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Kyoto University
SOME OBSERVATIONS OF APPROXIMANTS TO FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

Shin-ya Matsushita
Daishi Kuroiwa

Abstract

Let $E$ be a Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. In this paper, we study the convergence of the two sequences defined by

$$x_1 = x \in C, x_{n+1} = \alpha_n x + (1 - \alpha_n) QTx_n,$$

$$y_1 = y \in C, y_{n+1} = Q(\alpha_n y + (1 - \alpha_n) Ty_n), \quad n = 1, 2, \ldots,$$

where $0 \leq \alpha_n \leq 1$, and $Q$ is a sunny nonexpansive retraction from $E$ onto $C$.

1 Introduction

Let $E$ be a Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$ such that the set $F(T)$ of all fixed points of $T$ is nonempty. In 1998, Takahashi and Kim[8] defined two contraction mappings $S_t$ and $U_t$ the following: For a given $u \in C$ and each $t \in (0, 1)$,

$$S_t x = tu + (1 - t) QTx \quad \text{for all } x \in C \quad (1.1)$$

and

$$U_t x = Q(tu + (1 - t) Tx) \quad \text{for all } x \in C, \quad (1.2)$$

where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$. Then by the Banach contraction principle, there exists a unique element $x_t \in F(S_t)$ (resp. $y_t \in F(U_t)$), i.e.

$$x_t = tu + (1 - t) QTx_t \quad (1.3)$$

and

$$y_t = Q(tu + (1 - t) Ty_t). \quad (1.4)$$

Also, Takahashi and Kim[8] proved that if $E$ is a reflexive Banach space, $C$ is a nonempty closed convex subset of $E$ which has normal structure, $T$ is a nonexpansive nonself-mapping from $C$ into $E$ satisfying the weak inwardness condition. Suppose that $C$ is a sunny nonexpansive retract. Then $\{x_t\}$ (resp. $\{y_t\}$) defined by (1.3) (resp. (1.4)) converges strongly as $t \to 0$ to an element of $F(T)$. On the other hand, Shioji and Takahashi[7] studied the convergence of the iteration

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Sx_n \quad \text{for } n \geq 1.$$
where \( x, x_1 \) are elements of \( C \), \( S \) is a nonexpansive mapping from \( C \) into itself such that \( F(S) \) is nonempty. They proved \( \{ x_n \} \) converges strongly to an element of \( F(S) \).

In this paper, we deal with the strong convergence to fixed points of nonexpansive nonself-mapping \( T \), which satisfies new boundary condition. At first, We define a new boundary condition and obtain some results with respect to new boundary condition. Further we consider two iteration schemes for \( T \). Then we prove that the iterates converge strongly to fixed points of \( T \).

\section{Preliminaries}

Throughout this paper, we denote the set of all positive integer by \( \mathbb{N} \). Let \( E \) be a real Banach space with norm \( \| \cdot \| \), \( E^* \) a dual space of \( E \). The value of \( x^* \in E^* \) at \( x \in E \) will be denote by \( \langle x, x^* \rangle \). Let \( C \) be a closed convex subset of \( E \), and \( T \) a nonexpansive nonself-mapping from \( C \) into \( E \). We denote the set of all fixed points of \( T \) by \( F(T) \). Let \( D \) be a subset of \( C \). A mapping \( Q \) from \( C \) into \( D \) is said to be sunny if \( Q(Qx + t(x - Qx)) = Qx \) whenever \( Qx + t(x - Qx) \in C \) for \( x \in C \) and \( t \geq 0 \). A mapping \( Q \) from \( C \) into \( D \) is said to be retraction if \( Q^2 = Q \). A subset \( D \) of \( C \) is said to be a sunny nonexpansive retract if there exists sunny nonexpansive retraction of \( C \) onto \( D \). Concerning sunny nonexpansive retractions, The following lemma was proved by Bruck, Jr.[1], Reich[5]:

\textbf{Lemma 2.1} \textit{Let \( E \) be Banach space whose norm Gâteaux differentiable, \( C \) a convex subset of \( E \), \( D \) a nonempty subset of \( C \), and \( Q \) a retraction from \( C \) onto \( D \). Then \( Q \) is sunny nonexpansive if and only if}

\[ \langle x - Qx, J(y - Qx) \rangle \leq 0 \quad \text{for each} \quad x \in C \quad \text{and} \quad y \in D. \]

\[ \delta(\epsilon) = \inf \{ 1 - \frac{\| x + y \|}{2} : \| x \| \leq 1, \| x - y \| \geq \epsilon \} \]

for all \( \epsilon \) with \( 0 \leq \epsilon \leq 2 \). A Banach space \( E \) is said to be uniformly convex if \( \delta(\epsilon) > 0 \) for all \( \epsilon > 0 \). Let \( U = \{ x \in E : \| x \| = 1 \} \). The duality mapping \( J \) from \( E \) into \( 2^E \) is defined by

\[ J(x) = \{ y^* \in E^* : \langle x, y^* \rangle = \| x \|^2 = \| y \|^2 \}, \quad x \in E. \]

The norm of \( E \) is said to be Gâteaux differentiable norm if

\[ \lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} \quad \text{exists for each} \quad x, y \in U. \]

It is also said to be uniformly Gâteaux differentiable if for each \( y \in U \), the limit(2.5) is attained uniformly for \( x \in U \). It is well known that if the norm of \( E \) is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm weak star, uniformly continuous on each bounded subset of \( E \). A closed convex subset \( C \) of \( E \) is said to have normal structure, if for each bounded closed convex subset \( K \) of \( C \), which contains at least two points, there exists an element of \( K \) which is not a diametral point of \( K \). It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure.

Let \( \mu \) be a continuous, linear functional on \( l^\infty \) and let \( (a_1, a_2, \ldots) \in l^\infty \). We write \( \mu(a_n) \) instead of \( \mu((a_1, a_2, \ldots)) \). A function \( \mu \) is said to be Banach limit if

\[ \| \mu \| = \mu_n(1) = 1 \quad \text{and} \quad \mu_n(a_{n+1}) = \mu_n(a_n) \quad \text{for all} \quad (a_1, a_2, \ldots) \in l^\infty. \]
We know that if $\mu$ is Banach limit then

$$\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n$$

for all $a = (a_1, a_2 \ldots) \in \ell^\infty$. The following lemma was proved by Shioji and Takahashi[7].

**Lemma 2.2** Let $a$ be a real number, and $(a_1, a_2 \ldots) \in \ell^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limits $\mu$ and $\limsup_{n \to \infty} (a_{n+1} - a_n) \leq 0$. Then $\limsup_{n \to \infty} a_n \leq a$.

Next, we introduce several boundary conditions upon the nonself-mapping.

(i) **Rothe’s condition**: $T(\partial C) \subset C$, where $\partial C$ is boundary set of $C$;

(ii) **inwardness condition**: $Tx \in I_c(x)$ for all $x \in C$, where

$$I_c(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\};$$

(iii) **weak inwardness condition**: $Tx \in \text{cl } I_c(x)$ for all $x \in C$, where cl denotes the norm-closure; and

(iv) **nowhere normal-outward condition**: $Tx \in \{y \in E|y \neq x, Py = x\}^c$ where $P$ is the metric projection from $E$ onto $C$.

It is easily seen that there hold implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Now, we define a new boundary condition.

**Definition 2.1** (condition (C1)) $Tx \in S^c_x$ for all $x \in C$, where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$, $x \in C$, and $S^c_x = \{y \in E|y \neq x, Qy = x\}$.

**Remark 2.1** Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, and $T$ a nonexpansive nonself-mapping from $C$ into $H$. Then $T$ satisfies nowhere normal-outward condition if and only if $T$ satisfies condition (C1).

By using condition (C1), we obtain two propositions.

**Proposition 2.1** Let $E$ be a Banach space whose norm is uniformly Gâteaux differentiable, $C$ a nonempty closed convex subset of $E$, $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract, and $T$ satisfies weak inwardness condition then $T$ satisfies condition (C1).

**Proposition 2.2** Let $E$ be a Banach space, $C$ a nonempty closed convex subset of $E$, $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract, and $T$ satisfies condition (C1). Then $F(T) = F(QT)$, where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$.

This proposition is very simple, but very useful. By using this proposition, we can extend all fixed point theorems with respect to nonexpansive self-mappings in Banach space, because when $C$ is a sunny nonexpansive retract, $T$ is a nonexpansive nonself-mapping from $C$ into $E$ which satisfies condition (C1), by applying fixed point theorems to $QT$ where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$, we can obtain results concerned with fixed points of $QT$, then we have theorems concerned with fixed points of $T$. On the other hand, we follow the two corollaries, the proof mainly due to Takahashi and Kim[8].
Corollary 2.1 Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$ which has normal structure, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0$ and in this case, $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of $T$.

Corollary 2.2 Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$ which has normal structure, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then $T$ has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 0$ and in this case, $\{y_t\}$ converges strongly as $t \to 0$ to a fixed point of $T$.

Also, by using Reich[6]'s result, and propositino 2.2, we obtain two corollaries.

Corollary 2.3 Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0$ and in this case, $\{x_t\}$ converges strongly as $t \to 0$ to $Q_2u \in F(T)$ where $Q_2$ is the unique sunny nonexpansive retraction from $C$ onto $F(T)$.

Corollary 2.4 Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then $T$ has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 0$ and in this case, $\{y_t\}$ converges strongly as $t \to 0$ to $Q_2u \in F(T)$ where $Q_2$ is the unique sunny nonexpansive retraction from $C$ onto $F(T)$.

3 Main Results

In this section, we study two type strong convergence of nonexpansive nonself-mappings which satisfies condition (C1). The proof mainly due to Wittmann[10], and Shioji and Takahashi[7].

Theorem 3.1 Let $E$ be a uniformly convex Banach space whose norm is uniform Gâteaux differentiable, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$ such that $F(T) \neq \emptyset$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1). Let $Q_1$ be a sunny nonexpansive retraction from $E$ onto $C$, $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Q_1Tx_n$$

for $n \geq 1$.

Then, $\{x_n\}$ converges strongly to $Q_2x \in F(T)$, where $Q_2$ is a sunny nonexpansive retraction from $C$ onto $F(T)$. 

Theorem 3.2 Let $E$ be a uniformly convex Banach space whose norm is uniformly Gateaux differentiable, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$ such that $F(T) \neq \phi$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1). Let $Q_1$ be a sunny nonexpansive retraction from $E$ onto $C$, $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_1 = y \in C$ and

$$y_{n+1} = Q_1(\alpha_n y + (1 - \alpha_n)Ty_n)$$

for $n \geq 1$.

Then, $\{y_n\}$ converges strongly to $Q_2 y \in F(T)$, where $Q_2$ is a sunny nonexpansive retraction from $C$ onto $F(T)$.

References


