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Eigenvalue Problem related to Euler-Bernoulli Equation with Joint Boundary Condition

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Abstract

We consider the control system of two vibrating beams which are coupled at a joint. The displacement of the beam is described by an Euler-Bernoulli equation with control applied at a coupled point. Our purpose is to argue the controllability of the system. To this purpose, we discuss the eigenvalue problem related to this system.

1 Introduction

Let us consider the controllability problem for a system coupled by Euler-Bernoulli beams. For \( m \in (0, 1) \), we put \( x_0 = 0, x_1 = m \) and \( x_2 = 1 \). The displacement of each beam at time \( t \) is described by \( y_i(x, t) \) on \( I_i = (x_{i-1}, x_i) \), \( i = 1, 2 \), and satisfies the Euler-Bernoulli equation:

\[
\rho_i \ddot{y}_i + T_i \dot{y}_i^{(4)} = 0 \quad \text{on } I_i \times (0, T)
\]

where \( \dot{y}_i(x, t) = \partial y_i(x, t)/\partial t \), \( y_i^{(k)}(x, t) = \partial^k y_i(x, t)/\partial x^k \). \( \rho_i \) is mass density and \( T_i \) is flexural rigidity respectively on \( I_i \). Let both ends be clamped:

\[
(B_0 y)(t) := (y_1(0, t), y_1^{(1)}(0, t), y_2(1, t), y_2^{(1)}(1, t)) = 0.
\]

At the coupled point \( x = m \), we apply control \( F = (f_1, f_2, f_3, f_4) \) as follows:

\[
\begin{align*}
(B_1 y)(t) &:= y_1(m, t) - y_2(m, t) = f_1(t), \\
(B_2 y)(t) &:= y_1^{(1)}(m, t) - y_2^{(1)}(m, t) = f_2(t), \\
(B_3 y)(t) &:= T_1 y_1^{(2)}(m, t) - T_2 y_2^{(2)}(m, t) = f_3(t), \\
(B_4 y)(t) &:= T_1 y_1^{(3)}(m, t) - T_2 y_2^{(3)}(m, t) = f_4(t).
\end{align*}
\]

Initial condition is given as follows

\[
y_i(x, 0) = y_i^0(x), \quad \dot{y}_i(x, 0) = y_i^1(x), \quad x \in I_i, \ i = 1, 2.
\]

We assume that controls \( f_i \) belong to \( L^2(0, T) \), \( i = 1, 2, 3, 4 \). In this paper, we treat controllability of the above system. Roughly speaking, the system (1)(2)(3)(4) is controllable if for any initial value \( (y_i^0, y_i^1) \) and final value \( (z_i^0, z_i^1) \), \( i = 1, 2 \), there exists a control \( F = (f_1, f_2, f_3, f_4) \) such that the corresponding solution of the system (1)(2)(3)(4) satisfies the final condition \( (y_i(x, T), \dot{y}_i(x, T)) = (z_i^0(x), z_i^1(x)) \), \( i = 1, 2 \).
2 Eigenvalue Problem

Let us identify $v \in L^2(I)$ with $
abla \in H = L^2(I) = L^2(I_1) \times L^2(I_2)$ where $v_i = v|_{I_i}$, $i = 1, 2$, $I_1 = (0, m)$, $I_2 = (m, 1)$. Then $H$ becomes a Hilbert space with inner product

$$(v, w) = \rho_1(v_1, w_1)_{L^2(I_1)} + \rho_2(v_2, w_2)_{L^2(I_2)}$$

for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in H$.

We define an operator $A$ in $H$ by

$$Av = \begin{pmatrix} T_1 / \rho_1 v_1^{(4)} \\ T_2 / \rho_2 v_2^{(4)} \end{pmatrix}$$

for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in D(A) := H^4(I_1) \times H^4(I_2)$

and an operator $\mathcal{A}$ by restricting $A$ to

$$D(\mathcal{A}) := \{ v \in H^4(I_1) \times H^4(I_2); B_0v = 0, Bv := (B_1v, B_2v, B_3v, B_4v) = 0 \}.$$

For this operator $\mathcal{A}$, we have

**Lemma 1** The operator $\mathcal{A}$ is a selfadjoint operator in $H$ with compact resolvent.

The proof of this lemma is easy to verify.

Let $\lambda$ be an eigenvalue for $\mathcal{A}$ with corresponding eigenfunction $\phi$. Then we have

$$A\phi = \lambda \phi$$

with boundary conditions

$$B_0\phi = 0, \quad B\phi = 0.$$

We introduce functions $C_\pm, S_\pm$ by

$$C_\pm(\theta) := \frac{\cosh \theta \pm \cos \theta}{2}, \quad S_\pm(\theta) := \frac{\sinh \theta \pm \sin \theta}{2}$$

for $\theta \in \mathbb{R}$.

Let $\phi_i = \phi|_{I_i}$, $\alpha_i = (\rho_i/T_i)^{\frac{1}{4}}$, $i = 1, 2$. A system of fundamental solutions to (1) in each $I_i$ is given by $\{ C_\pm(\alpha_i \omega(x - x_{i-1}))$, $S_\pm(\alpha_i \omega(x - x_{i-1})) \}$ and we have

$$\begin{align*}
\phi_i(x) &= (p_1 C_+ + p_2 S_+ + p_3 C_- + p_4 S_-)(\alpha_i \omega(x - x_{i-1})), \\
(\phi_i)^{(1)}(x) &= \alpha_i \omega(p_1 C_+ + p_2 S_+ + p_3 C_- + p_4 S_-)(\alpha_i \omega(x - x_{i-1})), \\
(\phi_i)^{(2)}(x) &= \alpha_i^2 \omega^2(p_1 C_+ + p_2 S_+ + p_3 C_- + p_4 S_-)(\alpha_i \omega(x - x_{i-1})), \\
(\phi_i)^{(3)}(x) &= \alpha_i^3 \omega^3(p_1 C_+ + p_2 S_+ + p_3 C_- + p_4 S_-)(\alpha_i \omega(x - x_{i-1}))
\end{align*}$$

for $x \in I_i$ where $\omega = \lambda^{\frac{1}{4}}$. By (2), we have $p_1 = p_2 = 0$ and therefore

$$\begin{align*}
\phi_i(x) &= (p_3 C_- + p_4 S_-)(\alpha_i \omega x), \\
(\phi_i)^{(1)}(x) &= \alpha_i \omega(p_3 C_- + p_4 S_-)(\alpha_i \omega x), \\
(\phi_i)^{(2)}(x) &= \alpha_i^2 \omega^2(p_3 C_- + p_4 S_-)(\alpha_i \omega x), \\
(\phi_i)^{(3)}(x) &= \alpha_i^3 \omega^3(p_3 C_- + p_4 S_-)(\alpha_i \omega x).
\end{align*}$$

(3)
By (2), we have
\[
\gamma_2 \alpha_2 p^2 = \gamma_2 \alpha_2 \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} = \begin{pmatrix} \gamma_2 \alpha_2 C_+^1(\omega) & \gamma_2 \alpha_2 S_+^1(\omega) \\ \gamma_2 \alpha_1 S_+^1(\omega) & \gamma_2 \alpha_1 C_+^1(\omega) \\ \gamma_1 \alpha_2 C_+^1(\omega) & \gamma_1 \alpha_2 S_+^1(\omega) \\ \gamma_1 \alpha_1 S_+^1(\omega) & \gamma_1 \alpha_1 C_+^1(\omega) \end{pmatrix} \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix}
\]
where \( \gamma_i = T_i \alpha_i^2 \), \( i = 1, 2 \), \( \beta_1 = \alpha_1 m \), \( S_{\pm}^2(\omega) = S_{\pm}(\beta_1 \omega) \), \( C_{\pm}^2(\omega) = C_{\pm}(\beta_1 \omega) \). By (2), we see
\[
\begin{pmatrix} C_+^2(\omega) & S_+^2(\omega) & C_+^2(\omega) & S_+^2(\omega) \\ S_+^2(\omega) & C_+^2(\omega) & S_+^2(\omega) & C_+^2(\omega) \end{pmatrix} \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} = 0,
\]
where \( \beta_2 = \alpha_2(1 - m) \), \( S_{\pm}^2(\omega) = S_{\pm}(\beta_2 \omega) \), \( C_{\pm}^2(\omega) = C_{\pm}(\beta_2 \omega) \). Let
\[
D(\omega) = \begin{pmatrix} D_{11}(\omega) & D_{12}(\omega) \\ D_{21}(\omega) & D_{22}(\omega) \end{pmatrix} := \begin{pmatrix} C_+^2(\omega) & S_+^2(\omega) & C_+^2(\omega) & S_+^2(\omega) \\ S_+^2(\omega) & C_+^2(\omega) & S_+^2(\omega) & C_+^2(\omega) \end{pmatrix} \begin{pmatrix} \gamma_2 \alpha_2 C_+^1(\omega) & \gamma_2 \alpha_2 S_+^1(\omega) \\ \gamma_2 \alpha_1 S_+^1(\omega) & \gamma_2 \alpha_1 C_+^1(\omega) \\ \gamma_1 \alpha_2 C_+^1(\omega) & \gamma_1 \alpha_2 S_+^1(\omega) \\ \gamma_1 \alpha_1 S_+^1(\omega) & \gamma_1 \alpha_1 C_+^1(\omega) \end{pmatrix}.
\]
Then
\[
D_{11}(\omega) = (\gamma_2 \alpha_2 C_+^2 \cdot C_+^1 + \gamma_2 \alpha_1 S_+^2 \cdot S_+^1 + \gamma_1 \alpha_2 C_+^2 \cdot C_+^1 + \gamma_1 \alpha_1 S_+^2 \cdot S_+^1)(\omega),
\]
\[
D_{12}(\omega) = (\gamma_2 \alpha_2 C_+^2 \cdot S_+^1 + \gamma_2 \alpha_1 S_+^2 \cdot C_+^1 + \gamma_1 \alpha_2 C_+^2 \cdot S_+^1 + \gamma_1 \alpha_1 S_+^2 \cdot C_+^1)(\omega),
\]
\[
D_{21}(\omega) = (\gamma_2 \alpha_2 S_+^2 \cdot C_+^1 + \gamma_2 \alpha_1 C_+^2 \cdot S_+^1 + \gamma_1 \alpha_2 S_+^2 \cdot C_+^1 + \gamma_1 \alpha_1 C_+^2 \cdot S_+^1)(\omega),
\]
\[
D_{22}(\omega) = (\gamma_2 \alpha_2 S_+^2 \cdot S_+^1 + \gamma_2 \alpha_1 C_+^2 \cdot C_+^1 + \gamma_1 \alpha_2 S_+^2 \cdot S_+^1 + \gamma_1 \alpha_1 C_+^2 \cdot C_+^1)(\omega).
\]
We put
\[
d(\omega) := 4 \det D(\omega)
\]
\[
= 4\gamma_2^2 \alpha_2 \alpha_1 (S_+^2 \cdot S_+^2 - C_+^2 \cdot C_+^2) \cdot (S_+^1 \cdot S_+^1 - C_+^1 \cdot C_+^1)(\omega)
+ 4\gamma_2 \gamma_1 \alpha_2 (S_+^2 \cdot C_+^2 - S_+^2 \cdot C_+^2) \cdot (S_+^1 \cdot C_+^1 - C_+^1 \cdot S_+^1)(\omega)
+ 8\gamma_2 \gamma_1 \alpha_2 \alpha_1 (S_+^2 \cdot S_+^2 - C_+^2 \cdot C_+^2) \cdot (S_+^1 \cdot S_+^1 - C_+^1 \cdot C_+^1)(\omega)
+ 4\gamma_2 \gamma_1 \alpha_1 \alpha_2 (S_+^2 \cdot C_+^2 - C_+^2 \cdot S_+^2) \cdot (S_+^1 \cdot C_+^1 - C_+^1 \cdot S_+^1)(\omega)
+ 4\gamma_1^2 \alpha_1 \alpha_2 (S_+^2 \cdot S_+^2 - C_+^2 \cdot C_+^2) \cdot (S_+^1 \cdot S_+^1 - C_+^1 \cdot C_+^1)(\omega).
\]
By (4), (5), we have
\[
D(\omega) \begin{pmatrix} p_3^1 \\ p_4^1 \end{pmatrix} = 0.
\]
Since \( \phi \) is an eigenfunction if and only if \((p_3^1, p_4^1) \neq 0\), we see that \( \lambda = \omega^4 \) is an eigenvalue of \( \mathcal{A} \) if \( d(\omega) = 0 \), \( \omega > 0 \). Let \( \omega_n, n \in \mathbb{N} \), is the \( n \)-th positive zero of \( d(\omega) \). Then \( \lambda_n := \omega_n^4 \),
$0 < \omega_1 < \omega_2 < \cdots$, is the $n$-th eigenvalue of $A$. We can verify that $\lambda_n$ is a simple eigenvalue. Let $\phi^n$ be an eigenfunction corresponding to $\lambda_n$, normalized in $H$. In the following, let $\varphi(\omega)$ be a function defined by

$$\varphi(\omega) = A \cos \beta_1 \omega \cos \beta_2 \omega - B \sin \beta_1 \omega \sin \beta_2 \omega + C \sin (\beta_1 - \beta_2) \omega \quad (7)$$

where $A = (\gamma_1 \alpha_1 + \gamma_2 \alpha_2)(\gamma_1 \alpha_2 + \gamma_2 \alpha_1)$, $B = \gamma_1 \gamma_2 (\alpha_1 + \alpha_2)^2$, $C = \gamma_1 \gamma_2 (\alpha_1^2 - \alpha_2^2)$. We denote the $n$-th positive zero of $\varphi$ by $\mu_n$.

**Lemma 2** $d(\omega)$ is written as

$$d(\omega) = e^{(\beta_1 + \beta_2) \omega} (\varphi(\omega) - h(\omega)), \quad \omega \in \mathbb{R}$$

where $h(\omega) \in C^1(\mathbb{R})$ and $h(\omega) \to 0$, $h'(\omega) \to 0$ exponentially as $\omega \to \infty$.

**Proof** By (6), $h(\omega) = \varphi(\omega) - e^{-(\beta_1 + \beta_2) \omega} d(\omega)$ and $h'(\omega)$ converge to 0 exponentially as $\omega \to \infty$.

To discuss controllability, we treat the moment problem on the system (1)(2)(3)(4). According to Krabs [4] or Russell [12], to solve the moment problem, we need the following conditions:

$$\lim_{n \to \infty} (\omega_{n+1}^2 - \omega_n^2) > \frac{2\pi}{T}, \quad (8)$$

$$\lim_{y \to \infty} \limsup_{x \to \infty} \frac{d(x+y) - d(x)}{y} \leq \frac{T}{2\pi}, \quad (9)$$

where $d(x) =$ number of $\omega_j$ with $\omega_j < x^2$. The aim of this paper is to prove the following

**Theorem 1** We have

1. There exist $M$ and $N$ such that $\omega_{M+n} - \mu_{N+n} \to 0$,
2. $0 < \frac{1}{a} (\pi - \sin^{-1} k) \leq \liminf_{n \to \infty} (\omega_{n+1}^2 - \omega_n^2) \leq \limsup_{n \to \infty} (\omega_{n+1}^2 - \omega_n^2) < \infty$,
3. $\lim_{n \to \infty} (\omega_{n+1}^2 - \omega_n^2) = \infty$.

By this theorem, it is clear that $\{\omega_n\}_{n \in \mathbb{N}}$ satisfies the codition (8). Moreover, the condition (8) verify the condition (9).

Some simple facts for $\varphi(\omega)$ are given in the following

**Lemma 3** In the formula (7), we have

1. $A \geq B > 0$,
2. $A = B$ if and only if $\rho_1 T_1 = \rho_2 T_2$.
3. $C = 0$ if and only if $\rho_1 T_2 = \rho_2 T_1$.
4. $A = B$, $C = 0$ if and only if $(\rho_1, T_1) = (\rho_2, T_2)$.
5. $A > B$ or $C \neq 0$ if and only if $(\rho_1, T_1) \neq (\rho_2, T_2)$. 


of positive the put all assertions even any increasing We zero increasing on every decreasing monotone we have exist which there simplicity. 2For any proved that we small write since there are and odd and $(10)$ similarly.

We assume $\beta_1 \geq \beta_2 > 0$ for simplicity. So we have $a > b \geq 0$. We put $f_k(\omega) = \cos a\omega + k \sin(b\omega + \tau)$. Since $f_k(\omega) = 0$ implies $|\cos a\omega| = |k \sin(b\omega + \tau)| \leq k$, all the positive zeros of $\varphi(\omega)$ are in the set $\{\omega > 0; |\cos a\omega| \leq k\} = \bigcup_{n=1}^{\infty} I_n(k)$, $I_n(k) = [s_n(k), t_n(k)] \subseteq J_n = [(n-1)\pi/a, n\pi/a]$, $n = 1, 2, \ldots$ where $s_n(k) = (2n-1)\pi/2a - \sin^{-1}k/a$, $t_n(k) = (2n-1)\pi/2a + \sin^{-1}k/a$. We write $I_n = I_n(k)$, $s_n = s_n(k)$, $t_n = t_n(k)$ and $f(\omega) = f_k(\omega)$. In Theorem 2 below, we prove that there exists exactly one zero of $f$ in each $I_n \subset J_n$.

**Theorem 2** For each $n \in \mathbb{N}$, there exist $u_n, v_n \in I_n$ such that

1. $f(u_n) = 1 - k = -f(v_n)$ and $f(\omega)$ is monotone decreasing on $[u_n, v_n]$ for odd $n$,
2. $f(u_n) = k - 1 = f(v_n)$ and $f(\omega)$ is monotone increasing on $[u_n, v_n]$ for even $n$,
3. $|f(\omega)| \geq 1 - k$ for $\omega \in J_n \setminus [u_n, v_n]$
4. only zero of $f$ exists in $(u_n, v_n)$ for any $n$, which implies that $\mu_n \in I_n$ for any $n \in \mathbb{N}$.

First, we show, for sufficiently small $k$, $\mu_n \in I_n$ for every $n \in \mathbb{N}$.

**Lemma 4** Let $k \in [0, 1/\sqrt{2}]$. Then we have

1. for odd $n$, $f(s_n) \geq 0 \geq f(t_n)$ and $f(\omega)$ is monotone decreasing on $[s_n, t_n]$,
2. for even $n$, $f(t_n) \geq 0 \geq f(s_n)$ and $f(\omega)$ is monotone increasing on $[s_n, t_n]$.

Consequently, in $J_n$, $f(\omega)$ has only one zero in $[s_n, t_n]$.

**Proof** We only show (1). (2) is proved similarly.

\[
\begin{align*}
f(s_n) &= \cos \left(\frac{(2n-1)\pi}{2} - \sin^{-1}k\right) - k \sin(bs_n + \tau) \\
&= (-1)^{n+1}k - k \sin(bs_n + \tau) = k - k \sin(bs_n + \tau) \geq 0 \\
f(t_n) &= (-1)^n k - k \sin(bs_n + \tau) = -k - k \sin(bs_n + \tau) \leq -k + k = 0.
\end{align*}
\]

In $[s_n, t_n]$, we have $\sin a\omega \geq 1/\sqrt{2}$ and

\[
f'(\omega) = -a \sin a\omega + kb \cos(b\omega + \tau) \leq -a/\sqrt{2} + b/\sqrt{2} < 0.
\]
In the following, we put \( k = \frac{kb^2}{a^2} \), \( s_n = s_n(k) \), \( \overline{t}_n = t_n(k) \), \( \overline{I}_n = I_n(k) \), and \( \overline{\mu}_n = \mu_n(k) \), \( \hat{f}(\omega) = f_k(\omega) \) and \( S = \{ k \in [0, 1); \mu_n(k) \in I_n(k) \subset J_n \} \) for each \( n \in \mathbb{N} \).

**Lemma 5** For \( \overline{k} \in S \), the conclusion of Theorem 2 is valid.

**Proof** We have

\[
\begin{align*}
    f''(\omega) &= f''_k(\omega) = -a^2 \cos aw - kb^2 \sin(bw + \tau) \\
    &= -a^2(\cos aw + k \sin(bw + \tau)) = -a^2 f_k(\omega) = -a^2 \hat{f}(\omega).
\end{align*}
\]

Let \( n \) be odd. The case where \( n \) is even is also treated similarly. Then \( \hat{f}((n - 1)\pi/a) \geq 1 - k > 0 > k - 1 \geq \hat{f}(\pi/a) \). Since \( \overline{\mu}_n \) is the only zero of \( \hat{f} \) in \((n - 1)\pi/a, n\pi/a)\), we have \( f''(\omega) < 0 \) for \( \omega \in ((n - 1)\pi/a, \overline{\mu}_n) \), \( f''(\omega) > 0 \) for \( \omega \in (\overline{\mu}_n, n\pi/a) \). Thus \( f(\omega) \) is concave on \((n - 1)\pi/a, \mu_n)\) and convex on \((\mu_n, n\pi/a)\). Let \( y_n, z_n \in J_n \) with \( f(y_n) = \max_{\omega \in I_n} f(\omega) \geq 1 - k \) and \( f(z_n) = \min_{\omega \in I_n} f(\omega) \leq k - 1 \). Then, we find \( u_n, v_n \) with \( s_n \leq y_n < \mu_n < v_n \leq z_n \leq t_n \) such that \( f(u_n) = 1 - k \) and \( f(v_n) = k - 1 \). Thus, \( f \) is monotone decreasing on \([u_n, v_n] \subset [y_n, z_n] \).

**Proof of Theorem 2** There exists \( N \in \mathbb{N} \) such that \( 0 \leq (b/a)^{2N} \leq 1/2 \). Let \( k \in [0, 1) \) and \( k_i = k(b/a)^{2i}, i = 0, 1, 2, \ldots, N \). Then \( k_N \in S \) by Lemma 4. Therefore, by Lemma 5, \( k_i \in S, i = 1, 2, \ldots, N - 1 \). In particular, \( k_1 = k(b/a)^2 = \overline{k} \in S \). Thus, by using Lemma 5 again, we can prove Theorem 2.

Next, we want to show that, for sufficiently large \( n \), there exists only one zero of \( d(\omega) \) in each \( J_n \). More precisely, we have

**Theorem 3** There exists \( M, N \in \mathbb{N} \) such that \( \omega_{M+n} \in J_{N+n} \) for \( n = 0, 1, \ldots \).

To prove the above theorem, we prepare Lemma 6 and Lemma 7 given below:

**Lemma 6** For any \( n \in \mathbb{N} \), the following inequality holds:

\[
    |f'(\mu_n)| \geq \delta = \sqrt{(1 - k^2)(a^2 - b^2)}.
\]

**Proof** Since \( \mu_n, n \in \mathbb{N} \), are zeros of \( f \), we have

\[
    f(\mu_n) = \cos a\mu_n + k \sin(b\mu_n + \tau) = 0, \quad f'(\mu_n) = -a \sin a\mu_n + kb \cos(b\mu_n + \tau).
\]

If \( b = 0 \), then \( (f'(\mu_n))^2 = a^2 \sin^2 a\mu_n = a^2(1 - \cos^2 \tau) \geq a^2(1 - k^2) \). Hence we have (11). If \( b \neq 0 \), then

\[
    f'(\mu_n)^2 \geq \frac{(a^2 - b^2)}{b^2} \left( \sin a\mu_n - \frac{a f'(\mu_n)}{a^2 - b^2} \right)^2 + (1 - k^2)(a^2 - b^2)
\]

\[
    \geq (1 - k^2)(a^2 - b^2).
\]

Thus we have (11).
Lemma 7 There exists an interval \([a_n, b_n] \subset [u_n, v_n]\) and \(l \in [0, 1 - k]\) such that
\[
|f(\omega)| \leq l, \quad |f'(\omega)| \geq \delta / 2 \quad \text{for} \ \omega \in [a_n, b_n], \tag{14}
\]
\[
|f(\omega)| \geq l \quad \text{for} \ \omega \in J_n \setminus [a_n, b_n]. \tag{15}
\]

Proof By uniform continuity of \(f'(\omega)\) and Lemma 6, there exists \(c > 0\) with \(l = \delta c / 2 < 1 - k\) such that
\[
|f'(\omega)| \geq \delta / 2 \quad \text{for} \ \omega \in [\mu_n - c, \mu_n + c]. \tag{16}
\]
Therefore, we have \(|f(\omega)| \geq (\delta / 2)|\omega - \mu_n|\) on \([\mu_n - c, \mu_n + c]\). If \(n\) is odd (resp. even), we define \(a_n, b_n\) with \(\mu_n - c < a_n < b_n < \mu_n + c\) by \(f(a_n) = l\) (resp. \(-l\)) and \(f(b_n) = -l\) (resp. \(l\)). Hence we have
\[
\{\omega \in [\mu_n - c, \mu_n + c]; |f(\omega)| \leq l\} = \{a_n, b_n\}. \tag{17}
\]

By (16) and (17), we see (14), and by Theorem 3, (15).

We put \(g(\omega) = \varphi(\omega) - h(\omega) = Df(\omega) - h(\omega)\). Since \(h(\omega), h'(\omega) \to 0\) as \(\omega \to \infty\), there exists \(N \in \mathbb{N}\) such that \(|h(\omega)| < Dl\) and \(|h'(\omega)| < D\delta / 2\) for \(\omega > (N - 1)\pi / a\).

Let \(n\) be odd with \(n > N\). Then, by Lemma 7, we have \(f(a_n) = l, f(b_n) = -l\) and \(f'(\omega) < -\delta / 2\) for \(\omega \in [a_n, b_n]\). Hence \(g(a_n) = Df(a_n) - h(a_n) = Dl - h(a_n) > Dl - Dl = 0\) and \(g(b_n) = Df(b_n) - h(b_n) = Dl - h(b_n) < -Dl + Dl = 0\). Thus, for \(\omega \in [a_n, b_n]\), \(\omega = Df'(\omega) - h'(\omega) \leq -D\delta / 2 + |h'(\omega)| < -D\delta / 2 + D\delta / 2 = 0\) which implies that \(g(\omega)\) has a unique zero in \((a_n, b_n)\). For \(\omega \in J_n \setminus [a_n, b_n]\), by (14), we have \(|g(\omega)| \geq |Df(\omega)| - |h(\omega)| \geq Dl - Dl = 0\). Therefore, \(g(\omega)\) has a unique zero in \(J_n\). The case with even \(n \geq N\) is also similarly proved. Let \(\omega_M\) be a zero of \(g(\omega)\) in \(J_n\). Thus \(\omega_{M+n} \in J_{N+n}\) for \(n = 0, 1, 2, \ldots\).

Proof of Theorem 1 Since \(f(\mu_{N+n}) = g(\omega_{M+n}) = 0\), we have
\[
h(\omega_{M+n}) = Df(\omega_{M+n}) - g(\omega_{M+n}) = D(\omega_{M+n} - f(\mu_{N+n})). \tag{18}
\]
By Mean Value Theorem, there exists \(\theta \in (0, 1)\) such that \(f(\omega_{M+n}) - f(\mu_{N+n}) = (\omega_{M+n} - \mu_{N+n})f'(\mu_{N+n} + \theta(\omega_{M+n} - \mu_{N+n})).\) Thus, by (14) and (18),
\[
|\omega_{M+n} - \mu_{N+n}| \leq \frac{|f(\omega_{M+n}) - f(\mu_{N+n})|}{|f'(\mu_{N+n} + \theta(\omega_{M+n} - \mu_{N+n}))|} \leq \frac{2}{D\delta}|h(\omega_{M+n})| \to 0
\]
as \(n \to \infty\). Therefore,
\[
\liminf_{n \to \infty}(\omega_{M+n} - \omega_{M+n}) = \liminf_{n \to \infty}(\omega_{M+n+1} - \omega_{M+n})
\]
\[
= \liminf_{n \to \infty}(\omega_{M+n+1} - \mu_{N+n+1} + \mu_{N+n+1} - \mu_{N+n} + \mu_{N+n} - \omega_{M+n})
\]
\[
= \liminf_{n \to \infty}(\mu_{N+n+1} - \mu_{N+n}) = \liminf_{n \to \infty}(\mu_{N+n+1} - \mu_{N+n})
\]

By \(s_n < \mu_n < t_n < s_{n+1} < \mu_{n+1} < t_{n+1}\), we have
\[
\mu_{n+1} - \mu_n \geq s_{n+1} - t_n = \left(\frac{2n+1}{2a}\pi - \frac{\sin^{-1}k}{a}\right) - \left(\frac{2n-1}{2a}\pi + \frac{\sin^{-1}k}{a}\right)
\]
\[
= \pi - \frac{2\sin^{-1}k}{a} = \frac{1}{a}(\pi - 2\sin^{-1}k)
\]
The above theorem implies that
\[
\liminf_{n \to \infty}(\omega_{n+1}^2 - \omega_n^2) \geq \liminf_{n \to \infty}(\omega_{n+1} + \omega_n) \liminf_{n \to \infty}(\omega_{n+1} - \omega_n) = \infty. \tag{19}
\]
3 Concluding Remarks

This paper is only a first step to the controllability theory for the Euler-Bernoulli equation using the moment problem method [4], [12].

References


