数理解析研究所講究録 1246 巻 2002 年 56-65

UNIFORM TIGHTNESS FOR TRANSITION PROBABILITIES ON NUCLEAR SPACES

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Abstract. The aim of this paper is to give a notion of uniform tightness for transition probabilities on topological spaces, which assures the uniform tightness of compound probability measures. Then the upper semicontinuity of set-valued mappings are used in essence. As an important example, the uniform tightness for Gaussian transition probabilities on the strong dual of a nuclear real Fréchet space is studied. It is also shown that some of our results contain well-known results concerning the uniform tightness and the weak convergence of probability measures.

1. Introduction

Let X and Y be topological spaces. In this paper, we present a notion of uniform tightness for transition probabilities on $X \times Y$ which assures the uniform tightness for compound probability measures $\mu \circ \lambda$ defined by

$$\mu \circ \lambda(D) = \int_X \lambda(x, D_x)\mu(dx)$$

for a measure μ on X and a transition probability λ on $X \times Y$. We may consider that the compound probability measure is a generalization of the product measure or the convolution measure, and have to notice that the weak convergence of convolution measures has been looked into in great details by Csiszár [2, 3] and Kallianpur [6]. In Section 2 we recall notation and necessary definitions and results concerning probability measures on topological spaces, and then give a necessary and sufficient condition for a probability measure-valued mapping to be a transition probability in terms of the measurability of its characteristic functional.

In Section 3 we present a notion of uniform tightness for transition probabilities, using the upper semicontinuity of set-valued mappings, so that the corresponding set of compound probability measures is uniformly tight. We also give a sufficient condition for the weak convergence of a net of compound probability measures.

2000 Mathematics Subject Classification: Primary 28A33, 28C15; Secondary 60B05, 60B10.

Key words and phrases: transition probability, compound probability measure, upper semicontinuous, set-valued mapping, uniform tightness, Gaussian, nuclear space.

Research supported by Grant-in-Aid for General Scientific Research No. 13640162, Ministry of Education, Culture, Sports, Science and Technology, Japan.

In Section 4 we study Gaussian transition probabilities on the strong dual of a nuclear real Fréchet space as an important example of the uniform tightness for transition probabilities. We also show that some of the results in this section contain well-known results concerning the uniform tightness and the weak convergence of probability measures.

Throughout this paper, we suppose that all the topological spaces and all the topological linear spaces are Hausdorff.

2. Preliminaries

Let $(X, \mathcal{A})$ be a measurable space and Y a topological space. We denote by $\mathcal{B}(Y)$ the σ-algebra of all Borel subsets of Y. By a *Borel measure* on Y we mean a finite measure defined on $\mathcal{B}(Y)$ and we denote by $\mathcal{P}(Y)$ the set of all Borel probability measures on Y.

If Y is completely regular, we equip $\mathcal{P}(Y)$ with the weakest topology for which the functionals

$$\nu \in \mathcal{P}(Y) \mapsto \int_Y g(y)\nu(dy), \quad g \in C_b(Y),$$

are continuous. Here $C_b(Y)$ denotes the set of all bounded continuous real-valued functions on Y. This topology on $\mathcal{P}(Y)$ is called the *weak topology*, and we say that a net $\{\nu_\alpha\}$ in $\mathcal{P}(Y)$ *converges weakly* to a Borel probability measure ν and we write $\nu_\alpha \xrightarrow{w} \nu$, if

$$\lim_\alpha \int_Y g(y)\nu_\alpha(dy) = \int_Y g(y)\nu(dy)$$

for every $g \in C_b(Y)$.

A *transition probability* λ on $X \times Y$ is defined to be a mapping from X into $\mathcal{P}(Y)$ which satisfies

(T1) for every $B \in \mathcal{B}(Y)$, the function $x \in X \mapsto \lambda_x(B) \equiv \lambda(x, B)$ is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(\mathbb{R})$.

In case X is also a topological space we always take $\mathcal{A} = \mathcal{B}(X)$.

Denote by $C(Y)$ the set of all continuous real-valued functions on Y. For each transition probability λ on $X \times Y$ and each $h \in C(Y)$, we can define a measurable function

$$x \in X \mapsto \chi[\lambda, h](x) \equiv \int_Y e^{ih(y)}\lambda(x, dy).$$

In the rest of this section we give a condition for a mapping λ from X into $\mathcal{P}(Y)$ to be a transition probability on $X \times Y$ in terms of the measurability of the above function $\chi[\lambda, h](x)$. Denote by $\mathbb{R}^N$ be the N-dimensional Euclidian space. For $u =$

$(u_1, u_2, \cdots, u_N)$, $v = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$, we set $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots + u_Nv_N$ and $\|u\| = \sqrt{\langle u, u\rangle}$. We denote by $K(\mathbb{R}^N)$ the set of all continuous complex-valued functions on $\mathbb{R}^N$ with compact supports.

Lemma 1. *Let $(X, \mathcal{A})$ be a measurable space and let λ be a mapping from X into $\mathcal{P}(\mathbb{R}^N)$. Then λ is a transition probability on $X \times \mathbb{R}^N$ if and only if for each $u = (u_1, u_2, \cdots, u_N) \in \mathbb{R}^N$, the function*

$$x \in X \mapsto \hat{\lambda}_x(u) \equiv \int_{\mathbb{R}^N} e^{i\langle u,v\rangle}\lambda(x, dv)$$

is measurable.

Recall that a topological space is called a *Suslin space* if it is the continuous image of some Polish space and recall that a subset H of $C(Y)$ is said to *separate points* of Y if for each $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there exists a function $h \in H$ such that $h(y_1) \neq h(y_2)$.

Proposition 1. *Let $(X, \mathcal{A})$ be a measurable space and Y a completely regular Suslin space, and let λ be a mapping from X into $\mathcal{P}(Y)$. Assume that a linear subspace H of $C(Y)$ separates points of Y. Then λ is a transition probability on $X \times Y$ if and only if for each $h \in H$, the function $x \in X \mapsto \chi[\lambda, h](x)$ is measurable.*

3. Uniform Tightness for Transition Probabilities

Let X and Y be topological spaces. Let us denote by $\mathcal{T}(X, Y)$ the set of all transition probabilities on $X \times Y$ and denote by $\mathcal{T}^*(X, Y)$ the set of all $\lambda \in \mathcal{T}(X, Y)$ which satisfy the condition

(T2) for each $D \in \mathcal{B}(X \times Y)$, the function $x \in X \mapsto \lambda(x, D_x)$ is Borel measurable.
Here for a subset D of $X \times Y$ and $x \in X$, D_x denotes the section determined by x, that is, $D_x = \{y \in Y : (x, y) \in D\}$.

Let $\mu \in \mathcal{P}(X)$ and $\lambda \in \mathcal{T}^*(X, Y)$. Then we can define a Borel probability measure $\mu \circ \lambda$ on $X \times Y$, which is called the *compound probability measure* of μ and λ, by

$$\mu \circ \lambda(D) = \int_X \lambda(x, D_x)\mu(dx) \quad \text{for all } D \in \mathcal{B}(X \times Y).$$

Denote by $\mu\lambda$ the projection of $\mu \circ \lambda$ onto Y, that is, $\mu\lambda(B) = \mu \circ \lambda(X \times B)$ for all $B \in \mathcal{B}(Y)$. By a standard argument, we can show that the Fubini's theorem remains valid for all Borel measurable and $\mu \circ \lambda$-integrable functions f on $X \times Y$;

$$\int_{X\times Y} f(x, y)\mu \circ \lambda(dx, dy) = \int_X \int_Y f(x, y)\lambda(x, dy)\mu(dx).$$

It is obvious that (T2) implies (T1), and (T2) is satisfied, for instance, if the product σ-algebra $\mathcal{B}(X) \times \mathcal{B}(Y)$ coincides with $\mathcal{B}(X \times Y)$ (this is satisfied if X and Y are Suslin spaces; see [13], page 105). We also know that (T2) is satisfied for any *continuous* τ*-smooth* transition probability on an arbitrary topological space (see Proposition 1 of Kawabe [7]). In what follows, for $P \subset \mathcal{P}(X)$ and $Q \subset \mathcal{T}^*(X,Y)$, we set $P \circ Q = \{\mu \circ \lambda : \mu \in P \text{ and } \lambda \in Q\}$ and $PQ = \{\mu\lambda : \mu \in P \text{ and } \lambda \in Q\}$.

Recall that a subset P of $\mathcal{P}(X)$ is said to be *uniformly tight* if for each $\varepsilon > 0$, there exists a compact subset K_ε of X such that

$$\mu(X - K_\varepsilon) < \varepsilon \quad \text{for all } \mu \in P$$

(see Prokhorov [11]). It is easy to see that $P \circ Q$ is uniformly tight if and only if P and PQ are uniformly tight. However PQ and $P \circ Q$ are *not* necessarily uniformly tight even if P is uniformly tight and $Q[x] = \{\lambda_x : \lambda \in Q\}$ is uniformly tight for each $x \in X$ as is seen in the following example. In what follows, δ_x denotes the Dirac measure concentrated on x, that is, $\delta_x(B) = 1$ if $x \in B$; $\delta_x(B) = 0$ if $x \notin B$.

Example. *Let $X = Y = \mathbb{R}$. For each $n \geq 1$, put*

$$s_n^2(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ n^2 x & \text{for } 0 < x \leq 1/n \\ 2n - n^2 x & \text{for } 1/n < x \leq 2/n \\ 0 & \text{for } 2/n < x \end{cases}$$

and define a transition probability λ_n on $\mathbb{R} \times \mathbb{R}$ by $\lambda_n(x, \cdot) = \mathcal{N}[0, s_n^2(x)]$, where $\mathcal{N}[m, \sigma^2]$ denotes the Gaussian measure on $\mathbb{R}$ with the mean m and the variance σ^2. We also put $P = \{\delta_{1/n}\}$ and $Q = \{\lambda_n\}$. Then P is uniformly tight and $Q[x]$ is uniformly tight for each $x \in \mathbb{R}$, but PQ and $P \circ Q$ are not uniformly tight.

We now present a notion of uniform tightness for transition probabilities. We say that $Q \subset \mathcal{T}(X,Y)$ is *uniformly tight* if each $\varepsilon > 0$, we can find an upper semicontinuous compact-valued mapping $\Lambda_\varepsilon : X \twoheadrightarrow Y$ such that

$$\lambda(x, Y - \Lambda_\varepsilon(x)) < \varepsilon$$

for all $x \in X$ and $\lambda \in Q$. Recall that a set-valued mapping $\Lambda : X \twoheadrightarrow Y$ is *upper semicontinuous* if $\Lambda^w(F) \equiv \{x \in X : \Lambda(x) \cap F \neq \emptyset\}$ is a closed subset of X for every closed subset F of Y. For the reader's convenience, we collect some well-known facts about upper semicontinuous set-valued mappings which will be used later (see [9], pages 89 and 90).

Proposition 2. *Let Γ and Λ be upper semicontinuous set-valued mappings from X to Y. Then one has:*

(1) *If Λ is compact-valued then $\Lambda(K) = \bigcup_{x\in K} \Lambda(x)$ is compact for every compact subset K of X.*

(2) *If Y is a topological linear space, and Γ and Λ are compact-valued, then the mapping $x \in X \mapsto \Gamma(x) + \Lambda(x)$ is compact-valued and upper semicontinuous.*

The following theorem asserts that our notion of uniform tightness for transition probabilities assures the uniform tightness of compound probability measures.

Theorem 1. *Let X and Y be topological spaces. If $P \subset \mathcal{P}(X)$ and $Q \subset \mathcal{T}^*(X,Y)$ are uniformly tight, then $P \circ Q \subset \mathcal{P}(X \times Y)$ is uniformly tight.*

Let X be a topological space. Denote by $C(X)$ the set of all continuous real-valued functions on X. We say that a subset F of $C(X)$ is *equicontinuous on a set* A of X if the set of all restrictions of functions of F to A is equicontinuous on A.

A Borel measure μ on X is said to be *τ-smooth* if for every increasing net $\{G_\alpha\}$ of open subsets of X, we have $\mu(\bigcup_\alpha G_\alpha) = \sup_\alpha \mu(G_\alpha)$. Denote by $\mathcal{P}_\tau(X)$ the set of all τ-smooth probability measures on X. Every Radon measure is tight and τ-smooth, and if X is regular, every τ-smooth measure is regular (see [15], Proposition I.3.1). Conversely every tight and regular Borel measure is Radon. The proof of the following lemma is an easy modification of the proof of Theorem 2 in [2], and so we omit its proof.

Lemma 2. *Let X be a completely regular topological space and let $\{\mu_\alpha\}$ be a net in $\mathcal{P}(X)$ which is uniformly tight. Assume that a net $\{\varphi_\alpha\}$ in $C_b(X)$ satisfies*

(a) $\{\varphi_\alpha\}$ is uniformly bounded;

(b) $\{\varphi_\alpha\}$ is equicontinuous on every compact subset of X.

If $\mu \in \mathcal{P}_\tau(X)$ and $\mu_\alpha \xrightarrow{w} \mu$, and if $\varphi \in C_b(X)$ and $\varphi_\alpha(x) \to \varphi(x)$ for each $x \in X$, then we have

$$\lim_\alpha \int_X \varphi_\alpha(x)\mu_\alpha(dx) = \int_X \varphi(x)\mu(dx).$$

We give a sufficient condition for the weak convergence of a net of compound probability measures.

Theorem 2. *Let X and Y be completely regular Suslin spaces. Let H be a linear subspace of $C(Y)$ which separates points of Y. Assume that a net $\{\lambda_\alpha\}$ in $\mathcal{T}(X,Y)$ and $\lambda \in \mathcal{T}(X,Y)$ satisfy* (*a*) $\{\lambda_\alpha\}$ *is uniformly tight;*

(*b*) *for each* $h \in H$, *the set* $\{\chi[\lambda_\alpha, h]\}$ *of the functions* $x \in X \mapsto \chi[\lambda_\alpha, h](x)$ *is equicontinuous on every compact subset of* X;

(*c*) $\chi[\lambda_\alpha, h](x) \to \chi[\lambda, h](x)$ *for each* $x \in X$ *and* $h \in H$.

Then for any uniformly tight net $\{\mu_\alpha\}$ *in* $\mathcal{P}(X)$ *converging weakly to* $\mu \in \mathcal{P}(X)$, *we have* $\mu_\alpha \circ \lambda_\alpha \xrightarrow{w} \mu \circ \lambda$.

We have typical and somewhat trivial examples of uniformly tight transition probabilities below. We study non-trivial examples in the following section.

Proposition 3. *Let* X *be a topological space and* Y *a completely regular topological space.*

(1) *For each* α, *put* $\lambda_\alpha(x, B) = \nu_\alpha(B)$ *for all* $x \in X$ *and* $B \in \mathcal{B}(Y)$, *where* $\{\nu_\alpha\} \subset \mathcal{P}_\tau(Y)$ *is uniformly tight. Then the* λ_α*'s satisfy* (T2), *and* $\{\lambda_\alpha\}$ *is uniformly tight.*

(2) *Let* $X = Y = G$ *be a topological group. For each* α, *put* $\lambda_\alpha(x, B) = \nu_\alpha(Bx^{-1})$ *for all* $x \in G$ *and* $B \in \mathcal{B}(G)$, *where* $\{\nu_\alpha\} \subset \mathcal{P}_\tau(G)$ *is uniformly tight. Then the* λ_α*'s satisfy* (T2), *and* $\{\lambda_\alpha\}$ *is uniformly tight.*

4. Gaussian Transition Probabilities on Nuclear Spaces

In this section we study Gaussian transition probabilities on nuclear spaces, such as the strong dual of the space of all rapidly decreasing functions, which are important and non-trivial examples of uniformly tight transition probabilities.

Let Ψ be a nuclear real Fréchet space, Ψ' the dual of Ψ and $\langle\cdot,\cdot\rangle$ the bilinear form on $\Psi \times \Psi'$. Let us denote by Ψ'_σ and Ψ'_β the weak and strong dual of Ψ with the weak topology $\sigma(\Psi', \Psi)$ and the strong topology $\beta(\Psi', \Psi)$, respectively. For the following properties which Ψ'_β enjoys the reader will find more details and proofs in Schaefer [12] and Fernique [4].

Proposition 4.

(1) Ψ'_β *is a Montel space, that is, it is a barreled space which every closed, bounded subset is compact.*

(2) Ψ'_β *is a completely regular Suslin space, in fact, Lusin space.*

(3) *Every closed, bounded subset of* Ψ'_σ *is a compact and sequentially compact subset of* Ψ'_β.

A seminorm p on Ψ is called *Hilbertian* (H-seminorm) if p has the form $p(u) = \sqrt{p(u,u)}$, where $p(u,v)$ is a symmetric, non-negative definite, bilinear functional on $\Psi \times \Psi$. Then the p-completion of $\Psi/\ker p$, denoted by Ψ_p, is a separable Hilbert space, and its dual Ψ_p' is also a separable Hilbert space with the norm $p'(\eta) = \sup\{|\langle u,\eta\rangle| : p(u) \le 1\}$.

Let p and q be H-seminorms on Ψ. Following Itô [5], we say that p is said to be *bounded* by q, written $p \prec q$, if

$$(p:q) = \sup\{p(u) : q(u) \le 1\} < \infty.$$

We also say that p is said to be *Hilbert-Schmidt bounded* by q, written $p \prec_{HS} q$, if $p \prec q$ and

$$(p:q)_{HS} = \left(\sum_{j=1}^{\infty} p(e_j)^2\right)^{1/2} < \infty \quad \text{for some CONS } \{e_j\} \text{ in } (\Psi, q).$$

It is well-known and is easily verified that $P \subset \mathcal{P}(\Psi_\beta')$ is uniformly tight if and only if for each $\varepsilon > 0$, there exists a continuous H-seminorm p_ε on Ψ such that

$$\mu(\{\eta \in \Psi' : |\langle u,\eta\rangle| \le p_\varepsilon(u) \text{ for all } u \in \Psi\}) \ge 1 - \varepsilon$$

for all $\mu \in P$. For the uniform tightness for transition probabilities we have:

Theorem 3. *Let X be a topological space which satisfies the first axiom of countability and Q a subset of $\mathcal{T}(X, \Psi_\beta')$. Assume that for each $\varepsilon > 0$ there exists a mapping $p_\varepsilon : X \times \Psi \to [0,\infty)$ satisfying*

(a) for each $u \in \Psi$, the mapping $x \in X \mapsto p_\varepsilon(x,u)$ is upper semicontinuous on X;

(b) for each $x \in X$, $p_\varepsilon(x)(\cdot) \equiv p_\varepsilon(x,\cdot)$ is a continuous H-seminorm on Ψ;

(c) $\lambda(x, \{\eta \in \Psi' : |\langle u,\eta\rangle| \le p_\varepsilon(x,u)$ for all $u \in \Psi\}) \ge 1 - \varepsilon$ for all $x \in X$ and $\lambda \in Q$.

Then Q is uniformly tight. Moreover in case $\Psi = \mathbb{R}^N$, the assumption that X satisfies the first axiom of countability is superfluous.

A Borel probability measure μ on Ψ_β' is said to be *Gaussian* if for each $u \in \Psi$, the function $\eta \in \Psi' \mapsto \langle u,\eta\rangle$ is a real (possibly degenerate) Gaussian random variable on the probability measure space $(\Psi', \mathcal{B}(\Psi_\beta'), \mu)$. For a Gaussian measure μ on Ψ_β', we define its *mean functional* m and *covariance seminorm* s of μ by

$$\langle u, m\rangle = \int_{\Psi'} \langle u,\eta\rangle \mu(d\eta)$$

$$s(u,v) = \int_{\Psi'} \langle u, \eta - m\rangle \langle v, \eta - m\rangle \mu(d\eta)$$

for all $u, v \in \Psi$ and we put $s^2(u) = s(u,u)$. We know that $m \in \Psi'$ and s is a continuous H-seminorm on Ψ (see e.g., [5], Theorem 2.6.2).

Let $(X, \mathcal{A})$ be a measurable space. A transition probability λ on $X \times \Psi'_\beta$ is said to be *Gaussian* if for each $x \in X$, $\lambda_x(\cdot) \equiv \lambda(x, \cdot)$ is a Gaussian measure on Ψ'_β. For a Gaussian transition probability λ on $X \times \Psi'_\beta$ we define for each $x \in X$ and each $u, v \in \Psi$,

$$m(x,u) = \int_{\Psi'} \langle u, \eta\rangle \lambda(x, d\eta)$$

and

$$s(x,u,v) = \int_{\Psi'} \{\langle u, \eta\rangle - m(x,u)\} \{\langle v, \eta\rangle - m(x,v)\} \lambda(x, d\eta),$$

and we put $s^2(x,u) = s(x,u,u)$. We say that the functions $m : x \in X \mapsto m(x, \cdot)$ and $s : x \in X \mapsto s(x, \cdot, \cdot)$ are the *mean function* and the *covariance function* of λ, respectively. Since a Gaussian measure is uniquely determined by its mean functional and covariance seminorm (see [5], Theorem 2.6.3), it is easily verified that a Gaussian transition probability λ is also uniquely determined by its mean function m and covariance function s, and hence we write $\lambda = \mathcal{TN}[m, s^2]$.

The following proposition asserts that a Gaussian transition probability can be characterized in terms of the measurability of its mean and covariance functions.

Proposition 5. *Let λ be a mapping from X into $\mathcal{P}(\Psi'_\beta)$ such that for each $x \in X$, λ_x is a Gaussian measure on Ψ'_β with its mean functional $m(x, \cdot)$ and covariance seminorm $s(x, \cdot, \cdot)$. Then λ is a transition probability on $X \times \Psi'_\beta$ and $\lambda = \mathcal{TN}[m, s^2]$ if and only if for each $u \in \Psi$, the functions $x \in X \mapsto m(x,u)$ and $x \in X \mapsto s^2(x,u)$ are measurable.*

The following theorem gives a sufficient condition under which a set of Gaussian transition probabilities on $X \times \Psi'_\beta$ is uniformly tight, in terms of mean and covariance functions.

Theorem 4. *Let X be as in Theorem 3 and Q a set of Gaussian transition probabilities on $X \times \Psi'_\beta$ with $\lambda = \mathcal{TN}[m_\lambda, s^2_\lambda]$, $\lambda \in Q$. Assume that there exists a mapping $q : X \times \Psi \to [0, \infty)$ satisfying*

(a) for each $u \in \Psi$, the mapping $x \in X \mapsto q(x,u)$ is upper semicontinuous on X;

(*b*) *for each* $x \in X$, $q_x(\cdot) = q(x, \cdot)$ *is a continuous* H*-seminorm on* Ψ.
Further, assume that there exist non-negative upper semicontinuous functions $M(x)$ *and* $S(x)$ *on* X *such that for every* $x \in X$,

$$\sup_{\lambda \in Q} q'_x(m_\lambda(x)) \leq M(x) \quad \textit{and} \quad \sup_{\lambda \in Q} (s_\lambda(x) : q_x)_{HS} \leq S(x).$$

Then Q *is uniformly tight.*

In case $\Psi = \mathbb{R}^N$ we have:

Corollary 1. *Let* X *be a topological space and* Q *a set of Gaussian transition probabilities on* $X \times \mathbb{R}^N$ *with* $\lambda = \mathcal{TN}[m_\lambda, s_\lambda^2]$, $\lambda \in Q$. *Assume that there exist non-negative functions* $M(x, u)$ *and* $S(x, u)$ *defined on* $X \times \mathbb{R}^N$ *which satisfy*

(*a*) *for each* $u \in \mathbb{R}^N$, *the functions* $x \in X \mapsto M(x, u)$ *and* $x \in X \mapsto S(x, u)$ *are upper semicontinuous on* X;

(*b*) $\sup_{\lambda \in Q} |\langle u, m_\lambda(x) \rangle| \leq M(x, u)$ *and* $\sup_{\lambda \in Q} s_\lambda(x, u) \leq S(x, u)$ *for all* $x \in X$ *and* $u \in \mathbb{R}^N$.
Then Q *is uniformly tight.*

In the case when X is a one point set we have the following well-known result.

Corollary 2. *Let* P *be a set of Gaussian measures on* Ψ'_β *with mean functionals* m_μ *and covariance seminorms* s_μ, $\mu \in P$. *Assume that* $\sup_{\mu \in P} |\langle u, m_\mu \rangle| < \infty$ *and* $\sup_{\mu \in P} s_\mu(u) < \infty$ *for each* $u \in \Psi$. *Then* P *is uniformly tight.*

Let Φ be a nuclear real Fréchet space. In case $X = \Phi'_\beta$, combined Theorem 1 and Corollary 1 with a well-known criterion for uniform tightness of probability measures on nuclear spaces, we have:

Theorem 5. *Let* Q *be a set of Gaussian transition probabilities on* $\Phi'_\beta \times \Psi'_\beta$ *with* $\lambda = \mathcal{TN}[m_\lambda, s_\lambda^2]$, $\lambda \in Q$. *Assume that there exist non-negative functions* $M(\xi, u)$ *and* $S(\xi, u)$ *defined on* $\Phi' \times \Psi$ *which satisfy*

(*a*) *for each* $u \in \Psi$, *the functions* $\xi \in \Phi'_\beta \mapsto M(\xi, u)$ *and* $\xi \in \Phi'_\beta \mapsto S(\xi, u)$ *are upper semicontinuous on* Φ'_β;

(*b*) $\sup_{\lambda \in Q} |m_\lambda(\xi, u)| \leq M(\xi, u)$ *and* $\sup_{\lambda \in Q} s_\lambda(\xi, u) \leq S(\xi, u)$ *for all* $\xi \in \Phi'$ *and* $u \in \Psi$.
Then $P \circ Q$ *is uniformly tight for any uniformly tight subset* P *of* $\mathcal{P}(\Phi'_\beta)$.

For the weak convergence of compound probability measures we have:

Theorem 6. *Let $\lambda_\alpha = \mathcal{TN}[m_\alpha, s_\alpha^2]$ be a net of Gaussian transition probabilities on $\Phi'_\beta \times \Psi'_\beta$ and $\lambda = \mathcal{TN}[m, s^2]$ a Gaussian transition probability on $\Phi'_\beta \times \Psi'_\beta$. Assume that in addition to assumptions (a) and (b) of Theorem 5,*

(c) for each $x \in X$, the sets $\{m_\alpha(\cdot, u)\}$ and $\{s_\alpha^2(\cdot, u)\}$ are equicontinuous on every compact subset of Φ'_β;

(d) $\lim_\alpha m_\alpha(\xi, u) = m(\xi, u)$ and $\lim_\alpha s_\alpha^2(\xi, u) = s^2(\xi, u)$ for each $\xi \in \Phi'$ and $u \in \Psi$.

Then for any uniformly tight net $\{\mu_\alpha\}$ in $\mathcal{P}(\Phi'_\beta)$ converging weakly to $\mu \in \mathcal{P}(\Phi'_\beta)$, we have $\mu_\alpha \circ \lambda_\alpha \xrightarrow{w} \mu \circ \lambda$.

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