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Remarks on a New Existence Theorem for Generalized Vector Equilibrium Problems and its Applications

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We consider a generalized vector equilibrium problem, which is the following set-valued vector version of Ky Fan’s minimax inequality:

Find $\bar{x} \in C$ such as to satisfy $\varphi(\bar{x}, y) \not\subset K(\bar{x})$ for all $y \in C$, \hspace{1cm} (GVEP)

where
$\bullet$ $X$ and $E$ are topological vector spaces,
$\bullet$ $C$ is a nonempty closed convex subset of $X$,
$\bullet$ $\varphi : C \times C \to 2^E$ is a set-valued map, and
$\bullet$ $K$ is a set-valued map from $C$ to $E$.

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Using a particular case of the extended version of Fan-KKM theorem [6, Theorem 2.1], we can formulate the following general existence theorem for (GVEP) in topological vector spaces.

First, we need to recall the following definitions. Let $\psi : C \times C \to 2^E$ and $L : C \to 2^E$ be two other set-valued maps, and denote by $\mathcal{F}(C)$ the set of all finite subsets of $C$.

**Definition 1.** We say that $\psi$ is diagonally quasi-convex in its first argument relatively to $L$, in short $L$-diagonally quasi-convex in $x$, if for any $A$ in $\mathcal{F}(C)$ and any $y$ in $\text{co}(A)$, we have $\psi(A, y) \not\subseteq L(y)$.

**Definition 2.** We say that $\varphi$ is $K$-transfer semicontinuous in $y$ if for any $(x, y) \in C \times X$ with $\varphi(x, y) \subseteq K(y)$, there exist an element $x' \in C$ and an open $V \subset X$ containing $y$ such that $\varphi(x', v) \subseteq K(v)$ for all $v \in V$.

**Theorem 1.** ([7, Theorem 2.1]) Suppose that

(A0) $\psi(x, y) \not\subseteq L(y) \implies \varphi(x, y) \not\subseteq K(y) \forall x, y \in C$;

(A1) $\psi$ is $L$-diagonally quasi-convex in $x$;

(A2) $\varphi$ is $K$-transfer semicontinuous in $y$;

(A3) there is a nonempty compact subset $B$ in $X$ such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset X$ containing $A$ such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with

$$y \in \text{int}_X \{v \in X : \psi(x, v) \subseteq L(v)\}.$$

Then there exists $\overline{y} \in B$ such that $\varphi(x, \overline{y}) \not\subseteq K(\overline{y})$ for all $x \in C$.

Theorem 1 generalizes [2, Theorem 2.1], which is proved by means of a Fan-Browder fixed point theorem - an immediate consequence of the Fan-KKM theorem. As we will mention in the 'Assumptions analysis' subsection, our hypotheses are more general than those used in [2]. Besides, the scalar version of this result extends [10, Theorem 4] (we take $C_A = \text{co}(A \cup R) \cap X$ where $R$ is the convex compact which contains $C$ in [10, Theorem 4, (4iii)]). Other particular cases are [1, Theorem 2], [12, Theorem 2.1], [13, Theorem 2.11], [11, Theorem 1], [8, Corollary 2.4], [9, Lemma 2.1] and [3, Theorem 2]. The origin of this kind of results goes back to Ky Fan [5]. His classical
minimax inequality can be deduced from our result by setting $E = \mathbb{R}, K(x) = \mathbb{R}_+^*$ and $\varphi(x, y) = \psi(x, y) = f(x, y) - \sup_{x \in C} f(x, x)$ for all $x, y \in C$.

Let us turn to Theorem 1 and analyze its requirements by presenting different situations where assumptions (A0)-(A3) hold true. Let $(P(y))_{y \in C}$ a family of proper convex closed cones on $E$ with $int P(y) \neq \emptyset$ for all $y \in C$.

- **Pseudomonotonicity**

**Remark 1.** (A0) holds provided one of the following statements is satisfied.

(a) $\varphi = \psi$ and $K = L$.

(b) $x = C$, $K(y) = -L(y) = -int P(y)$, $\psi(x, y) = \varphi(y, x)$ for all $x, y \in C$, and $\varphi$ is $P_x$-pseudomonotone, that is,

\[ \varphi(x, y) \not\subset int P(x) \implies \varphi(y, x) \not\subset -int P(x) \quad \forall x, y \in C. \]

- **Convexity.**

**Remark 2.** (A1) holds provided that, for every $y \in C$, one has either

(a) $\psi(y, y) \not\subset L(y)$, and

(b) the set \{ $x \in C : \psi(x, y) \subseteq L(y)$ \} is convex,

or

(i) $L(y) = -int P(y)$ and $P(y)$ is w-pointed \(^1\),

(ii) $\psi(y, y) \subseteq P(y)$, and

(iii) $\psi$ is left $P_y$-quasiconvex, that is, for all $x_1, x_2, y \in C$ and all $\lambda \in [0, 1]$, one has either

\[ \psi(x_1, y) \subseteq \psi(\lambda x_1 + (1 - \lambda)x_2, y) + P(y) \]

or

\[ \psi(x_2, y) \subseteq \psi(\lambda x_1 + (1 - \lambda)x_2, y) + P(y). \]

- **Continuity**

\(^1\)A cone $P$ is w-pointed if $P \cap -int P = \emptyset.$
Remark 3. (A2) holds provided that one of the following statements is satisfied.

(a) \( \varphi \) is (transfer) u.s.c in \( y \) with compact values and if \( K \) has an open graph.
(b) \( \varphi \) is (transfer) u.s.c in \( y \) and \( K(x) = O \) for all \( x \in C \), where \( O \) is an open subset of \( E \).
(c) For each \( x \in C \), the set \( \{ y \in X : \varphi(x,y) \not\subseteq K(y) \} \) is closed in \( C \).

- Coercivity.

Remark 4. (A3) holds if one of the following statements is satisfied.

(a) \( C \) is compact.
(b) There is \( x_0 \in C \) such that \( \psi(x_0,.) \) is \( K \)-compact.
(c) There is a nonempty compact subset \( B \) in \( C \) such that for each \( y \in C \setminus B \) there exists \( x \in B \cap C \) such that \( \psi(x,y) \subseteq L(y) \).
(d) There is a nonempty compact subset \( B \) of \( C \) and a compact convex subset \( B' \subseteq C \) such that for each \( y \in C \setminus B \) there exists \( x \in B' \cap C \) with

\[
y \in \text{int} \{ v \in X : \psi(x,v) \subseteq L(v) \}.
\]

Besides, when the classical assumption (c) of Remark 3 is satisfied, (A3) holds provided that
(e) there is a nonempty compact subset \( B \) in \( X \) such that for each \( A \in \mathcal{F}(C) \) there is a compact convex \( B_A \subseteq X \) containing \( A \) such that, for every \( y \in C \setminus B \), there exists \( x \in B_A \cap C \) with \( \varphi(x,y) \subseteq K(y) \).

Applications

a) Generalized vector variational like-inequalities

Let us consider a set-valued operator \( T \) from \( C \) into \( L(X,E) \), and a bifunction \( \eta \) from \( C \) to itself. We write for \( \Pi \subset L(X,E) \) and \( x \in C \), \( \langle \Pi, x \rangle = \)
\{(\pi, x) : \pi \in \Pi\}, where \langle \pi, x \rangle denotes the evaluation of the linear mapping \pi at x which is supposed to be continuous on \(L(X, E) \times X^2\).

The generalized vector variational inequality problem (GVVLI\(P\)) takes the following form:

Find \(\overline{x} \in C\) such that \(\langle T\overline{x}, \eta(y, \overline{x}) \rangle \not\subset \text{int} \, P(\overline{x}) \forall y \in C\).

Thus (GVVLI\(P\)) is a particular case of (GVEP) if we take

\[\varphi(x, y) = \{(t, \eta(y, x)) : t \in Tx\}.

For the reader's convenience, we recall the following definitions.

**Definition 3.**

1) \(T\) is said to be \(\eta\)-pseudomonotone if, for all \(x, y \in C\),

\[\langle Tx, \eta(y, x) \rangle \not\subset \text{int} \, P(x) \Rightarrow \langle Ty, \eta(y, x) \rangle \not\subset \text{int} \, P(x)\]

2) \(T\) is said to be \(V\)-hemicontinuous if for any \(x, y \in C\) and \(t \in ]0,1[\), \(T(tx + (1-t)y) \rightarrow T(y)\) as \(t \rightarrow 0_+\) (i.e. for any \(z_t \in T(tx + (1-t)y)\) there exists \(z \in Ty\) such that for any \(a \in C\), \(\langle z_t, a \rangle \rightarrow \langle z, a \rangle\) as \(t \rightarrow 0_+\)).

It has to be observed that when \(T\) is single-valued, we recover the hemi-continuity used in [4], if \(\eta(x, y) = x - y\) for all \(x, y \in C\), \(\eta\) is dropped from the definition of pseudomonotonicity.

The linearization lemma plays a significant role in variational inequalities. Chen [4] extended this lemma to the single-valued vector case. For our need in this subsection, we state it in the set-valued case by using standard Minty's argument. Consider the following problem, which may be seen as a dual problem of (GVVLI\(P\)),

Find \(\overline{x} \in C\) such that \(\langle Ty, \eta(y, \overline{x}) \rangle \not\subset \text{int} \, P(\overline{x}) \forall y \in C\).  

(**GVVLI\(P^*\)**)

**Lemma 1.** Suppose that \(\eta(\cdot, x)\) is affine and \(\eta(x, x) = 0\) for each \(x \in C\). If \(T\) is \(\eta\)-pseudomonotone and \(V\)-hemicontinuous then (GVVLI\(P\)) and (GVVLI\(P^*\)) are equivalent.

As an application of Theorem 1, we are now in position to formulate the following existence result for (GVVLI\(P\)).

\(^2\)A typical situation when \(X\) is a reflexive Banach and \(E\) is a Banach
Theorem 2. Suppose that

(i) the mapping \( \text{int} \ P(\cdot) \) has an open graph in \( C \times L(X,E) \);

(ii) for each \( x \in C \), \( \eta(\cdot, x) \) is affine, \( \eta(x, \cdot) \) is continuous and \( \eta(x, x) = 0 \);

(iii) \( T \) is compact valued, \( \eta \)-pseudomonotone and \( V \)-hemicontinuous;

(iv) there is a nonempty compact subset \( B \) in \( C \) such that for each \( A \in \mathcal{F}(C) \) there is a compact convex \( B_A \subset C \) containing \( A \) such that, for every \( y \in B_A \setminus B \), there exists \( x \in B_A \cap C \) with

\[
y \in \text{int}_C \{ v \in C : \langle Tv, \eta(x, v) \rangle \subseteq -\text{int} \ P(v) \}.
\]

Then \( (GVVLIP) \) has at least one solution, which is in \( B \).

Proof. Set \( \varphi(x,y) = \langle Tx, \eta(x,y) \rangle \), \( \psi(x,y) = \langle Ty, \eta(x,y) \rangle \) and \( K(x) = -\text{int} \ P(x) \) for all \( x, y \in C \). We can show that the assumptions of Theorem 1 are satisfied; see the proof of Theorem 4.1 in [7]. Therefore, from Theorem 1, there exists \( \overline{x} \in B \) such that

\[
\langle Ty, \eta(y, \overline{x}) \rangle \not\subseteq -\text{int} \ P(\overline{x}) \forall y \in C.
\]

Hence \( (GVVLIP^*) \) has a solution in \( B \), which completes the proof of the theorem according to Lemma 1. \[ \square \]

b) Vector complementarity problems

A natural extension of the classical nonlinear complementarity problem, \( (CP) \) for short, is considered as follows. Let \( T \) be a single-valued operator from \( C \), which is supposed to be a convex closed cone, to \( L(X,E) \). The vector complementarity problem considered in this subsequent, \( (VCP) \) for short, is to find \( \overline{x} \in C \) such that

\[
\langle T(\overline{x}), \overline{x} \rangle \notin \text{int} \ P(\overline{x}), \text{ and } \langle T(\overline{x}), y \rangle \notin -\text{int} \ P(\overline{x}) \forall y \in C.
\]

This problem collapses to \( (CP) \) when \( E = \mathbb{R} \) and \( P(x) = \mathbb{R}_+ \) for all \( x \in C \).

By means of vector variational inequalities, we can formulate the following existence theorem for \( (VCP) \).

Theorem 3. Suppose that
(i) the set-valued map \( \text{int} P(\cdot) \) has an open graph in \( C \times L(X, E) \);

(ii) \( T \) is pseudomonotone and hemicontinuous;

(iv) there is a nonempty compact subset \( B \) in \( C \) such that for each \( A \in \mathcal{F}(C) \) there is a compact convex \( B_A \subset C \) containing \( A \) such that, for every \( y \in B_A \setminus B \), there exists \( x \in B_A \cap C \) with

\[
y \in \text{int}_C \{ v \in C : \langle Tv, x - v \rangle \in -\text{int} P(v) \}.
\]

Then (VCP) has at least one solution, which is in \( B \).

**Proof.** It is clear that all the assumptions of Theorem 2 are satisfied with \( \eta(x, y) = x - y \) for all \( x, y \in C \). Therefore there exists \( \bar{x} \in B \) such that

\[
\langle T\bar{x}, z - \bar{x} \rangle \notin -\text{int} P(\bar{x}) \quad \forall z \in C.
\]

(1)

Since \( C \) is a convex cone, then setting in (1), \( z = 0 \) and \( z = y + \bar{x} \) for an arbitrary \( y \in C \), we get respectively

\[
\langle T\bar{x}, \bar{x} \rangle \notin \text{int} P(\bar{x}) \quad \text{and} \quad \langle T\bar{x}, y \rangle \notin -\text{int} P(\bar{x}).
\]

Hence we conclude that \( \bar{x} \) is also a solution to (VCP).

\[\blacksquare\]

**c) Vector optimization**

Here, to convey an idea about the use of vector variational-like inequalities in vector optimization which involves smooth vector mappings, we prove the existence of solutions to weak vector optimization problems, (WVOP) for short, by considering the concept of invexity. Let us state the problem as follows.

Find \( \bar{x} \in C \) such that \( \phi(y) - \phi(\bar{x}) \notin -\text{int} P \) for all \( y \in C \), \quad (WVOP)

where \( \phi : C \to E \) is a given vector-valued function and \( P \) is a given convex cone in \( E \).

Let \( \eta : C \times C \to X \) be a given function, and denote by \( \nabla \phi \) the Fréchet derivative of \( \phi \) once the latter is assumed to be Fréchet differentiable.
Theorem 4. Suppose that $P$ is a convex cone in $E$ with $\text{int } P \neq \emptyset$, and let $\phi : C \rightarrow E$ be a Fréchet differentiable mapping. Assume that

(i) $\langle \nabla \phi(x), \eta(y, x) \rangle \notin -\text{int } P$ implies $\langle \nabla \phi(y), \eta(y, x) \rangle \notin -\text{int } P$ for all $x, y \in C$;

(ii) $\phi$ is $P$-invex with respect to $\eta$, that is,

$$\phi(y) - \phi(x) - \langle \nabla \phi(x), \eta(y, x) \rangle \in P \quad \forall x, y \in C.$$  

(iii) $\nabla \phi$ is hemicontinuous;

(iv) for each $x \in C$, $\eta(., x)$ is affine, $\eta(x, .)$ is continuous and $\eta(x, x) = 0$;

(v) there is a compact subset $B$ in $C$ such that for every finite subset $A$ in $C$ there is a compact convex $C_A \subset X$ containing $A$ such as to satisfy, for every $y \in C \setminus B$, there exists $x \in C_A \cap C$ with $\langle \nabla \phi(x), \eta(x, y) \rangle \in -\text{int } P$.

Then (WVOP) has at least one solution.

**Proof.** First, by virtue of Theorem 2 with $T := \nabla \phi$, we get

$$\langle \nabla \phi(x), \eta(y, x) \rangle \notin -\text{int } P \quad \forall y \in C.$$  

Then the $P$-invexity of $\phi$ allows us to conclude. \hfill \blacksquare

**References**


