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On a New Existence Result for Cone Saddle Point Problems

新潟大学自然科学研究科 〒 950–2181, 新潟市五十嵐 2 の町 8050
木村 健志 (KENJI KIMURA)
kenji@ekeland.gs.niigata-u.ac.jp

Visiting Professor of the Graduate School of Science and Technology, Niigata University.
Mostafa Kalmoun (MOSTAFA KALMOUN)
kalmoun@toki.gs.niigata-u.ac.jp

新潟大学自然科学研究科 〒 950–2181, 新潟市五十嵐 2 の町 8050
田中 環 (TAMAKI TANAKA)
tamaki@mathweb.sc.niigata-u.ac.jp

1 Introduction

Studies on vector-valued minimax theorems or vector saddle point problems have been extended widely; see [6] and references cited therein. Existence results for cone saddle points are based on some fixed point theorems or scalar minimax theorems; see [5]. Recently, this kind of problems is solved by a different approach in [3], in which a vector variational inequality problem is treated in a finite dimensional vector space. In this paper, we consider its generalization to vector problems involving the concept of moving cone in the general setting of a normed space.

2 Problem Formulation and Existence Result

Let K and E be nonempty subsets of a normed space X and a topological vector space Y, respectively, and let Z be a normed space.
Given a vector-valued function $L : K \times E \to Z$ and a pointed convex cone C on Z with $\mathrm{int}C \neq \phi$, Vector Saddle Point Problem(in short, VSPP) is to find $x_0 \in X$ and $y_0 \in Y$ such that

$$\begin{aligned} L(x_0, y_0) - L(x, y_0) &\notin \mathrm{int}\, C, \quad \forall x \in K, \\ L(x_0, y) - L(x_0, y_0) &\notin \mathrm{int}\, C, \quad \forall y \in E. \end{aligned}$$

A solution (x_0, y_0) of (VSPP) is called a weak C-saddle point of the function L.

On the other hand, Vector Variational Inequality Problem(in short, VVIP) is to find $x_0 \in K$ and $y_0 \in T(x_0)$ such that

$$\langle L'(x_0, y_0), x - x_0 \rangle \notin -\mathrm{int}C, \ \forall x \in K,$$

where $T : X \to Y$ is a multifunction defined by

$$T(x) := \{ y \in C \mid L(x, v) - L(x, y) \notin \mathrm{int}\, C, \ \forall v \in E \},$$

and $L'(x_0, y_0)$ denotes the Fréchet derivative of L with respect to the first argument at (x_0, y_0).

Definition 2.1 *A function $f : K \to Z$, where K is convex set, is called C-convex if for each $x, y \in K$ and $\lambda \in [0, 1]$,*

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \in C.$$

Definition 2.2 *A function $f : K \to Z$ is called Fréchet differentiable if for every $x \in K$ and $\varepsilon > 0$, there exists $f'_x \in L(K, Z)$ and $\delta > 0$ such that*

$$\|f(x+h) - f(x) - f'_x(h)\| < \varepsilon \text{ for all } h \in K;\ \|h\| < \delta,$$

where $L(K, Z)$ is the space of all linear continuous operators from K into Z.

First we show an equivalence condition between (VSPP) and (VVIP).

Theorem 2.1 *Suppose that K is convex and L is C-convex and Fréchet differentiable in the first argument. Then problems* (VSPP) *and* (VVIP) *have the same solution set.*

Proof. Assume that $(x_0, y_0) \in K \times E$ is a solution of (VSPP). Then

$$L(x_0, y_0) - L(x, y_0) \notin \operatorname{int} C, \tag{1}$$

for all $x \in K$.

$$L(x_0, y) - L(x_0, y_0) \notin \operatorname{int} C, \tag{2}$$

for all $y \in E$. Since K is convex, We have

$$x_0 + \alpha(x - x_0) \in K,$$

for all $x \in K$ and $\alpha \in [0, 1]$. Hence condition(1) implies

$$\alpha^{-1}[L(x_0 + \alpha(x - x_0), y_0) - L(x_0, y_0)] \notin -\operatorname{int} C,$$

for all $x \in K$ and $\alpha \in (0, 1]$. Since $Z \backslash (-\operatorname{int} C)$ is closed and L is Fréchet differentiable in the first argument, it follows that

$$\langle L'(x_0, y_0), x - x_0 \rangle \notin -\operatorname{int} C,$$

for all $x \in K$. $y_0 \in T(x_0)$ follows from (2).

Conversely, assume that $(x_0, y_0) \in K \times E$ is a solution of (VVIP). Then we have

$$\langle L'(x_0, y_0), x - x_0 \rangle \notin -\operatorname{int} C, \tag{3}$$

for all $x \in K$ and

$$L(x_0, y) - L(x_0, y_0) \notin \operatorname{int} C, \tag{4}$$

for all $y \in E$. Since L is C-convex with respect to the first argument, we have

$$\alpha L(x, y_0) + (1-\alpha)L(x_0, y_0) - L(x_0 + \alpha(x - x_0), y_0) \in C,$$

for all $x \in K$ and $\alpha \in (0, 1)$, and since C is cone, we have

$$L(x, y_0) - L(x_0, y_0) - \frac{L(x_0 + \alpha(x - x_0), y_0) - L(x_0, y_0)}{\alpha} \in C,$$

for all $x \in K$ and $\alpha \in (0,1)$. Since L is Fréchet differentiable with respect to the first argument, if α converge to 0, then we have

$$L(x, y_0) - L(x_0, y_0) - \langle L'(x_0, y_0), x - x_0 \rangle \in C,$$

for all $x \in K$. From condition(3), it follows

$$L(x_0, y_0) - L(x, y_0) \notin \operatorname{int} C$$

for all $x \in K$. Hence $(x_0, y_0) \in K \times E$ is also a solution of (VSPP). ∎

Now, we introduce Fan-KKM theorem, which is important in the field related to (VVIP), for theorem 2.3.

Theorem 2.2 (Fan-KKM Theorem see;[4]) *Let X be a subset of a topological vector space. For each $x \in X$, let a closed set $F(x)$ in X be given such that $F(x)$ is compact for at least one $x \in X$. If the convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$, then $\bigcap_{x \in X} F(x) \neq \phi$.*

Next we show an existence result of (VSPP) by using (VVIP).

Theorem 2.3 *Let K and E be a nonempty closed convex subset of a normed space X and a nonempty compact subset of a topological vector space Y, respectively. Assume that the vector-velued function L is continuously differentiable and C-convex in the first argument and L' is continuous in both x and y, and let $T : K \to E$ be the multifunction defined by*

$$T(x) := \{\, y \in E \mid L(x, v) - L(x, y) \notin \operatorname{int} C, \quad \forall v \in E \,\}.$$

If there exists a nonempty compact subset B of X and $\bar{x} \in B \cap K$ such that for any $x \in K \backslash B$ and $y \in T(x)$,

$$\langle L'(x, y), x_0 - x \rangle \in -\operatorname{int} C,$$

then problem (VSPP) *has at least one solution.*

Proof. In order to proof the theorem, it is sufficient to show that (VVIP) has at least one solution $x_0 \in K,\ y_0 \in T(x_0)$. Define a multifunction $F : K \to K$ by

$$F(u) = \{\, x \in K \mid \langle L'(x, y), u - x \rangle \notin -\operatorname{int} C, \quad \text{for some } y \in T(x) \,\}, \quad u \in K.$$

First, we prove that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K is contained in the corresponding union $\bigcup_{i=1}^{m} F(x_i)$, that is, $\operatorname{Co}\{x_1, x_2, \cdots, x_m\} \subset \bigcup_{i=1}^{m} F(x_i)$. Suppose to the contrary that there exist $x_1, x_2, \ldots, x_m$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\hat{x} := \sum_{i=1}^{m} \alpha_i x_i \notin \bigcup_{i=1}^{m} F(x_i), \quad \sum_{i=1}^{m} \alpha_i = 1.$$

Then, $\hat{x} \notin F(x_i)$ for all $i = 1, \ldots, n$, and hence for any $y \in T(\hat{x})$,

$$\langle L'(\hat{x}, y), x_i - \hat{x} \rangle \in -\operatorname{int} C,$$

for all $i = 1, \dots, m$. Since $\operatorname{int} C$ is convex, we have

$$\sum_{i=1}^{m} \alpha_i \langle L'(\hat{x}, y), x_i - \hat{x} \rangle \in -\operatorname{int} C.$$

Since $L'(\hat{x}, y)$ is a linear operater, we have

$$\langle L'(\hat{x}, y), \sum_{i=1}^{m} \alpha_i x_i \rangle - \sum_{i=1}^{m} \alpha_i \langle L'(\hat{x}, y), \hat{x} \rangle \in -\operatorname{int} C.$$

Hence

$$\langle L'(\hat{x}, y), \hat{x} \rangle - \langle L'(\hat{x}, y), \hat{x} \rangle = 0 \in -\operatorname{int} C,$$

which is inconsistent. Thus, we deduce that

$$\operatorname{Co}\{x_1, x_2, \dots, x_m\} \subset \bigcup_{i=1}^{m} F(x_i).$$

Next, we show the multifunction T satisfied Hogan's upper semi-continuity. Let $\{x_n\}$ be a sequence in K such that $x_n \to x \in K$ and let $\{y_n\}$ be a sequence such that $y_n \in T(x_n)$. Since $y_n \in T(x_n)$, we have

$$L(x_n, v) - L(x_n, y_n) \notin \operatorname{int} C,$$

for all $v \in E$. Since $\{y_n\} \subset E$ and E is compact we can assume that there exists $y \in E$ such that $y_n \to y$, without loss of generality. Now the continuity of L and the closedness of $(Z \backslash \operatorname{int} C)$ gives that

$$L(x, v) - L(x, y) \in (Z \backslash \operatorname{int} C)$$

for all $v \in E$, which implies that $y \in T(x)$. Thus the multifunction T is upper semicontinuous.

Next, we show that $F(u)$ is a closed set for each $u \in K$. Let $\{x_n\} \subset F(u)$ such that $x_n \to x \in K$. Since $x_n \in F(u)$ for all n, there exists $y_n \in T(x_n)$ such that

$$\langle L'(x_n, y_n), u - x_n \rangle \in (Z \backslash - \operatorname{int} C)$$

for all $u \in K$. As $\{y_n\} \subset E$, without loss of generality, we can assume that there exists $y \in E$ such that $y_n \to y$. Since L' is continuous, T is upper semicontinous and $(Z \backslash - \operatorname{int} C)$ is closed, we have

$$\langle L'(x_n, y_n), u - x_n \rangle \to \langle L'(x, y), u - x \rangle \in (Z \backslash - \operatorname{int} C).$$

Hence $x \in F(u)$.

Finally, we prove that for $\bar{x} \in B \cap K$, $F(\bar{x})$ is compact. Since $F(\hat{u})$ is closed and B is compact, it is sufficient to show that $F(\hat{u}) \subset B$. Suppose to the contrary that there exists $\hat{x} \in F(\hat{u})$ such that $\hat{x} \notin B$. Since $\hat{x} \in F(\hat{u})$, there exists $\hat{y} \in T(\hat{x})$ such that

$$\langle L'(\hat{x}, \hat{y}), \hat{u} - \hat{x} \rangle \notin -\operatorname{int} C. \tag{5}$$

Since $\hat{x} \notin B$, by the hypothesis, for any $y \in T(\hat{x})$,

$$\langle L'(\hat{x}, y), \hat{u} - \hat{x} \rangle \in -\operatorname{int} C,$$

which contradicts condition(5). Hence $F(\bar{x}) \subset B$. Since B is compact and $F(\bar{x})$ is also closed, $F(\bar{x})$ is compact. Consequently by Theorem 2.2, it follows that $\bigcap_{x \in K} F(x) \neq \phi$. Thus, there exists $x_0 \in K$ and $y_0 \in T(y_0)$ such that

$$\langle L'(x_0, y_0), x - x_0 \rangle \notin -\operatorname{int} C,$$

for all $x \in K$. ∎

3 An Extension based on Moving Cone

We can extension concepts (VSPP) and (VVIP) by considering a moveing cone. To begin with, we introduce some parameterized concepts for the extension. Assume that the multifunction $C : X \to 2^Z$ has solid pointed convex cone values.

Definition 3.1 (Parameterized Cone Convexity)
A vector valued function $f : K \to Z$ is said to be $C(x)$-convex if

$$\alpha f(x_1) + (1-\alpha)f(x_2) - f(\alpha x_1 + (1-\alpha)x_2) \in C(\alpha x_1 + (1-\alpha)x_2),$$

for all x_1, $x_2 \in K$ and $\alpha \in [0,1]$.

Definition 3.2 Parameterized Vector Saddle Point Problem
The Parameterized Vector Saddle Point Problem, (PVSPP) *for short, is to find $x_0 \in K$ and $y_0 \in T(x_0)$ such that*

$$L(x_0, y_0) - L(x, y_0) \notin \operatorname{int} C(x_0), \quad \forall x \in K,$$
$$L(x_0, y) - L(x_0, y_0) \notin \operatorname{int} C(x_0), \quad \forall y \in E.$$

A solution $(x_0, y_0) \in K \times E$ of (PVSPP) is called a weak $C(x)$-saddle point of function L.

Definition 3.3 Parameterized Vector Variational Inequality Problem
The Parameterized Vector Variational Inequality Problem, (PVVIP) for short, is to find $x_0 \in K$ and $y_0 \in T(x_0)$ such that

$$\langle L'(x_0, y_0), x - x_0 \rangle \notin -\operatorname{int} C(x), \ \forall x \in K,$$

where $T : X \to 2^Y$ is a multifunction defined by

$$T(x) := \{\, y \in C \mid L(x, v) - L(x, y) \notin \operatorname{int} C(x), \ \forall v \in E \,\}.$$

Definition 3.4 *A multifunction $F : K \to 2^Z$ is called upper-semicontinuous if for every $x \in K$ and $U_x \subset Z$; neighborhood of $F(x)$ there exists $V_x \subset K$; neighborhood of x such that $F(y) \subset U_x$ for all $y \in V_x$.*

Definition 3.5 *A multifunction $F : K \to 2^Z$ is called lower-semicontinuous if for every $x \in K$ there exists $V_x \subset K$; neighborhood of x such that $F(y) \cap V_x \neq \phi$ for all $V_x \subset Z$, where V_x is an open set satisfying $F(x) \cap V_x \neq \phi$.*

Definition 3.6 *A multifunction $F : K \to 2^Z$ is called continuous if F satisfy upper-semicontinuous and lower-semicontinuous.*

Definition 3.7 *A multifunction $F : K \to 2^Z$ is called closed if $\{x_n\} \subset K$ converging to x, and $\{z_n\} \subset Z$, with $z_n \in F(x_n)$, converging to z, implies $z \in F(x)$.*

Remma 3.1 *Assume that the multifuncion $C : K \to 2^Z$ is continuous. Then the multifunction C and W are closed, where $W : K \to 2^Z$ is a mulitfunction defined by*

$$W(x) := Z \backslash \operatorname{int} C(x)$$

Now, we extend the results of Section 2 by using these concepts.

Theorem 3.1 *Let K and E be a convex subset of a normed space X and an arbitrary subset of a topological vector space Y. Assume that the multifunction $C : X \to 2^Z$ has solid pointed convex cone values and it is continuous, and L is $C(x)$-convex and Fréchet differentiable in the first argument. Then problems* (PVSPP) *and* (PVVIP) *have the same solution set.*

Proof. Assume that $(x_0, y_0) \in K \times E$ is a solution of (PVSPP). Then

$$L(x_0, y_0) - L(x, y_0) \notin \operatorname{int} C(x_0), \tag{6}$$

for all $x \in K$.

$$L(x_0, y) - L(x_0, y_0) \notin \operatorname{int} C(x_0), \tag{7}$$

for all $y \in E$. Since K is convex, We have

$$x_0 + \alpha(x - x_0) \in K,$$

for all $x \in K$ and $\alpha \in [0, 1]$. Hence condition(6) implies

$$\alpha^{-1}[L(x_0 + \alpha(x - x_0), y_0) - L(x_0, y_0)] \notin -\operatorname{int} C(x_0 + \alpha(x - x_0)),$$

for all $x \in K$ and $\alpha \in (0, 1]$. Since $Z\backslash(-\operatorname{int} C(x))$ is continuous and L is Fréchet differentiable in the first argument, it follows that

$$\langle L'(x_0, y_0), x - x_0\rangle \notin -\operatorname{int} C(x_0),$$

for all $x \in K$. $y_0 \in T(x_0)$ follows from (7).

Conversely, assume that $(x_0, y_0) \in K \times E$ is a solution of (PVVIP). Then we have

$$\langle L'(x_0, y_0), x - x_0\rangle \notin -\operatorname{int} C(x_0), \tag{8}$$

for all $x \in K$.

$$L(x_0, y) - L(x_0, y_0) \notin \operatorname{int} C(x_0), \tag{9}$$

for all $y \in E$. Since L is C-convex with respect the first argument, we have

$$\alpha L(x, y_0) + (1 - \alpha)L(x_0, y_0) - L(x_0 + \alpha(x - x_0), y_0) \in C(x_0 + \alpha(x - x_0)),$$

for all $x \in K$ and $\alpha \in (0, 1)$, and since $C(x)$ is cone, we have

$$L(x, y_0) - L(x_0, y_0) - \frac{L(x_0 + \alpha(x - x_0), y_0) - L(x_0, y_0)}{\alpha} \in C(x_0),$$

for all $x \in K$ and $\alpha \in (0, 1)$. Since L is Fréchet differentiable with respect to the first argument, if α converges to 0, then we have

$$L(x, y_0) - L(x_0, y_0) - \langle L'(x_0, y_0), x - x_0\rangle \in C(x_0),$$

for all $x \in K$. From (8), it follows

$$L(x_0, y_0) - L(x, y_0) \notin \operatorname{int} C(x_0),$$

for all $x \in K$. Hence $(x_0, y_0) \in K \times E$ is also a solution of (PVSPP). ∎

Theorem 3.2 *Let K and E be a nonempty closed convex subset of a normed space X and a nonempty compact subset of a topological vector space Y, respectively. Assume that the multifunction $C : X \to 2^Z$ has solid pointed convex cone values and it is continuous. Assume that the vector valued function L is $C(x)$-convex and Fréchet differentiable in the first argument, L' is a continuous function in both x and y, and let T, $K \to E$ be the multifunction defined by*

$$T(x) := \{\, y \in E \mid L(x,v) - L(x,y) \notin \operatorname{int} C(x), \quad \forall v \in E \,\}.$$

If there exist a nonempty compact subset B of X and $x_0 \in B \cap K$ such that for any $x \in K \backslash B$, $y \in T(x)$,

$$\langle L'(x,y), x_0 - x \rangle \in -\operatorname{int} C(x),$$

then problem (PVSPP) *has at least one solution.*

Proof. It is sufficient to show that the (PVVIP) has at least one solution $x_0 \in K$ and $y_0 \in T(x_0)$. Define a multifunction $F : K \to K$ by

$$F(u) = \{\, x \in K \mid \langle L'(x,y), u - x \rangle \notin -\operatorname{int} C(x), \quad \text{for some } y \in T(x) \,\}, \quad u \in K.$$

We first prove that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K is contained in the corresponding union $\bigcup_{i=1}^{m} F(x_i)$, that is, $\mathrm{Co}\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} F(x_i)$. Suppose that there exists $x_1, x_2, \ldots, x_m$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\hat{x} = \sum_{i=1}^{m} \alpha_i x_i \notin \bigcup_{i=1}^{m} F(x_i), \quad \sum_{i=1}^{m} \alpha_i = 1.$$

Then for any $y \in T(\hat{x})$,

$$\langle L'(\hat{x},y), x_i - \hat{x} \rangle \in -\operatorname{int} C(\hat{x}),$$

for all $i = 1, \ldots, m$. Since $\operatorname{int} C(x)$ is convex, we have

$$\sum_{i=1}^{m} \alpha_i \langle L'(\hat{x},y), x_i - \hat{x} \rangle \in -\operatorname{int} C(\hat{x}).$$

Since $L'(\hat{x},y)$ is a linear operater, we have

$$\langle L'(\hat{x},y), \sum_{i=1}^{m} \alpha_i x_i \rangle - \sum_{i=1}^{m} \alpha_i \langle L'(\hat{x},y), \hat{x} \rangle \in -\operatorname{int} C(\hat{x}).$$

Hence

$$\langle L'(\hat{x},y), \hat{x} \rangle - \langle L'(\hat{x},y), \hat{x} \rangle = 0 \in -\operatorname{int} C(\hat{x}),$$

which is inconsistent. Thus, we deduce that

$$\mathrm{Co}\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} F(x_i).$$

Next, we show the multifunction T satisfied Hogan's upper semi-continuity. Let $\{x_n\}$ be a sequence in K such that $x_n \to x \in K$ and let $\{y_n\}$ be a sequence such that $y_n \in T(x_n)$. Since $y_n \in T(x_n)$, we have

$$L(x_n, v) - L(x_n, y_n) \notin \operatorname{int} C(x_n)$$

for all $v \in E$. Since $\{y_n\} \subset E$ and E is compact we can assume that there exists $y \in E$ such that $y_n \to y$, without loss of generality. Now the continuity of L and the closedness of $(Z \backslash \operatorname{int} C(x))$ gives that

$$L(x, v) - L(x, y) \in (Z \backslash \operatorname{int} C(x))$$

for all $v \in E$, which implies that $y \in T(x)$. Thus the multifunction T is upper semicontinuous.

Next, we show that $F(u)$ is closed for each $u \in K$. Indeed, let $\{x_n\} \subset F(u)$ such that $x_n \to x \in K$. Since $x_n \in F(u)$ for all n, there exists $y_n \in T(x_n)$ such that

$$\langle L'(x_n, y_n), u - x_n \rangle \in (Z \backslash -\operatorname{int} C(x_n))$$

for all $u \in K$. As $\{y_n\} \subset E$ we can assume that there exists $y \in E$ such that $y_n \to y$, without loss of generality. Since L' is continuous, T is upper semicontinous and $(Z \backslash -\operatorname{int} C(x))$ is closed, we have

$$\langle L'(x_n, y_n), u - x_n \rangle \to \langle L'(x, y), u - x \rangle \in (Z \backslash -\operatorname{int} C(x)).$$

Hence $x \in F(u)$.

Finally, we prove that for $\bar{x} \in B \cap K$, $F(\bar{x})$ is compact. Since $F(\hat{u})$ is closed and B is compact, it is sufficient to show that $F(\hat{u} \subset B)$. Suppose that there exists $\hat{x} \in F(\hat{u})$ such that $\hat{x} \notin B$. Since $\hat{x} \in F(\hat{u})$, there exists $\hat{y} \in T(\hat{x})$ such that

$$\langle L'(\hat{x}, \hat{y}), \hat{u} - \hat{x} \rangle \notin -\operatorname{int} C(\hat{x}). \tag{10}$$

Since $\hat{x} \notin B$, by hypothesis, for any $y \in T(\hat{x})$,

$$\langle L'(\hat{x}, y), \hat{u} - \hat{x} \rangle \in -\operatorname{int} C(\hat{x}),$$

which contradicts (10). Hence $F(\bar{x}) \subset B$. Since B is compact and $F(\bar{x})$ is closed, $F(\bar{x})$ is compact. By Theorem 2.2, it follows that $\bigcap_{x \in K} F(x) \neq \phi$. Thus, there exists $x_0 \in K$, $y_0 \in T(y_0)$ such that

$$\langle L'(x_0, y_0), x - x_0 \rangle \notin -\operatorname{int} C(x_0),$$

for all $x \in K$. ∎

4 Conclusions

In this paper, we have extended an existence theorem established Kazmi and Khan to a more generalized one. We have also extended the theorem by using a concept of moving cone, which first entered in game theory to cope with turning the purpose of a situation.

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