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On some generalizations of q-uniform convexity inequalities

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Abstract. This is an announcement of some recent results of the authors concerning the q-uniform convexity and p-uniform smoothness inequalities.

We shall consider some generalizations of p-uniform smoothness and q-uniform convexity inequalities. In particular we shall characterize these two geometric notions by type- and cotype-like inequalities which are stronger than those of type and cotype, respectively.

1. p-uniformly smooth and q-uniformly convex spaces

Let X be a Banach space with $\dim X \geq 2$. The *modulus of convexity* of X is

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\right\}, \quad 0 \leq \varepsilon \leq 2.$$

X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$, and *q-uniformly convex* $(2 \leq q < \infty)$ if there exists a constant $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for all $\varepsilon > 0$. The *modulus of smoothness* of X is

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \ \|x\| = \|y\| = 1\right\}, \quad \tau > 0.$$

X is called *uniformly smooth* if $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$, and *p-uniformly smooth* $(1 < p \leq 2)$ if there exists a constant $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau > 0$. These moduli have the best values with a Hilbert space H (cf. [8, p. 68]): For any Banach space X

$$\begin{aligned} \delta_X(\varepsilon) &\leq \delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}, \\ \rho_X(\tau) &\geq \rho_H(\tau) = \sqrt{1+\tau^2} - 1. \end{aligned}$$

In view of these facts *no Banach space is q-uniformly convex for $q < 2$ and p-uniformly smooth for $p > 2$.* In fact, if $q < 2$, since

$$\frac{\delta_X(\varepsilon)}{\varepsilon^q} \leq \frac{1 - \sqrt{1 - \varepsilon^2/4}}{\varepsilon^q} = \frac{\varepsilon^{2-q}}{4(1 + \sqrt{1 - \varepsilon^2/4})},$$

we have $\lim_{\varepsilon \to +0} \delta_X(\varepsilon)/\varepsilon^q = 0$. When $p > 2$,

$$\frac{\rho_X(\tau)}{\tau^p} \geq \frac{\sqrt{1+\tau^2}-1}{\tau^p} = \frac{1}{\tau^{p-2}(\sqrt{1+\tau^2}+1)} \to \infty \quad \text{as } \tau \to 0.$$

Also every Banach space is 1-uniformly smooth as $\rho_X(\tau) \leq \tau$ for all $\tau > 0$. It is clear that p-uniformly smooth spaces are r-uniformly smooth if $1 < r \leq p \leq 2$, and q-uniformly convex spaces are r-uniformly convex if $2 \leq q \leq r < \infty$.

p-uniformly smooth and q-uniformly convex spaces are characterized by the following *p-uniform smoothness* and *q-uniform convexity inequalities*:

Lemma 1 ([1], [2]). (i) Let $1 < p \leq 2$. Then X is p-uniformly smooth if and only if there exists $K > 0$ such that

$$\text{(1)} \qquad \frac{\|x+y\|^p + \|x-y\|^p}{2} \leq \|x\|^p + \|Ky\|^p \quad \text{for all } x, y \in X.$$

(ii) Let $2 \leq q < \infty$. Then X is q-uniformly convex if and only if there exists $C > 0$ such that

$$\text{(2)} \qquad \frac{\|x+y\|^q + \|x-y\|^q}{2} \geq \|x\|^q + \|Cy\|^q \quad \text{for all } x,\ y \in X.$$

Remark 1. (i) The validity of the inequality (1) implies $K \geq 1$. Thus (1) with the best constant $K = 1$ is the following *Clarkson inequality*

$$\text{(3)} \qquad \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \leq (\|x\|^p + \|y\|^p)^{1/p} \qquad (1 < p \leq 2)$$

(ii) In (2) we have necessarily $0 < C \leq 1$ (indeed put $x = 0$), and the inequality (2) with the best constant $C = 1$ is the following *Clarkson inequality*

$$\text{(4)} \qquad \left(\frac{\|x+y\|^q + \|x-y\|^q}{2}\right)^{1/q} \geq (\|x\|^q + \|y\|^q)^{1/q} \qquad (2 \leq q < \infty)$$

2. Generalizations of p-uniform smoothness and q-uniform convexity inequalities

We shall present some generalizations of p-uniform smoothness and q-uniform convexity inequalities which hold to characterize these smoothness and convexity. More prescisely, in the first sense we shall give two-element inequalities sharper than (1) and (2) respectively, and in the secondary sense we shall characterize p-uniform smoothness and q-uniform convexity by type-, cotype-like inequalities which are stronger than type, cotype inequalities respectively.

The notions of type and cotype were introduced by Hoffman-Jørgensen [3] (cf. [9]) in the context of the law of large numbers for random variables with values in a Banach space. A Banach space X is called *of type* p, $1 \le p \le 2$, if there is $M > 0$ (necessarily $M \ge 1$) such that

$$(5) \qquad \left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|^p\right)^{1/p} \le M\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p}$$

for all finite systems $x_1, \cdots, x_n \in X$. X is called *of cotype* q, $2 \le q < \infty$, if there is $M > 0$ (necessarily $M \ge 1$) such that

$$(6) \qquad \left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^{q}\theta_j x_j\right\|^q\right)^{1/q} \ge \frac{1}{M}\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{1/q}$$

for all finite systems $x_1, \cdots, x_n \in X$.

These probabilistic properties are characterized by Clarkson's inequalities which are of geometric nature. Namely, in 1997 the first and second authors [6] showed that X is of type p with $M = 1$ if and only if Clarkson's inequality (3) holds in X and the corresponding fact for cotype and Clarkson's inequality (4) (their presentations are more general). On the other hand it is well known that

(i) p-uniformly smooth spaces are of type p,
(ii) q-uniformly convex spaces are of cotype q

and there is no converse of these assertions. Indeed there exisits a non-reflexive space X having type 2 (James [4]). Then X is of type p for any $1 < p \le 2$, whereas X is not p-uniformly smooth because uniformly smooth spaces must be reflexive. Also its dual space X^* is of cotype q for any $2 \le q < \infty$, but not q-uniformly convex as X^* is not reflexive.

Theorem 1 (p-uniform smoothness). Let $1 < p \le 2$ and $1 \le s < \infty$. The following are equivalent.

(i) X is p-uniformly smooth.
(ii) There exists $K \ge 1$ such that

$$(7) \qquad \left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \le (\|x\|^p + \|Ky\|^p)^{1/p} \quad \forall x, y \in X.$$

If $p \le s < \infty$, in addition:

(iii) There exists $K \ge 1$ such that

$$(8) \qquad \left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|^s\right)^{1/s} \le \left(\|x_1\|^p + \sum_{j=2}^{n}\|Kx_j\|^p\right)^{1/p}$$

for all finite systems $x_1, \cdots, x_n \in X$.

Remark 2. (i) The inequality (7) is sharper than (1) of Lemma 1 if $p \le s$. Indeed in this case by Lemma 2

$$\left(\frac{\|x+y\|^p+\|x-y\|^p}{2}\right)^{1/p} \le \left(\frac{\|x+y\|^s+\|x-y\|^s}{2}\right)^{1/s} \le (\|x\|^p+\|Ky\|^p)^{1/p}.$$

(ii) For the case $K = 1$ the equivalence of the inequalities (7) and (8) is proved in Kato-Takahashi [6].

(iii) The inequality (8) is stronger than the type p inequality (5). Indeed, the space X of James stated above is of type p, whereas (8) fails to hold in X. So we refer to (8) as *strong type p inequality.*

Theorem 2 (q-uniform convexity). Let $2 \le q < \infty$ and $1 < t \le \infty$. The following are equivalent.

(i) X is q-uniformly convex.

(ii) There exists $0 < C \le 1$ such that

$$\text{(9)} \qquad \left(\frac{\|x+y\|^t+\|x-y\|^t}{2}\right)^{1/t} \ge (\|x\|^q+\|Cy\|^q)^{1/q} \quad \forall x,\, y \in X.$$

If $1 < t \le q$, in addition:

(iii) There exists $0 < C \le 1$ such that

$$\text{(10)} \qquad \left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^n \theta_j x_j\right\|^t\right)^{1/t} \ge \left(\|x_1\|^q+\sum_{j=2}^n\|Cx_j\|^q\right)^{1/q}$$

for all finite systems $x_1, \cdots, x_n \in X$.

Remark 3. (i) The inequality (9) is sharper than (2) of Lemma 1 if $q \ge t$. Indeed we have

$$\left(\frac{\|x+y\|^q+\|x-y\|^q}{2}\right)^{1/q} \ge \left(\frac{\|x+y\|^t+\|x-y\|^t}{2}\right)^{1/t} \ge (\|x\|^q+\|Cy\|^q)^{1/q}.$$

(ii) For the case $C = 1$ the equivalence of the inequalities (9) and (10) is proved in Kato-Takahashi [6].

(iii) The inequality (10) is stronger than the cotype q inequality (6). Indeed the dual space X^* of the space X of James is of cotype q, but (10) fails to hold in X. L_1 is also a counter example, since it is of cotype 2 and non-reflexive. So we refer to (10) as *strong cotype q inequality.*

It is well known that if X is of type p, then X^* is of cotype q, where $1/p+1/q = 1$, and the converse is not true ([2, pp. 309-310]). Indeed, $l_1 = (c_0)^*$ has cotype 2, whereas c_0 has no non-trivial type. Our next theorem asserts that for our strong type and cotype inequalities (8) and (10) the converse is also true if $p \le s < \infty$.

Theorem 3 (duality). Let $1 \leq p \leq 2$, $1 < s < \infty$ and $1/p+1/q = 1/s+1/t = 1$. Let $1 \leq K < \infty$. Then if

$$\left(\frac{1}{2^n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|^s\right)^{1/s} \leq \left(\|x_1\|^p + \sum_{j=2}^{n}\|Kx_j\|^p\right)^{1/p} \tag{8}$$

holds in X,

$$\left(\frac{1}{2^n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j^*\right\|^t\right)^{1/t} \geq \left(\|x_1^*\|^q + \sum_{j=2}^{n}\|K^{-1}x_j^*\|^q\right)^{1/q} \tag{10*}$$

holds in X^*. If $p \leq s < \infty$ the converse is true.

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