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<th>Title</th>
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</thead>
<tbody>
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Eigenvalue Problem related to Euler-Bernoulli Equation with Joint Boundary Condition

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Abstract
We consider the control system of two vibrating beams which are coupled at a joint. The displacement of the beam is described by an Euler-Bernoulli equation with control applied at a coupled point. Our purpose is to argue the controllability of the system. To this purpose, we discuss the eigenvalue problem related to this system.

1 Introduction
Let us consider the controllability problem for a system coupled by Euler-Bernoulli beams. For $m \in (0, 1)$, we put $x_0 = 0$, $x_1 = m$ and $x_2 = 1$. The displacement of each beam at time $t$ is described by $y_i(x, t)$ on $I_i = (x_{i-1}, x_i)$, $i = 1, 2$, and satisfies the Euler-Bernoulli equation:

$$\rho_i \ddot{y}_i + T_i y_i^{(4)} = 0 \quad \text{on } I_i \times (0, T)$$

where $\dot{y}_i(x, t) = \partial y_i(x, t)/\partial t$, $y_i^{(k)}(x, t) = \partial^k y_i(x, t)/\partial x^k$. $\rho_i$ is mass density and $T_i$ is flexural rigidity respectively on $I_i$. Let both ends be clamped:

$$(B_0 y)(t) := (y_1(0, t), y_1^{(1)}(0, t), y_2(1, t), y_2^{(1)}(1, t)) = 0.$$ 

At the coupled point $x = m$, we apply control $F = (f_1, f_2, f_3, f_4)$ as follows:

$$\begin{align*}
(B_1 y)(t) &:= y_1(m, t) - y_2(m, t) = f_1(t), \\
(B_2 y)(t) &:= y_1^{(1)}(m, t) - y_2^{(1)}(m, t) = f_2(t), \\
(B_3 y)(t) &:= T_1 y_1^{(2)}(m, t) - T_2 y_2^{(2)}(m, t) = f_3(t), \\
(B_4 y)(t) &:= T_1 y_1^{(3)}(m, t) - T_2 y_2^{(3)}(m, t) = f_4(t).
\end{align*}$$

Initial condition is given as follows

$$y_i(x, 0) = y_i^0(x), \quad \dot{y}_i(x, 0) = y_i^1(x), \quad x \in I_i, \ i = 1, 2.$$ 

We assume that controls $f_i$ belong to $L^2(0, T)$, $i = 1, 2, 3, 4$. In this paper, we treat controllability of the above system. Roughly speaking, the system (1)(2)(3)(4) is controllable if for any initial value $(y_i^0, y_i^1)$ and final value $(z_i^0, z_i^1)$, $i = 1, 2$, there exists a control $F = (f_1, f_2, f_3, f_4)$ such that the corresponding solution of the system (1)(2)(3)(4) satisfies the final condition $(y_i(x, T), \dot{y}_i(x, T)) = (z_i^0(x), z_i^1(x))$, $i = 1, 2$. 
2 Eigenvalue Problem

Let us identify $v \in L^2(I)$ with $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H = L^2(I) = L^2(I_1) \times L^2(I_2)$ where $v_i = v|_{I_i}$, $i = 1, 2$, $I_1 = (0, m)$, $I_2 = (m, 1)$. Then $H$ becomes a Hilbert space with inner product

$$(v, w) = \rho_1(v_1, w_1)_{L^2(I_1)} + \rho_2(v_2, w_2)_{L^2(I_2)}$$

for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in H$.

We define an operator $A$ in $H$ by

$$Av = \begin{pmatrix} (T_1/\rho_1)v_1^{(4)} \\ (T_2/\rho_2)v_2^{(4)} \end{pmatrix}$$

for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in D(A) := H^4(I_1) \times H^4(I_2)$ and an operator $\mathcal{A}$ by restricting $A$ to

$$D(\mathcal{A}) := \{ v \in H^4(I_1) \times H^4(I_2); \mathcal{B}_0v = 0, \mathcal{B}v := (\mathcal{B}_1v, \mathcal{B}_2v, \mathcal{B}_3v, \mathcal{B}_4v) = 0 \}.$$

For this operator $\mathcal{A}$, we have

**Lemma 1** The operator $\mathcal{A}$ is a selfadjoint operator in $H$ with compact resolvent.

The proof of this lemma is easy to verify.

Let $\lambda$ be an eigenvalue for $A$ with corresponding eigenfunction $\phi$. Then we have

$$A\phi = \lambda\phi$$

with boundary conditions

$$\mathcal{B}_0\phi = 0, \quad \mathcal{B}\phi = 0.$$  

We introduce functions $C_\pm, S_\pm$ by

$$C_\pm(\theta) := \frac{\cosh \theta \pm \cos \theta}{2}, \quad S_\pm(\theta) := \frac{\sinh \theta \pm \sin \theta}{2} \quad \text{for} \quad \theta \in \mathbb{R}.$$  

Let $\phi_i = \phi|_{I_i}$, $\alpha_i = (\rho_i/T_i)^{\frac{1}{4}}$, $i = 1, 2$. A system of fundamental solutions to (1) in each $I_i$ is given by $\{ C_\pm(\alpha_i \omega(x - x_{i-1}))$, $S_\pm(\alpha_i \omega(x - x_{i-1})) \}$ and we have

$$\phi_i(x) = (p_i^1 C_+ + p_i^1 S_+ + p_i^1 C_- + p_i^1 S_-)(\alpha_i \omega(x - x_{i-1})),
\begin{align*}
(\phi_i)^{(1)}(x) &= \alpha_i \omega(p_i^1 S_+ + p_i^1 C_+ + p_i^1 S_+ + p_i^1 C_-)(\alpha_i \omega(x - x_{i-1})),
(\phi_i)^{(2)}(x) &= \alpha_i^2 \omega^2(p_i^1 C_- + p_i^1 S_- + p_i^1 C_+ + p_i^1 S_+)(\alpha_i \omega(x - x_{i-1})),
(\phi_i)^{(3)}(x) &= \alpha_i^3 \omega^3(p_i^1 S_+ + p_i^1 C_+ + p_i^1 S_- + p_i^1 C_-)(\alpha_i \omega(x - x_{i-1}))
\end{align*}$$

for $x \in I_i$ where $\omega = \lambda^{\frac{1}{4}}$. By (2), we have $p_i^1 = p_i^2 = 0$ and therefore

$$\phi_1(x) = (p_2^1 C_- + p_2^1 S_+)(\alpha_1 \omega x),
\begin{align*}
(\phi_1)^{(1)}(x) &= \alpha_1 \omega(p_2^1 S_- + p_2^1 C_+)(\alpha_1 \omega x),
(\phi_1)^{(2)}(x) &= \alpha_1^2 \omega^2(p_2^1 C_+ + p_2^1 S_-)(\alpha_1 \omega x),
(\phi_1)^{(3)}(x) &= \alpha_1^3 \omega^3(p_2^1 S_- + p_2^1 C_+)(\alpha_1 \omega x).
\end{align*}$$

(3)
By (2), we have
\[
\gamma_2 \alpha_2 p^2 = \gamma_2 \alpha_2 \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} = \begin{pmatrix} \gamma_2 \alpha_2 C^1_1(\omega) & \gamma_2 \alpha_2 S^1_1(\omega) \\ \gamma_2 \alpha_1 S^1_1(\omega) & \gamma_2 \alpha_1 C^1_1(\omega) \\ \gamma_1 \alpha_2 C^1_1(\omega) & \gamma_1 \alpha_2 S^1_1(\omega) \\ \gamma_1 \alpha_1 S^1_1(\omega) & \gamma_1 \alpha_1 C^1_1(\omega) \end{pmatrix} \begin{pmatrix} p_1^3 \\ p_4^3 \end{pmatrix}
\]
(4)

where \( \gamma_i = T_i \alpha_i^2 \), \( i = 1, 2 \), \( \beta_1 = \alpha_1 m \), \( S^2_\pm(\omega) = S_\pm(\beta_1 \omega) \), \( C^2_\pm(\omega) = C_\pm(\beta_1 \omega) \).

By (2), we see
\[
\{ p_1^2, p_2^2, p_3^2, p_4^2 \} = 0,
\]
(5)

where \( \beta_2 = \alpha_2 (1 - m) \), \( S^2_\pm(\omega) = S_\pm(\beta_2 \omega) \), \( C^2_\pm(\omega) = C_\pm(\beta_2 \omega) \).

Let
\[
D(\omega) = \begin{pmatrix} D_{11}(\omega) & D_{12}(\omega) \\ D_{21}(\omega) & D_{22}(\omega) \end{pmatrix}
\]

Then
\[
D_{11}(\omega) = (\gamma_2 \alpha_2 C^2_+ \cdot C^-_1 + \gamma_2 \alpha_2 S^2_+ \cdot S^1_+ + \gamma_1 \alpha_2 C^2_- \cdot C^1_+ + \gamma_1 \alpha_1 S^2_- \cdot S^-_1)(\omega),
\]
\[
D_{12}(\omega) = (\gamma_2 \alpha_2 C^2_+ \cdot S^-_1 + \gamma_2 \alpha_1 S^2_+ \cdot C^-_1 + \gamma_1 \alpha_2 C^2_- \cdot S^1_+ + \gamma_1 \alpha_1 C^2_- \cdot C^1_+)(\omega),
\]
\[
D_{21}(\omega) = (\gamma_2 \alpha_2 S^2_+ \cdot C^1_+ + \gamma_2 \alpha_1 C^2_+ \cdot S^1_+ + \gamma_1 \alpha_2 S^2_- \cdot C^-_1 + \gamma_1 \alpha_1 C^2_- \cdot S^-_1)(\omega),
\]
\[
D_{22}(\omega) = (\gamma_2 \alpha_2 S^2_- \cdot S^-_1 + \gamma_1 \alpha_2 C^2_+ \cdot C^-_1 + \gamma_1 \alpha_2 S^2_- \cdot S^1_+ + \gamma_1 \alpha_1 C^2_- \cdot C^-_1)(\omega).
\]

We put
\[
d(\omega) := 4 \det D(\omega)
\]
\[
= 4 \gamma^2_2 \alpha_2 \alpha_1 (S^2_+ \cdot S^2_- - C^2_+ \cdot C^2_-) \cdot (S^1_+ \cdot S^1_- - C^1_+ \cdot C^1_-)(\omega)
\]
\[
+ 4 \gamma \gamma_1 \alpha_2 (S^2_+ \cdot C^2_- - S^2_- \cdot C^2_+) \cdot (S^1_+ \cdot C^1_- - C^1_+ \cdot S^1_-)(\omega)
\]
\[
+ 8 \gamma \gamma_1 \alpha_2 \alpha_1 (S^2_+ \cdot S^2_- - C^2_+ \cdot C^2_-) \cdot (S^1_+ \cdot S^1_- - C^1_+ \cdot C^1_-)(\omega)
\]
\[
+ 4 \gamma \gamma_1 \alpha_2 (S^2_+ \cdot C^2_- - C^2_+ \cdot S^2_-) \cdot (S^1_+ \cdot C^1_- - S^1_- \cdot C^1_+)(\omega)
\]
\[
+ 4 \gamma \alpha_1 \alpha_2 (S^2_+ \cdot S^2_- - C^2_+ \cdot C^2_-) \cdot (S^1_+ \cdot S^1_- - C^1_+ \cdot C^1_-)(\omega).
\]
(6)

By (4), (5), we have
\[
D(\omega) \begin{pmatrix} p_3^1 \\ p_4^1 \end{pmatrix} = 0.
\]

Since \( \phi \) is an eigenfunction if and only if \( (p_3^1, p_4^1) \neq 0 \), we see that \( \lambda = \omega^4 \) is an eigenvalue of \( \mathcal{A} \) if \( d(\omega) = 0, \omega > 0 \). Let \( \omega_n, n \in \mathbb{N} \), is the \( n \)-th positive zero of \( d(\omega) \). Then \( \lambda_n := \omega_n^4 \),
$0 < \omega_1 < \omega_2 < \cdots$, is the $n$-th eigenvalue of $A$. We can verify that $\lambda_n$ is a simple eigenvalue. Let $\phi^n$ be an eigenfunction corresponding to $\lambda_n$, normalized in $H$. In the following, let $\varphi(\omega)$ be a function defined by

$$\varphi(\omega) = A \cos \beta_1 \omega \cos \beta_2 \omega - B \sin \beta_1 \omega \sin \beta_2 \omega + C \sin(\beta_1 - \beta_2) \omega \quad (7)$$

where $A = (\gamma_1 \alpha_1 + \gamma_2 \alpha_2)(\gamma_1 \alpha_2 + \gamma_2 \alpha_1)$, $B = \gamma_1 \gamma_2 (\alpha_1 + \alpha_2)^2$, $C = \gamma_1 \gamma_2 (\alpha_1^2 - \alpha_2^2)$. We denote the $n$-th positive zero of $\varphi$ by $\mu_n$.

**Lemma 2** $d(\omega)$ is written as

$$d(\omega) = e^{(\beta_1 + \beta_2) \omega}(\varphi(\omega) - h(\omega)), \quad \omega \in \mathbb{R}$$

where $h(\omega) \in C^1(\mathbb{R})$ and $h(\omega) \to 0$, $h'(\omega) \to 0$ exponentially as $\omega \to \infty$.

**Proof** By (6), $h(\omega) = \varphi(\omega) - e^{-(\beta_1 + \beta_2) \omega}d(\omega)$ and $h'(\omega)$ converge to 0 exponentially as $\omega \to \infty$.

To discuss controllability, we treat the moment problem on the system (1)(2)(3)(4). According to Krabs [4] or Russell [12], to solve the moment problem, we need the following conditions:

$$\lim_{n \to \infty} \inf_{\omega} (\omega_{n+1}^2 - \omega_n^2) > \frac{2\pi}{T}, \quad \lim_{\omega \to \infty} \frac{d(x+y) - d(x)}{y} < \frac{T}{2\pi} \quad (9)$$

where $d(x) =$ number of $\omega_j$ with $\omega_j < x^2$. The aim of this paper is to prove the following

**Theorem 1** We have

(1) There exist $M$ and $N$ such that $\omega_{M+n} - \mu_{N+n} \to 0$,

(2) $0 < \frac{1}{a}(\pi - \sin^{-1} k) \leq \lim_{n \to \infty} (\omega_{n+1} - \omega_n) \leq \lim_{n \to \infty} (\omega_{n+1} - \omega_n) < \infty$,

(3) $\lim_{n \to \infty} (\omega_{n+1}^2 - \omega_n^2) = \infty$.

By this theorem, it is clear that $\{\omega_n\}_{n \in \mathbb{N}}$ satisfies the condition (8). Moreover, the condition (8) verify the condition (9).

Some simple facts for $\varphi(\omega)$ are given in the following

**Lemma 3** In the formula (7), we have

(1) $A \geq B > 0$,

(2) $A = B$ if and only if $\rho_1 T_1 = \rho_2 T_2$.

(3) $C = 0$ if and only if $\rho_1 T_2 = \rho_2 T_1$.

(4) $A = B$, $C = 0$ if and only if $(\rho_1, T_1) = (\rho_2, T_2)$.

(5) $A > B$ or $C \neq 0$ if and only if $(\rho_1, T_1) \neq (\rho_2, T_2)$. 


(6) \( \varphi \) is written as
\[
\varphi(\omega) = D(\cos a\omega + k \sin(b\omega + \tau)), \quad 0 \leq k < 1
\]  
where \( D = (A+B)/2, \ a = \beta_1 + \beta_2, \ b = \beta_1 - \beta_2, \ k = R/D, \ R = \sqrt{((A-B)/2)^2 + C^2}, \)
\[
\tau = \cos^{-1}(C/R) \in [0, \pi] \text{ for } R \neq 0 \text{ and } \tau = 0 \text{ for } R = 0.
\]

**Proof** We see (1) from \( A = B + \alpha_1 \alpha_2 (\gamma_1 - \gamma_2)^2 \geq B > 0 \). The assertions (2), (3), (4) and (5) are clear. We have (6) since
\[
\varphi(\omega) = \frac{A + B}{2} \cos(\beta_1 + \beta_2)\omega + \frac{A - B}{2} \cos(\beta_1 - \beta_2)\omega + C \sin(\beta_1 - \beta_2)\omega
\]
\[
= D \cos a\omega + R \sin(b\omega + \tau) = D(\cos a\omega + k \sin(b\omega + \tau)).
\]
where \( k \geq 0 \) satisfies
\[
k^2 = \frac{R^2}{D^2} = \frac{(A + B)^2 - 4(AB - C^2)}{(A + B)^2} = 1 - \frac{4(\gamma_1 + \gamma_2)^2 \gamma_1 \gamma_2 (\alpha_1 + \alpha_2)^2 \alpha_1 \alpha_2}{(A + B)^2} < 1.
\]

We assume \( \beta_1 \geq \beta_2 > 0 \) for simplicity. So we have \( a > b \geq 0 \). We put \( f_k(\omega) = \cos a\omega + k \sin(b\omega + \tau) \). Since \( f_k(\omega) = 0 \) implies \(|\cos a\omega| = |k \sin(b\omega + \tau)| \leq k\), all the positive zeros of \( \varphi(\omega) \) are in the set \( \{ \omega > 0; |\cos a\omega| \leq k \} = \bigcup_{n=1}^{\infty} \mathcal{I}_n(k), \ \mathcal{I}_n(k) = [s_n(k), \ t_n(k)] \subset \mathcal{J}_n = [(n - 1)\pi/a, n\pi/a], \ n = 1, 2, \cdots \) where \( s_n(k) = (2n - 1)\pi/2a - \sin^{-1}k/a, \ t_n(k) = (2n - 1)\pi/2a + \sin^{-1}k/a \). We write \( \mathcal{I}_n = \mathcal{I}_n(k), \ s_n = s_n(k), \ t_n = t_n(k) \) and \( f(\omega) = f_k(\omega) \). In Theorem 2 below, we prove that there exists exactly one zero of \( f \) in each \( \mathcal{I}_n \subset \mathcal{J}_n \).

**Theorem 2** For each \( n \in \mathbb{N} \), there exist \( u_n, v_n \in \mathcal{I}_n \) such that
1. \( f(u_n) = 1 - k = -f(v_n) \) and \( f(\omega) \) is monotone decreasing on \([u_n, v_n]\) for odd \( n \),
2. \( f(u_n) = k - 1 = -f(v_n) \) and \( f(\omega) \) is monotone increasing on \([u_n, v_n]\) for even \( n \),
3. \( |f(\omega)| \geq 1 - k \) for \( \omega \in \mathcal{J}_n \setminus [u_n, v_n] \)
4. only zero of \( f \) exists in \((u_n, v_n)\) for any \( n \), which implies that \( \mu_n \in \mathcal{I}_n \) for any \( n \in \mathbb{N} \).

First, we show, for sufficiently small \( k \), \( \mu_n \in \mathcal{I}_n \) for every \( n \in \mathbb{N} \).

**Lemma 4** Let \( k \in [0, 1/\sqrt{2}] \). Then we have
1. for odd \( n \), \( f(s_n) \geq 0 \geq f(t_n) \) and \( f(\omega) \) is monotone decreasing on \([s_n, t_n]\),
2. for even \( n \), \( f(t_n) \geq 0 \geq f(s_n) \) and \( f(\omega) \) is monotone increasing on \([s_n, t_n]\),

Consequently, in \( \mathcal{J}_n \), \( f(\omega) \) has only one zero in \([s_n, t_n]\).

**Proof** We only show (1). (2) is proved similarly.
\[
f(s_n) = \cos \left( \frac{(2n-1)\pi}{2} - \sin^{-1}k \right) - k \sin(bs_n + \tau)
\]
\[
= (-1)^{n+1}k - k \sin(bs_n + \tau) \geq 0
\]
\[
f(t_n) = (-1)^n k - k \sin(bs_n + \tau) = -k - k \sin(bs_n + \tau) \leq -k + k = 0.
\]
In \([s_n, t_n]\), we have \( \sin a\omega \geq 1/\sqrt{2} \) and
\[
f'(\omega) = -a \sin a\omega + k b \cos(b\omega + \tau) \leq -a/\sqrt{2} + b/\sqrt{2} < 0.
\]
In the following, we put \( \bar{k} = kb^{2}/a^{2}, \bar{s}_{n} = s_{n}(k), \bar{t}_{n} = t_{n}(k), \bar{I}_{n} = I_{n}(k), \) and \( \bar{\mu}_{n} = \mu_{n}(k), \)
\( \bar{f}(\omega) = f_{k}(\omega) \) and \( S = \{ k \in [0, 1); \mu_{n}(k) \in I_{n}(k) \subset J_{n} \) for each \( n \in N \}.

**Lemma 5** For \( \bar{k} \in S \), the conclusion of Theorem 2 is valid.

**Proof** We have
\[
f''(\omega) = f''(\omega) = -a^{2} \cos \omega - kb^{2} \sin(b \omega + \tau) = -a^{2}(\cos \omega + \bar{k} \sin(b \omega + \tau)) = -a^{2} f_{k}(\omega) = -a^{2} \bar{f}(\omega).
\]

Let \( n \) be odd. The case where \( n \) is even is also treated similarly. Then \( \bar{f}((n-1)\pi/a) \geq 1 - k > 0 \) \( k - 1 \geq \bar{f}(\pi/a) \). Since \( \bar{\mu}_{n} \) is the only zero of \( \bar{f} \) in \( ((n-1)\pi/a, n\pi/a) \), we have \( f''(\omega) < 0 \) for \( \omega \in ((n-1)\pi/a, \bar{\mu}_{n}) \), \( f''(\omega) > 0 \) for \( \omega \in (\bar{\mu}_{n}, n\pi/a) \). Thus \( f(\omega) \) is concave on \( ((n-1)\pi/a, \mu_{n}) \) and convex on \( (\mu_{n}, n\pi/a) \). Let \( y_{n}, z_{n} \in J_{n} \) with \( f(y_{n}) = \max_{\omega \in I_{n}} f(\omega) \geq 1 - k \) and \( f(z_{n}) = \min_{\omega \in I_{n}} f(\omega) \leq k - 1 \). Then, we find \( u_{n}, v_{n} \) with \( s_{n} \leq y_{n} < \mu_{n} < v_{n} \leq z_{n} \leq t_{n} \) such that \( f(u_{n}) = 1 - k \) and \( f(v_{n}) = k - 1 \). Thus, \( f \) is monotone decreasing on \( [u_{n}, v_{n}] \subset [y_{n}, z_{n}] \).

**Proof of Theorem 2** There exists \( N \in N \) such that \( 0 \leq (b/a)^{2N} \leq 1/2 \). Let \( k \in [0, 1) \) and \( k_{i} = k(b/a)^{2i}, i = 0, 1, 2, \ldots, N \). Then \( k_{N} \in S \) by Lemma 4. Therefore, by Lemma 5, \( k_{i} \in S, i = 1, 2, \ldots, N - 1 \). In particular, \( k_{1} = k(b/a)^{2} = \bar{k} \in S \). Thus, by using Lemma 5 again, we can prove Theorem 2.

Next, we want to show that, for sufficiently large \( n \), there exists only one zero of \( d(\omega) \) in each \( J_{n} \). More precisely, we have

**Theorem 3** There exists \( M, N \in N \) such that \( \omega_{M+n} \in J_{N+n} \) for \( n = 0, 1, \ldots \).

To prove the above theorem, we prepare Lemma 6 and lemma 7 given below:

**Lemma 6** For any \( n \in N \), the following inequality holds:
\[
|f'(\mu_{n})| \geq \delta = \sqrt{(1 - k^{2})(a^{2} - b^{2})}.
\]

**Proof** Since \( \mu_{n}, n \in N \), are zeros of \( f \), we have
\[
f(\mu_{n}) = \cos a \mu_{n} + k \sin(b \mu_{n} + \tau) = 0, \tag{12}
f'(\mu_{n}) = -a \sin a \mu_{n} + kb \cos(b \mu_{n} + \tau). \tag{13}
\]

If \( b = 0 \), then \( (f'(\mu_{n}))^{2} = a^{2} \sin^{2} a \mu_{n} = a^{2}(1 - \cos^{2} \tau) \geq a^{2}(1 - k^{2}) \). Hence we have (11). If \( b \neq 0 \), then
\[
f'(\mu_{n})^{2} = \left( \frac{a^{2} - b^{2}}{b^{2}} \right) \left( \sin a \mu_{n} - \frac{a f'(\mu_{n})}{a^{2} - b^{2}} \right)^{2} + (1 - k^{2})(a^{2} - b^{2}) \geq (1 - k^{2})(a^{2} - b^{2}).
\]

Thus we have (11).
Lemma 7 There exists an interval \([a_n, b_n] \subset [u_n, v_n]\) and \(l \in [0, 1 - k]\) such that
\[
|f(\omega)| \leq l, \quad |f'(\omega)| \geq \delta/2 \quad \text{for } \omega \in [a_n, b_n],
\]
\[
|f(\omega)| \geq l \quad \text{for } \omega \in \mathcal{J}_n \backslash [a_n, b_n].
\]

Proof By uniform continuity of \(f'(\omega)\) and Lemma 6, there exists \(c > 0\) with \(l = \delta c/2 < 1 - k\) such that
\[
|f'(\omega)| \geq \delta/2 \quad \text{for } \omega \in [\mu_n - c, \mu_n + c].
\]
Therefore, we have \(|f(\omega)| \geq (\delta/2)|\omega - \mu_n|\) on \([\mu_n - c, \mu_n + c]\). If \(n\) is odd (resp. even), we define \(a_n, b_n\) with \(\mu_n - c < a_n < b_n < \mu_n + c\) by \(f(a_n) = l\) (resp. \(-l\)) and \(f(b_n) = -l\) (resp. \(l\)). Hence we have
\[
\{\omega \in [\mu_n - c, \mu_n + c]; |f(\omega)| \leq l\} = [a_n, b_n].
\]
By (16) and (17), we see (14), and by Theorem 3, (15).

We put \(g(\omega) = \varphi(\omega) - h(\omega) = Df(\omega) - h(\omega)\). Since \(h(\omega), h'(\omega) \to 0\) as \(\omega \to \infty\), there exists \(N \in \mathbb{N}\) such that \(|h(\omega)| < DI\) and \(|h'(\omega)| < D\delta/2\) for \(\omega > (N - 1)\pi/a\).

Let \(n\) be odd with \(n > N\). Then, by Lemma 7, we have \(f(a_n) = l, f(b_n) = -l\) and \(f'(\omega) < -\delta/2\) for \(\omega \in [a_n, b_n]\). Hence \(g(a_n) = Df(a_n) - h(a_n) = DI - h(a_n) > DI - DI = 0\) and \(g(b_n) = Df(b_n) - h(b_n) = DI - h(b_n) < -DI + DI = 0\). Thus, for \(\omega \in [a_n, b_n]\), \(g'(\omega) = Df'(\omega) - h'(\omega) \leq -D\delta/2 + |h'(\omega)| < -D\delta/2 + D\delta/2 = 0\) which implies that \(g(\omega)\) has a unique zero in \((a_n, b_n)\). For \(\omega \in \mathcal{J}_n \backslash [a_n, b_n]\), by (14), we have \(|g(\omega)| \geq |Df(\omega)| - |h(\omega)| \geq DI - DI = 0\). Therefore, \(g(\omega)\) has a unique zero in \(\mathcal{J}_n\). The case with even \(n > N\) is similarly proved. Let \(\omega_M\) be a zero of \(g(\omega)\) in \(\mathcal{J}_n\). Thus \(\omega_{M+n} \in \mathcal{J}_{N+n}\) for \(n = 0, 1, 2, \ldots\).

Proof of Theorem 1 Since \(f(\mu_{N+n}) = g(\omega_{M+n}) = 0\), we have
\[
h(\omega_{M+n}) = Df(\omega_{M+n}) - g(\omega_{M+n}) = D(f(\omega_{M+n}) - f(\mu_{N+n})).
\]
By Mean Value Theorem, there exists \(\theta \in (0, 1)\) such that \(f(\omega_{M+n}) - f(\mu_{N+n}) = (\omega_{M+n} - \mu_{N+n})f'(\mu_{N+n} + \theta(\omega_{M+n} - \mu_{N+n}))\). Thus, by (14) and (18),
\[
|\omega_{N+n} - \mu_{N+n}| \leq \frac{|f(\omega_{M+n}) - f(\mu_{N+n})|}{|f'(\mu_{N+n} + \theta(\omega_{M+n} - \mu_{N+n}))|} \leq \frac{2}{D\delta}|h(\omega_{M+n})| \to 0
\]
as \(n \to \infty\). Therefore,
\[
\liminf_{n \to \infty}(\omega_{N+n} - \omega_M) = \liminf_{n \to \infty}(\omega_{M+n+1} - \omega_{M+n}) = \liminf_{n \to \infty}(\omega_{M+n+1} - \mu_{N+n+1} + \mu_{N+n+1} - \mu_{N+n} + \mu_{N+n} - \omega_{M+n}) = \liminf_{n \to \infty}(\mu_{N+n+1} - \mu_{N+n}) = \liminf_{n \to \infty}(\mu_{N+n+1} - \mu_{N+n}).
\]

By \(s_n < \mu_n < t_n < s_{n+1} < \mu_{n+1} < t_{n+1}\), we have
\[
\mu_{n+1} - \mu_n \geq s_{n+1} - t_n = \left(\frac{2n + 1}{2a}\pi - \frac{\sin^{-1}k}{a}\right) - \left(\frac{2n - 1}{2a}\pi + \frac{\sin^{-1}k}{a}\right) = \frac{\pi}{a} - \frac{2\sin^{-1}k}{a} = \frac{1}{a}(\pi - 2\sin^{-1}k).
\]
The above theorem implies that
\[
\liminf_{n \to \infty}(\omega_{n+1} - \omega_n) \geq \liminf_{n \to \infty}(\omega_{n+1} + \omega_n) \liminf_{n \to \infty}(\omega_{n+1} - \omega_n) = \infty.
\]
3 Concluding Remarks

This paper is only a first step to the controllability theory for the Euler-Bernoulli equation using the moment problem method [4], [12].

References


