not even in the case that $X$ and $Y$ are subsets of Banach spaces. Thus we have a conjecture:

Conjecture 1 Let $X$ be a subset of a Banach space, $Y$ a Banach space and $K : X \rightarrow Y$ a compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $u : X \times Y \rightarrow [0, 1]$ such that

(i) $K(x) = \{ y \in Y : u(x, y) = \max_{z \in Y} f(x, z) \}$ for any $x \in X$; and

(ii) $u(x, y)$ is quasi-concave in $y$ for any $x \in X$.

3 Market games

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. A game $v$ is a nonnegative real valued function, defined on the $\sigma$-field $\mathcal{F}$, which maps the empty set to zero. An outcome of a game $v$ is a finitely additive real valued function $\alpha$ on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome $\alpha$ of $v$, an integrable function $f$ satisfying $\int_S f \, d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an outcome density of $\alpha$ with respect to $\mu$. An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The core of $v$ is the set of outcomes $\alpha$ satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game $v$ we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_{\Omega} \right\},$$

where $n = 1, 2, \ldots$, $S_i \in \mathcal{F}$, $\lambda_i$ is a real number. The notation $\chi_A$ denotes the characteristic function of a subset $A$ of $\Omega$. For a game $v$ with $|v| < \infty$, ...
we define two games $\bar{v}$ and $\hat{v}$ by

$$\bar{v}(S) = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S \right\}, \quad S \in \mathcal{F}, \quad (4)$$

$$\hat{v}(S) = \min \left\{ \alpha(S) : \alpha \text{ is additive, } \alpha \geq v, \alpha(\Omega) = |v| \right\}, \quad S \in \mathcal{F}, \quad (5)$$

following [6]. A game $v$ is said to be balanced if $v(\Omega) = |v|$, totally balanced if $v = \bar{v}$ and exact if $v = \hat{v}$, respectively. It is proved in [6] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game $v$ is said to be monotone if $S \subset T$ implies $v(S) \leq v(T)$ for any $S$ and $T$ in $\mathcal{F}$. A game $v$ is said to be inner continuous at $S$ in $\mathcal{F}$ if it follows that $\lim_{n \to \infty} v(S_n) = v(S)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = S$. Similarly, a game $v$ is said to be outer continuous at $S$ in $\mathcal{F}$ if it follows that $\lim_{n \to \infty} v(S_n) = v(S)$ for any nonincreasing sequence $\{S_n\}$ of measurable sets such that $\bigcap_{n=1}^{\infty} S_n = S$. A game $v$ is continuous at $S$ in $\mathcal{F}$ if it is both inner and outer continuous at $S$.

We denote utilities of players by a Carathéodory type function $u$ defined on $\Omega \times R^l_+$ to $R_+$, where $R^l_+$ denotes the nonnegative orthant of the $l$-dimensional Euclidean space $R^l$, and $R_+$ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player $\omega$ getting goods $x$. We always use the ordinary coordinatewise order when having concern with an order in $R^l_+$. We suppose that the function $u : \Omega \times R^l_+ \to R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R^l_+$;

2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all $\omega$ in $\Omega$;

3. $\sigma \equiv \sup \{u(\omega, x) : (\omega, x) \in \Omega \times B_+ \} < \infty$, where $B_+ = \{x \in R^l_+ : \|x\| \leq 1\}$, and $\|x\|$ denotes the Euclidean norm of $x \in R^l_+$.

For any set $S$ in $\mathcal{F}$, the set of integrable functions on $S$ to $R^l_+$ is denoted by $L_1(S, R^l_+)$. We take an element $e$ of $L_1(\Omega, R^l_+)$ as the density of initial endowments for the players. For any $S$ in $\mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) \, d\mu(\omega) : x \in L_1(S, R^l_+), \int_S x \, d\mu = \int_S e \, d\mu \right\}.$$

(6)
The set function \( v \) defined above is called a market game derived from the market \( (\Omega, \mathcal{F}, \mu, u, e) \).

The following theorem is proved in [4]:

**Theorem 4** The market game defined above is totally balanced and inner continuous at every \( S \in \mathcal{F} \).

Every exact game which is continuous at \( \Omega \), equivalently inner continuous at \( \Omega \), is continuous at every \( S \) in \( \mathcal{F} \) according to [6], but it is known that a market game is not necessarily continuous at every \( S \in \mathcal{F} \). Thus we are interested in the following conjecture as an inverse problem of Theorem 4 to understand the difference between totally balanced games and exact games.

**Conjecture 2** A totally balanced game that is inner continuous at any \( S \) in \( \mathcal{F} \) is a market game, that is, a game derived from a market.

**References**


