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Eigenvalue Problem related to Euler-Bernoulli Equation with Joint Boundary Condition

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Abstract

We consider the control system of two vibrating beams which are coupled at a joint. The displacement of the beam is described by an Euler-Bernoulli equation with control applied at a coupled point. Our purpose is to argue the contorollability of the system. To this purpose, we discuss the eigenvalue problem related to this system.

1 Introduction

Let us consider the controllability problem for a system coupled by Euler-Bernoulli beams. For $m\in(0,1)$, we put $x_0=0$, $x_1=m$ and $x_2=1$. The displacement of each beam at time t is described by $y_i(x,t)$ on $I_i=(x_{i-1},x_i)$, $i=1,2$, and satisfies the Euler-Bernoulli equation:

$$\rho_i\ddot{y}_i+T_iy_i^{(4)}=0 \quad \text{on } I_i\times(0,T) \tag{1}$$

where $\dot{y}_i(x,t)=\partial y_i(x,t)/\partial t$, $y_i^{(k)}(x,t)=\partial^k y_i(x,t)/\partial x^k$. ρ_i is mass density and T_i is flexural rigidity respectively on I_i. Let both ends be clamped:

$$(B_0y)(t):=(y_1(0,t),\ y_1^{(1)}(0,t),\ y_2(1,t),\ y_2^{(1)}(1,t))=0. \tag{2}$$

At the coupled point $x=m$, we apply control $F=(f_1,f_2,f_3,f_4)$ as follows:

$$\left.\begin{array}{l}(B_1y)(t):=y_1(m,t)-y_2(m,t)=f_1(t),\\(B_2y)(t):=y_1^{(1)}(m,t)-y_2^{(1)}(m,t)=f_2(t),\\(B_3y)(t):=T_1y_1^{(2)}(m,t)-T_2y_2^{(2)}(m,t)=f_3(t),\\(B_4y)(t):=T_1y_1^{(3)}(m,t)-T_2y_2^{(3)}(m,t)=f_4(t).\end{array}\right\} \tag{3}$$

Initial condition is given as follows

$$y_i(x,0)=y_i^0(x),\quad \dot{y}_i(x,0)=y_i^1(x),\qquad x\in I_i,\ i=1,2. \tag{4}$$

We assume that controls f_i belong to $L^2(0,T)$, $i=1,2,3,4$. In this paper, we treat controllability of the above system. Roughly speaking, the system (1)(2)(3)(4) is controllable if for any initial value (y_i^0,y_i^1) and final value (z_i^0,z_i^1), $i=1,2$, there exists a control $F=(f_1,f_2,f_3,f_4)$ such that the corresponding solution of the system (1)(2)(3)(4) satisfies the final condition $(y_i(x,T),\dot{y}_i(x,T))=(z_i^0(x),z_i^1(x))$, $i=1,2$.

2 Eigenvalue Problem

Let us identify $v \in L^2(I)$ with $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H = L^2(I) = L^2(I_1) \times L^2(I_2)$ where $v_i = v|_{I_i}$, $i = 1, 2$, $I_1 = (0, m)$, $I_2 = (m, 1)$. Then H becomes a Hilbert space with inner product

$$(v, w) = \rho_1(v_1, w_1)_{L^2(I_1)} + \rho_2(v_2, w_2)_{L^2(I_2)} \quad \text{for } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in H.$$

We define an operator A in H by

$$Av = \begin{pmatrix} (T_1/\rho_1)v_1^{(4)} \\ (T_2/\rho_2)v_2^{(4)} \end{pmatrix} \quad \text{for } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(A) := H^4(I_1) \times H^4(I_2)$$

and an operator $\mathcal{A}$ by restricting A to

$$\mathcal{D}(\mathcal{A}) := \{v \in H^4(I_1) \times H^4(I_2);\ B_0 v = 0,\ Bv := (B_1 v, B_2 v, B_3 v, B_4 v) = 0\}.$$

For this operator $\mathcal{A}$, we have

Lemma 1 *The operator $\mathcal{A}$ is a selfadjoint operator in H with compact resolvent.*

The proof of this lemma is easy to verify.

Let λ be an eigenvalue for $\mathcal{A}$ with corresponding eigenfunction ϕ. Then we have

$$\mathcal{A}\phi = \lambda\phi \tag{1}$$

with boundary conditions

$$B_0\phi = 0, \quad B\phi = 0. \tag{2}$$

We introduce functions $\mathrm{C}_\pm$, $\mathrm{S}_\pm$ by

$$\mathrm{C}_\pm(\theta) := \frac{\cosh\theta \pm \cos\theta}{2}, \quad \mathrm{S}_\pm(\theta) := \frac{\sinh\theta \pm \sin\theta}{2} \quad \text{for} \quad \theta \in \mathbf{R}.$$

Let $\phi_i = \phi|_{I_i}$, $\alpha_i = (\rho_i/T_i)^{\frac{1}{4}}$, $i = 1, 2$. A system of fundamental solutions to (1) in each I_i is given by $\{\mathrm{C}_\pm(\alpha_i\omega(x - x_{i-1})),\ \mathrm{S}_\pm(\alpha_i\omega(x - x_{i-1}))\}$ and we have

$$\left.\begin{array}{lcl} \phi_i(x) & = & (p_1^i\mathrm{C}_+ + p_2^i\mathrm{S}_+ + p_3^i\mathrm{C}_- + p_4^i\mathrm{S}_-)(\alpha_i\omega(x - x_{i-1})), \\ (\phi_i)^{(1)}(x) & = & \alpha_i\omega(p_1^i\mathrm{S}_- + p_2^i\mathrm{C}_+ + p_3^i\mathrm{S}_+ + p_4^i\mathrm{C}_-)(\alpha_i\omega(x - x_{i-1})), \\ (\phi_i)^{(2)}(x) & = & \alpha_i^2\omega^2(p_1^i\mathrm{C}_- + p_2^i\mathrm{S}_- + p_3^i\mathrm{C}_+ + p_4^i\mathrm{S}_+)(\alpha_i\omega(x - x_{i-1})), \\ (\phi_i)^{(3)}(x) & = & \alpha_i^3\omega^3(p_1^i\mathrm{S}_+ + p_2^i\mathrm{C}_- + p_3^i\mathrm{S}_- + p_4^i\mathrm{C}_+)(\alpha_i\omega(x - x_{i-1})) \end{array}\right\}$$

for $x \in I_i$ where $\omega = \lambda^{\frac{1}{4}}$. By (2), we have $p_1^1 = p_2^1 = 0$ and therefore

$$\left.\begin{array}{lcl} \phi_1(x) & = & (p_3^1\mathrm{C}_- + p_4^1\mathrm{S}_-)(\alpha_1\omega x), \\ (\phi_1)^{(1)}(x) & = & \alpha_1\omega(p_3^1\mathrm{S}_+ + p_4^1\mathrm{C}_-)(\alpha_1\omega x), \\ (\phi_1)^{(2)}(x) & = & \alpha_1^2\omega^2(p_3^1\mathrm{C}_+ + p_4^1\mathrm{S}_+)(\alpha_1\omega x), \\ (\phi_1)^{(3)}(x) & = & \alpha_1^3\omega^3(p_3^1\mathrm{S}_- + p_4^1\mathrm{C}_+)(\alpha_1\omega x). \end{array}\right\} \tag{3}$$

By (2), we have

$$\gamma_2\alpha_2 p^2 = \gamma_2\alpha_2 \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} = \begin{pmatrix} \gamma_2\alpha_2 \mathrm{C}_-^1(\omega) & \gamma_2\alpha_2 \mathrm{S}_-^1(\omega) \\ \gamma_2\alpha_1 \mathrm{S}_+^1(\omega) & \gamma_2\alpha_1 \mathrm{C}_-^1(\omega) \\ \gamma_1\alpha_2 \mathrm{C}_+^1(\omega) & \gamma_1\alpha_2 \mathrm{S}_+^1(\omega) \\ \gamma_1\alpha_1 \mathrm{S}_-^1(\omega) & \gamma_1\alpha_1 \mathrm{C}_+^1(\omega) \end{pmatrix} \begin{pmatrix} p_3^1 \\ p_4^1 \end{pmatrix} \tag{4}$$

where $\gamma_i = T_i\alpha_i^2$, $i = 1, 2$, $\beta_1 = \alpha_1 m$, $\mathrm{S}_\pm^1(\omega) = \mathrm{S}_\pm(\beta_1\omega)$, $\mathrm{C}_\pm^1(\omega) = \mathrm{C}_\pm(\beta_1\omega)$. By (2), we see

$$\begin{pmatrix} \mathrm{C}_+^2(\omega) & \mathrm{S}_+^2(\omega) & \mathrm{C}_-^2(\omega) & \mathrm{S}_-^2(\omega) \\ \mathrm{S}_-^2(\omega) & \mathrm{C}_+^2(\omega) & \mathrm{S}_+^2(\omega) & \mathrm{C}_-^2(\omega) \end{pmatrix} \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} = 0, \tag{5}$$

where $\beta_2 = \alpha_2(1-m)$, $\mathrm{S}_\pm^2(\omega) = \mathrm{S}_\pm(\beta_2\omega)$, $\mathrm{C}_\pm^2(\omega) = \mathrm{C}_\pm(\beta_2\omega)$. Let

$$\begin{aligned} D(\omega) &= \begin{pmatrix} D_{11}(\omega) & D_{12}(\omega) \\ D_{21}(\omega) & D_{22}(\omega) \end{pmatrix} \\ &:= \begin{pmatrix} \mathrm{C}_+^2(\omega) & \mathrm{S}_+^2(\omega) & \mathrm{C}_-^2(\omega) & \mathrm{S}_-^2(\omega) \\ \mathrm{S}_-^2(\omega) & \mathrm{C}_+^2(\omega) & \mathrm{S}_+^2(\omega) & \mathrm{C}_-^2(\omega) \end{pmatrix} \begin{pmatrix} \gamma_2\alpha_2 \mathrm{C}_-^1(\omega) & \gamma_2\alpha_2 \mathrm{S}_-^1(\omega) \\ \gamma_2\alpha_1 \mathrm{S}_+^1(\omega) & \gamma_2\alpha_1 \mathrm{C}_-^1(\omega) \\ \gamma_1\alpha_2 \mathrm{C}_+^1(\omega) & \gamma_1\alpha_2 \mathrm{S}_+^1(\omega) \\ \gamma_1\alpha_1 \mathrm{S}_-^1(\omega) & \gamma_1\alpha_1 \mathrm{C}_+^1(\omega) \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} D_{11}(\omega) &= (\gamma_2\alpha_2\mathrm{C}_+^2\cdot\mathrm{C}_-^1 + \gamma_2\alpha_1\mathrm{S}_+^2\cdot\mathrm{S}_+^1 + \gamma_1\alpha_2\mathrm{C}_-^2\cdot\mathrm{C}_+^1 + \gamma_1\alpha_1\mathrm{S}_-^2\cdot\mathrm{S}_-^1)(\omega), \\ D_{12}(\omega) &= (\gamma_2\alpha_2\mathrm{C}_+^2\cdot\mathrm{S}_-^1 + \gamma_2\alpha_1\mathrm{S}_+^2\cdot\mathrm{C}_-^1 + \gamma_1\alpha_2\mathrm{C}_-^2\cdot\mathrm{S}_+^1 + \gamma_1\alpha_1\mathrm{S}_-^2\cdot\mathrm{C}_+^1)(\omega), \\ D_{21}(\omega) &= (\gamma_2\alpha_2\mathrm{S}_-^2\cdot\mathrm{C}_-^1 + \gamma_2\alpha_1\mathrm{C}_+^2\cdot\mathrm{S}_+^1 + \gamma_1\alpha_2\mathrm{S}_+^2\cdot\mathrm{C}_+^1 + \gamma_1\alpha_1\mathrm{C}_-^2\cdot\mathrm{S}_-^1)(\omega), \\ D_{22}(\omega) &= (\gamma_2\alpha_2\mathrm{S}_-^2\cdot\mathrm{S}_-^1 + \gamma_2\alpha_1\mathrm{C}_+^2\cdot\mathrm{C}_-^1 + \gamma_1\alpha_2\mathrm{S}_+^2\cdot\mathrm{S}_+^1 + \gamma_1\alpha_1\mathrm{C}_-^2\cdot\mathrm{C}_+^1)(\omega). \end{aligned}$$

We put

$$\begin{aligned} d(\omega) &:= 4\det D(\omega) \\ &= 4\gamma_2^2\alpha_2\alpha_1\,(\mathrm{S}_+^2{\cdot}\mathrm{S}_-^2 - \mathrm{C}_+^2{\cdot}\mathrm{C}_+^2){\cdot}(\mathrm{S}_+^1{\cdot}\mathrm{S}_-^1 - \mathrm{C}_-^1{\cdot}\mathrm{C}_-^1)(\omega) \\ &\quad + 4\gamma_2\gamma_1\alpha_2^2\,(\mathrm{S}_+^2{\cdot}\mathrm{C}_+^2 - \mathrm{S}_-^2{\cdot}\mathrm{C}_-^2){\cdot}(\mathrm{S}_+^1{\cdot}\mathrm{C}_-^1 - \mathrm{C}_+^1{\cdot}\mathrm{S}_-^1)(\omega) \\ &\quad + 8\gamma_2\gamma_1\alpha_2\alpha_1\,(\mathrm{S}_+^2{\cdot}\mathrm{S}_+^2 - \mathrm{C}_+^2{\cdot}\mathrm{C}_-^2){\cdot}(\mathrm{S}_+^1{\cdot}\mathrm{S}_+^1 - \mathrm{C}_+^1{\cdot}\mathrm{C}_-^1)(\omega) \\ &\quad + 4\gamma_2\gamma_1\alpha_1^2\,(\mathrm{S}_+^2{\cdot}\mathrm{C}_-^2 - \mathrm{C}_+^2{\cdot}\mathrm{S}_-^2){\cdot}(\mathrm{S}_+^1{\cdot}\mathrm{C}_+^1 - \mathrm{S}_-^1{\cdot}\mathrm{C}_-^1)(\omega) \\ &\quad + 4\gamma_1^2\alpha_1\alpha_2\,(\mathrm{S}_+^2{\cdot}\mathrm{S}_-^2 - \mathrm{C}_-^2{\cdot}\mathrm{C}_-^2){\cdot}(\mathrm{S}_+^1{\cdot}\mathrm{S}_-^1 - \mathrm{C}_+^1{\cdot}\mathrm{C}_+^1)(\omega). \end{aligned} \tag{6}$$

By (4), (5), we have

$$D(\omega)\begin{pmatrix} p_3^1 \\ p_4^1 \end{pmatrix} = 0.$$

Since ϕ is an eigenfunction if and only if $(p_3^1, p_4^1) \neq 0$, we see that $\lambda = \omega^4$ is an eigenvalue of $\mathcal{A}$ if $d(\omega) = 0$, $\omega > 0$. Let ω_n, $n \in \mathbf{N}$, is the n-th positive zero of $d(\omega)$. Then $\lambda_n := \omega_n^4$,

$0 < \omega_1 < \omega_2 < \cdots$, is the n-th eigenvalue of $\mathcal{A}$. We can verify that λ_n is a simple eigenvalue. Let ϕ^n be an eigenfunction corresponding to λ_n, normalized in H. In the following, let $\varphi(\omega)$ be a function defined by

$$\varphi(\omega) = A\cos\beta_1\omega\cos\beta_2\omega - B\sin\beta_1\omega\sin\beta_2\omega + C\sin(\beta_1 - \beta_2)\omega \tag{7}$$

where $A = (\gamma_1\alpha_1 + \gamma_2\alpha_2)(\gamma_1\alpha_2 + \gamma_2\alpha_1)$, $B = \gamma_1\gamma_2(\alpha_1 + \alpha_2)^2$, $C = \gamma_1\gamma_2(\alpha_1^2 - \alpha_2^2)$. We denote the n-th positive zero of φ by μ_n.

Lemma 2 *$d(\omega)$ is written as*

$$d(\omega) = e^{(\beta_1+\beta_2)\omega}(\varphi(\omega) - h(\omega)), \quad \omega \in \mathbf{R}$$

where $h(\omega) \in \mathrm{C}^1(\mathbf{R})$ and $h(\omega) \to 0$, $h'(\omega) \to 0$ exponentially as $\omega \to \infty$.

Proof By (6), $h(\omega) = \varphi(\omega) - e^{-(\beta_1+\beta_2)\omega}d(\omega)$ and $h'(\omega)$ converge to 0 exponenrially as $\omega \to \infty$.

To discuss controllability, we treat the moment problem on the system (1)(2)(3)(4). According to Krabs [4] or Russell [12], to solve the moment problem, we need the following conditions:

$$\liminf_{n\to\infty}({\omega_{n+1}}^2 - {\omega_n}^2) > \frac{2\pi}{T}, \tag{8}$$

$$\limsup_{y\to\infty}\limsup_{x\to\infty}\frac{d(x+y) - d(x)}{y} < \frac{T}{2\pi} \tag{9}$$

where $d(x) =$ number of ω_j with $\omega_j < x^2$. The aim of this paper is to prove the following

Theorem 1 *We have*

(1) *There exist M and N such that $\omega_{M+n} - \mu_{N+n} \to 0$,*

(2) $0 < \frac{1}{a}(\pi - \sin^{-1}k) \le \liminf_{n\to\infty}(\omega_{n+1} - \omega_n) \le \limsup_{n\to\infty}(\omega_{n+1} - \omega_n) < \infty$,

(3) $\lim_{n\to\infty}({\omega_{n+1}}^2 - {\omega_n}^2) = \infty$.

By this theorem, it is clear that $\{\omega_n\}_{n\in\mathbf{N}}$ satisfies the codition (8). Moreover, the condition (8) verify the condition (9).

Some simple facts for $\varphi(\omega)$ are given in the following

Lemma 3 *In the formula (7), we have*

(1) $A \ge B > 0$,

(2) $A = B$ *if and only if* $\rho_1 T_1 = \rho_2 T_2$.

(3) $C = 0$ *if and only if* $\rho_1 T_2 = \rho_2 T_1$.

(4) $A = B$, $C = 0$ *if and only if* $(\rho_1, T_1) = (\rho_2, T_2)$.

(5) $A > B$ *or* $C \ne 0$ *if and only if* $(\rho_1, T_1) \ne (\rho_2, T_2)$.

(6) *φ is written as*

$$\varphi(\omega) = D(\cos a\omega + k\sin(b\omega + \tau)), \quad 0 \le k < 1 \tag{10}$$

where $D = (A+B)/2$, $a = \beta_1+\beta_2$, $b = \beta_1-\beta_2$, $k = R/D$, $R = \sqrt{((A-B)/2)^2 + C^2}$, $\tau = \cos^{-1}(C/R) \in [0, \pi]$ *for* $R \neq 0$ *and* $\tau = 0$ *for* $R = 0$.

Proof We see (1) from $A = B + \alpha_1\alpha_2(\gamma_1 - \gamma_2)^2 \ge B > 0$. The assertions (2), (3), (4) and (5) are clear. We have (6) since

$$\begin{aligned}\varphi(\omega) &= \frac{A+B}{2}\cos(\beta_1+\beta_2)\omega + \frac{A-B}{2}\cos(\beta_1-\beta_2)\omega + C\sin(\beta_1-\beta_2)\omega \\ &= D\cos a\omega + R\sin(b\omega+\tau) = D(\cos a\omega + k\sin(b\omega+\tau)).\end{aligned}$$

where $k \ge 0$ satisfies

$$k^2 = \frac{R^2}{D^2} = \frac{(A+B)^2 - 4(AB - C^2)}{(A+B)^2} = 1 - \frac{4(\gamma_1+\gamma_2)^2\gamma_1\gamma_2(\alpha_1+\alpha_2)^2\alpha_1\alpha_2}{(A+B)^2} < 1.$$

We assume $\beta_1 \ge \beta_2 > 0$ for simplicity. So we have $a > b \ge 0$. We put $f_k(\omega) = \cos a\omega + k\sin(b\omega+\tau)$. Since $f_k(\omega) = 0$ implies $|\cos a\omega| = |k\sin(b\omega+\tau)| \le k$, all the positive zeros of $\varphi(\omega)$ are in the set $\{\omega > 0; |\cos a\omega| \le k\} = \bigcup_{n=1}^{\infty}\mathcal{I}_n(k)$, $\mathcal{I}_n(k) = [s_n(k), t_n(k)] \subset \mathcal{J}_n = [(n-1)\pi/a, n\pi/a]$, $n = 1, 2, \cdots$ where $s_n(k) = (2n-1)\pi/2a - \sin^{-1}k/a$, $t_n(k) = (2n-1)\pi/2a + \sin^{-1}k/a$. We write $\mathcal{I}_n = \mathcal{I}_n(k)$, $s_n = s_n(k)$, $t_n = t_n(k)$ and $f(\omega) = f_k(\omega)$. In Theorem 2 below, we prove that there exists exactly one zero of f in each $\mathcal{I}_n \subset \mathcal{J}_n$.

Theorem 2 *For each* $n \in \mathbf{N}$, *there exist* $u_n, v_n \in \mathcal{I}_n$ *such that*

(1) $f(u_n) = 1 - k = -f(v_n)$ *and* $f(\omega)$ *is monotone decreasing on* $[u_n, v_n]$ *for odd* n,

(2) $f(u_n) = k - 1 = -f(v_n)$ *and* $f(\omega)$ *is monotone increasing on* $[u_n, v_n]$ *for even* n,

(3) $|f(\omega)| \ge 1 - k$ *for* $\omega \in \mathcal{J}_n \backslash [u_n, v_n]$

(4) *only zero of* f *exists in* (u_n, v_n) *for any* n, *which implies that* $\mu_n \in \mathcal{I}_n$ *for any* $n \in \mathbf{N}$.

First, we show, for sufficiently small k, $\mu_n \in \mathcal{I}_n$ for every $n \in \mathbf{N}$.

Lemma 4 *Let* $k \in [0, 1/\sqrt{2}]$. *Then we have*

(1) *for odd* n, $f(s_n) \ge 0 \ge f(t_n)$ *and* $f(\omega)$ *is monotone decreasing on* $[s_n, t_n]$,

(2) *for even* n, $f(t_n) \ge 0 \ge f(s_n)$ *and* $f(\omega)$ *is monotone increasing on* $[s_n, t_n]$,

Consequently, in $\mathcal{J}_n$, $f(\omega)$ *has only one zero in* $[s_n, t_n]$.

Proof We only show (1). (2) is proved similarly.

$$\begin{aligned}f(s_n) &= \cos\left(\frac{(2n-1)\pi}{2} - \sin^{-1}k\right) - k\sin(bs_n+\tau) \\ &= (-1)^{n+1}k - k\sin(bs_n+\tau) = k - k\sin(bs_n+\tau) \ge 0 \\ f(t_n) &= (-1)^n k - k\sin(bs_n+\tau) = -k - k\sin(bs_n+\tau) \le -k + k = 0.\end{aligned}$$

In $[s_n, t_n]$, we have $\sin a\omega \ge 1/\sqrt{2}$ and

$$f'(\omega) = -a\sin a\omega + kb\cos(b\omega+\tau) \le -a/\sqrt{2} + b/\sqrt{2} < 0.$$

In the following, we put $\bar{k} = kb^2/a^2$, $\overline{s_n} = s_n(\bar{k})$, $\overline{t_n} = t_n(\bar{k})$, $\overline{\mathcal{I}_n} = \mathcal{I}_n(\bar{k})$, and $\overline{\mu_n} = \mu_n(\bar{k})$, $\bar{f}(\omega) = f_{\bar{k}}(\omega)$ and $\mathrm{S} = \{k \in [0,\, 1); \mu_n(k) \in \mathcal{I}_n(k) \subset \mathcal{J}_n$ for each $n \in \mathbf{N}\}$.

Lemma 5 *For $\bar{k} \in \mathrm{S}$, the conclusion of Theorem 2 is valid.*

Proof We have

$$\begin{aligned} f''(\omega) &= f''_k(\omega) = -a^2 \cos a\omega - kb^2 \sin(b\omega + \tau) \\ &= -a^2(\cos a\omega + \bar{k} \sin(b\omega + \tau)) = -a^2 f_{\bar{k}}(\omega) = -a^2 \bar{f}(\omega). \end{aligned}$$

Let n be odd. The case where n is even is also treated similarly. Then $\bar{f}((n-1)\pi/a) \geq 1 - k > 0 > k - 1 \geq \bar{f}(\pi/a)$. Since $\overline{\mu_n}$ is the only zero of $\bar{f}$ in $((n-1)\pi/a,\, n\pi/a)$, we have $f''(\omega) < 0$ for $\omega \in ((n-1)\pi/a,\, \overline{\mu_n})$, $f''(\omega) > 0$ for $\omega \in (\overline{\mu_n},\, n\pi/a)$. Thus $f(\omega)$ is concave on $((n-1)\pi/a,\, \mu_n)$ and convex on $(\mu_n,\, n\pi/a)$. Let $y_n,\, z_n \in \mathcal{J}_n$ with $f(y_n) = \max_{\omega \in \mathcal{I}_n} f(\omega) \geq 1 - k$ and $f(z_n) = \min_{\omega \in \mathcal{I}_n} f(\omega) \leq k - 1$. Then, we find $u_n,\, v_n$ with $s_n \leq y_n < \mu_n < v_n \leq z_n \leq t_n$ such that $f(u_n) = 1 - k$ and $f(v_n) = k - 1$. Thus, f is monotone decreasing on $[u_n,\, v_n] \subset [y_n,\, z_n]$.

Proof of Theorem 2 There exists $N \in \mathbf{N}$ such that $0 \leq (b/a)^{2N} \leq 1/2$. Let $k \in [0,\, 1)$ and $k_i = k(b/a)^{2i}$, $i = 0, 1, 2, \ldots, N$. Then $k_N \in \mathrm{S}$ by Lemma 4. Therefore, by Lemma 5, $k_i \in \mathrm{S}, i = 1, 2, \ldots, N-1$. In particular, $k_1 = k(b/a)^2 = \bar{k} \in \mathrm{S}$. Thus, by using Lemma 5 again, we can prove Theorem 2.

Next, we want to show that, for sufficiently large n, there exists only one zero of $d(\omega)$ in each $\mathcal{J}_n$. More precisely, we have

Theorem 3 *There exists M, $N \in \mathbf{N}$ such that $\omega_{M+n} \in \mathcal{J}_{N+n}$ for $n = 0, 1, \ldots$.*

To prove the above thoerem, we prepare Lemma 6 and lemma 7 given below:

Lemma 6 *For any $n \in \mathbf{N}$, the following inequality holds:*

$$|f'(\mu_n)| \geq \delta = \sqrt{(1-k^2)(a^2-b^2)}. \tag{11}$$

Proof Since μ_n, $n \in \mathbf{N}$, are zeros of f, we have

$$f(\mu_n) = \cos a\mu_n + k \sin(b\mu_n + \tau) = 0, \tag{12}$$

$$f'(\mu_n) = -a \sin a\mu_n + kb \cos(b\mu_n + \tau). \tag{13}$$

If $b = 0$, then $(f'(\mu_n))^2 = a^2 \sin^2 a\mu_n = a^2(1 - \cos^2 \tau) \geq a^2(1 - k^2)$. Hence we have (11). If $b \neq 0$, then

$$\begin{aligned} f'(\mu_n)^2 &= \frac{(a^2-b^2)}{b^2}\left(\sin a\mu_n - \frac{af'(\mu_n)}{a^2-b^2}\right)^2 + (1-k^2)(a^2-b^2) \\ &\geq (1-k^2)(a^2-b^2). \end{aligned}$$

Thus we have (11).

Lemma 7 *There exists an interval* $[a_n, b_n] \subset [u_n, v_n]$ *and* $l \in [0, 1-k)$ *such that*

$$|f(\omega)| \leq l, \ |f'(\omega)| \geq \delta/2 \quad \text{for } \omega \in [a_n, b_n], \tag{14}$$

$$|f(\omega)| \geq l \quad \text{for } \omega \in \mathcal{J}_n \backslash [a_n, b_n]. \tag{15}$$

Proof By uniform continuity of $f'(\omega)$ and Lemma 6, there exists $c > 0$ with $l = \delta c/2 < 1 - k$ such that

$$|f'(\omega)| \geq \delta/2 \quad \text{for } \omega \in [\mu_n - c, \mu_n + c]. \tag{16}$$

Therefore, we have $|f(\omega)| \geq (\delta/2)|\omega - \mu_n|$ on $[\mu_n - c, \mu_n + c]$. If n is odd (resp. even), we define a_n, b_n with $\mu_n - c < a_n < b_n < \mu_n + c$ by $f(a_n) = l$ (resp. $-l$) and $f(b_n) = -l$ (resp. l). Hence we have

$$\{\omega \in [\mu_n - c, \mu_n + c]; |f(\omega)| \leq l\} = [a_n, b_n]. \tag{17}$$

By (16) and (17), we see (14), and by Theorem 3, (15).

We put $g(\omega) = \varphi(\omega) - h(\omega) = Df(\omega) - h(\omega)$. Since $h(\omega)$, $h'(\omega) \to 0$ as $\omega \to \infty$, there exists $N \in \mathbf{N}$ such that $|h(\omega)| < Dl$ and $|h'(\omega)| < D\delta/2$ for $\omega > (N-1)\pi/a$.

Let n be odd with $n > N$. Then, by Lemma 7, we have $f(a_n) = l$, $f(b_n) = -l$ and $f'(\omega) < -\delta/2$ for $\omega \in [a_n, b_n]$. Hence $g(a_n) = Df(a_n) - h(a_n) = Dl - h(a_n) > Dl - Dl = 0$ and $g(b_n) = Df(b_n) - h(b_n) = Dl - h(b_n) < -Dl + Dl = 0$. Thus, for $\omega \in [a_n, b_n]$, $g'(\omega) = Df'(\omega) - h'(\omega) \leq -D\delta/2 + |h'(\omega)| < -D\delta/2 + D\delta/2 = 0$ which implies that $g(\omega)$ has a unique zero in (a_n, b_n). For $\omega \in \mathcal{J}_n \backslash [a_n, b_n]$, by (14), we have $|g(\omega)| \geq |Df(\omega)| - |h(\omega)| \geq Dl - Dl = 0$. Therefore, $g(\omega)$ has a unique zero in $\mathcal{J}_n$. The case with even $n \geq N$ is also similarly proved. Let ω_M be a zero of $g(\omega)$ in $\mathcal{J}_N$. Thus $\omega_{M+n} \in \mathcal{J}_{N+n}$ for $n = 0, 1, 2, \ldots$.

Proof of Theorem 1 Since $f(\mu_{N+n}) = g(\omega_{M+n}) = 0$, we have

$$h(\omega_{M+n}) = Df(\omega_{M+n}) - g(\omega_{M+n}) = D(f(\omega_{M+n}) - f(\mu_{N+n})). \tag{18}$$

By Mean Value Theorem, there exists $\theta \in (0, 1)$ such that $f(\omega_{M+n}) - f(\mu_{N+n}) = (\omega_{M+n} - \mu_{N+n})f'(\mu_{N+n} + \theta(\omega_{M+n} - \mu_{M+n}))$. Thus, by (14) and (18),

$$|\omega_{N+n} - \mu_{N+n}| \leq \frac{|f(\omega_{M+n}) - f(\mu_{N+n})|}{|f'(\mu_{N+n} + \theta(\omega_{M+n} - \mu_{M+n}))|} \leq \frac{2}{D\delta}|h(\omega_{M+n})| \to 0$$

as $n \to \infty$. Therefore,

$$\begin{aligned}
\liminf_{n\to\infty}(\omega_{n+1} - \omega_n) &= \liminf_{n\to\infty}(\omega_{M+n+1} - \omega_{M+n}) \\
&= \liminf_{n\to\infty}(\omega_{M+n+1} - \mu_{N+n+1} + \mu_{N+n+1} - \mu_{N+n} + \mu_{N+n} - \omega_{M+n}) \\
&= \liminf_{n\to\infty}(\mu_{N+n+1} - \mu_{N+n}) = \liminf_{n\to\infty}(\mu_{n+1} - \mu_n).
\end{aligned}$$

By $s_n < \mu_n < t_n < s_{n+1} < \mu_{n+1} < t_{n+1}$, we have

$$\begin{aligned}
\mu_{n+1} - \mu_n \geq s_{n+1} - t_n &= \left(\frac{2n+1}{2a}\pi - \frac{\sin^{-1}k}{a}\right) - \left(\frac{2n-1}{2a}\pi + \frac{\sin^{-1}k}{a}\right) \\
&= \frac{\pi}{a} - \frac{2\sin^{-1}k}{a} = \frac{1}{a}(\pi - 2\sin^{-1}k)
\end{aligned}$$

The above theorem implies that

$$\liminf_{n\to\infty}({\omega_{n+1}}^2 - {\omega_n}^2) \geq \liminf_{n\to\infty}(\omega_{n+1} + \omega_n)\liminf_{n\to\infty}(\omega_{n+1} - \omega_n) = \infty. \tag{19}$$

3 Concluding Remarks

This paper is only a first step to the controllability theory for the Euler-Bernoulli equation using the moment problem method [4], [12].

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