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ON THE STEFAN PROBLEM WITH SURFACE TENSION IN A VISCOS INCOMPRESSIBLE FLUID FLOW

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Abstract. A solidification/melting process with the supercooling near the interface is described by the Stefan problem with Gibbs-Thompson law at the interface and the initial-boundary value problem for the incompressible Navier-Stokes equations. This paper is devoted to prove that the set of classical solutions of the problem mentioned above converges to the solution of the problem without the supercooling as the surface tension coefficient tends to zero.

1. Introduction. Let a region $\Omega$ with outer boundary $\Sigma$ be separated by a moving boundary $\Gamma_t$ into the liquid region $\Omega_{t}^{(1)}$ and the solid region $\Omega_{t}^{(2)}$. Let $v$, $p$, and $\theta^{(1)}$ be the velocity, the pressure and the temperature of the liquid, respectively. They are assumed to satisfy the following equations:

\[(1.1) \quad \nabla \cdot v = 0,\]

\[(1.2) \quad \frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p - \nu \Delta v = f(\theta^{(1)}),\]

\[(1.3) \quad \frac{\partial \theta^{(1)}}{\partial t} + (v \cdot \nabla)\theta^{(1)} - \frac{1}{\rho C_{p}^{(1)}} \nabla \cdot (\kappa^{(1)}(\theta^{(1)}) \nabla \theta^{(1)}) = \frac{2\nu}{C_{p}^{(1)}} \mathrm{D}(v) \cdot \nabla (v) \quad \text{in } \bigcup_{0 < t \leq T} (\Omega_{t}^{(1)} \times \{t\}),\]

These are the Navier-Stokes equations and the heat equation with the transport and viscous dissipation terms, where $\nu$, $\rho$, $C_{p}^{(1)}$, and $\kappa^{(1)}$ are a kinematic viscosity, the density, the specific heat at the constant pressure and the heat conductivity of the liquid, respectively. In $\Omega_{t}^{(2)}$, we consider only the heat transfer:

\[(1.4) \quad \frac{\partial \theta^{(2)}}{\partial t} - \frac{1}{\rho_{e} C_{p}^{(2)}} \nabla \cdot (\kappa^{(2)}(\theta^{(2)}) \nabla \theta^{(2)}) = 0 \quad \text{in } \bigcup_{0 < t \leq T} (\Omega_{t}^{(2)} \times \{t\}),\]

where $\rho_{e}$, $C_{p}^{(2)}$, and $\kappa^{(2)}$ are the density, the specific heat at the constant pressure and the heat conductivity of the solid, respectively. On the liquid-solid interface $\Gamma_t$, we impose the following conditions:

\[(1.5) \quad v \cdot n = \left(1 - \frac{\rho_{e}}{\rho}\right) V,\]

\[(1.6) \quad 2\nu \Pi \mathrm{D}(v) n = \Pi [v (v - V n)^{*}] n,\]

\[(1.7) \quad l_{\rho_{e}} V = - \left(\kappa^{(1)}(\theta^{(1)}) \nabla \theta^{(1)} - \kappa^{(2)}(\theta^{(2)}) \nabla \theta^{(2)}\right) \cdot n,\]
\begin{equation}
\theta^{(1)} = \theta^{(2)} = \theta_1 \left(1 - \frac{\sigma}{l} H\right),
\end{equation}
or
\begin{equation}
\theta^{(1)} = \theta^{(2)} = \theta_1 \quad \text{on} \quad \bigcup_{0 < t \leq T} (\Gamma_t \times \{t\}).
\end{equation}

These conditions are derived by applying conservation laws of mass, momentum and energy across the interface. But here we impose thermal equilibrium conditions \((1.8)\) or \((1.9)\) instead of the normal component of momentum. Especially condition \((1.8)\) is called the Gibbs-Thompson’s law. Here \(\Pi, D(\nu),\) and \(H\) are a projection operator on \(\Gamma_t,\) the velocity deformation tensor and the twice mean curvature of \(\Gamma_t,\) respectively. \(l, \theta_1\) and \(\sigma\) are the latent heat, the equilibrium temperature and the surface tension, respectively. To complete the problem, we further impose the initial and boundary conditions on the rigid boundary \(\Sigma:\)

\begin{equation}
\begin{cases}
\nu = v_{\sigma,0} \quad \text{or} \quad \nu = v_0, \\
\theta^{(1)} = \theta^{(1)}_{\sigma,0} \quad \text{or} \quad \theta^{(1)} = \theta^{(1)}_{0}
\end{cases}
on \hat{\Omega}^{(1)} \equiv \hat{\Omega}^{(1)}_0,
\end{equation}

\begin{equation}
\begin{cases}
\nu = 0, \\
\theta^{(1)} = \theta_2
\end{cases}
on \Sigma_T.
\end{equation}

In the sequel, by \((P_\sigma)\) we mean problem \((1.1)-(1.8),\) \((1.10)-(1.12),\) and by \((P)\) problem \((1.1)-(1.7),(1.9)-(1.12).\) \((v_{\sigma,0}, \theta^{(1)}_{\sigma,0}, \theta^{(2)}_{\sigma,0})\) and \((v_0, \theta^{(1)}_{0}, \theta^{(2)}_{0})\) are initial data imposed on problems \((P_\sigma)\) and \((P),\) respectively.

In [5] and [7], we have proved the unique classical solvability of problems \((P_\sigma)\) and \((P),\) respectively. In this paper, we prove that the problem \((P_\sigma)\) is uniquely solvable on a certain finite time interval independent of \(\sigma \in (0, \sigma^*), \sigma^* << 1,\) and that problem \((P)\) is the limit case of problem \((P_\sigma)\) as \(\sigma\) tends to zero. This is done on the basis of a uniform estimate of the solution of problem \((P_\sigma)\) with respect to \(\sigma\) which is obtained in some wider space of functions than the space defined in [5]. Bazalií and Degtyarev [1] also studied such a limit problem of the Stefan problem with Gibbs-Thompson’s law involving only the process of heat transfer. They showed the convergence in a class that the space of the limit functions is compactly embedded. We prove this convergence holds in the same class of the limit functions.

We study the above problem in the function spaces defined as follows. Let \(D_T\) be a cylindrical domain \(D \times (0, T),\) where \(D\) is a domain in \(\mathbb{R}^n,\) and \(T > 0.\) Let \(l\) be a non-negative integer and \(\alpha \in (0, 1).\) By \(C^{l+\alpha, \frac{l+\alpha}{2}}(D_T)\) we denote anisotropic Hölder space of functions whose norm are defined by

\[ |f|^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T} = \sum_{2r+|m|=0}^{l} |\partial_t^r \partial_x^m f|^{(0)}_{D_T} + \langle f \rangle^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T}, \]

where

\[ \langle f \rangle^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T} = \sum_{2r+|m|=l-1}^{l} |\partial_t^r \partial_x^m f|^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T} + \sum_{2r+|m|=l}^{l} |\partial_t^r \partial_x^m f|^{(0)}_{D_T}. \]
\[
\begin{aligned}
|f|_{D_{T}}^{(0)} & \equiv \sup_{(x,t)\in D_{T}} |f(x,t)|, \\
|f|_{L,D_{T}}^{(\alpha)} & \equiv \sup_{(x,t),(x',t')\in D_{T}, t \neq t'} \frac{|f(x,t) - f(x',t')}{|t - t'|^{\frac{\alpha}{2}}}, \\
|f|_{x,D_{T}}^{(\alpha)} & \equiv \sup_{(x,t),(x',t')\in D_{T}, x \neq x'} \frac{|f(x,t) - f(x',t')}{|x - x'|^{\alpha}},
\end{aligned}
\]

and

\[
|m| = \sum_{i=1}^{n} m_{i}, \quad \partial_{x}^{m} = \frac{\partial^{m}}{\partial_{x_{1}}^{m_{1}}} \cdots \partial_{x_{n}}^{m_{n}}
\]

for a multi index \( m = (m_{i}) \) (\( m_{i} \geq 0, i = 1, \ldots, n \)).

By \( \tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \) and \( C_{\sigma}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \) we denote the function spaces

\[
\left\{ f \in C^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \mid \partial_{t}f \in C^{l-1+\alpha,\frac{l-1+\alpha}{2}}(D_{T}) \right\}
\]

equipped with the norm

\[
||f||_{D_{T}}^{(l+\alpha,\frac{l+\alpha}{2})} \equiv ||f||_{D_{T}}^{(l+\alpha,\frac{l+\alpha}{2})} + |\partial_{t}f|_{D_{T}}^{(l-1+\alpha,\frac{l-1+\alpha}{2})}
\]

and

\[
\left\{ f \in \tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \mid \partial_{x}^{m}f \in \tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}), \quad |m| = 2 \right\}
\]

equipped with the norm

\[
|f|_{\sigma,D_{T}}^{(l+\alpha,\frac{l+\alpha}{2})} \equiv ||f||_{D_{T}}^{(l+\alpha,\frac{l+\alpha}{2})} + \sigma \sum_{|m|=2} |\partial_{x}^{m}f|_{D_{T}}^{(l+\alpha,\frac{l+\alpha}{2})} \quad (\sigma > 0),
\]

respectively. By \( C_{0}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \), \( \tilde{C}_{0}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \) and \( C_{\sigma,0}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \) we denote the function spaces

\[
\left\{ f \in C^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \mid \partial_{t}^{k}f|_{t=0} = 0, \quad k = 0, 1, \ldots, \left[ \frac{l+\alpha}{2} \right] \right\},
\]

\[
\left\{ f \in \tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \mid \partial_{t}^{k}f|_{t=0} = 0, \quad k = 0, 1, \ldots, \left[ \frac{l+1+\alpha}{2} \right] \right\}
\]

and

\[
\left\{ f \in C_{\sigma}^{l+\alpha,\frac{l+\alpha}{2}}(D_{T}) \mid \partial_{t}^{k}f|_{t=0} = 0, \quad k = 0, 1, \ldots, \left[ \frac{l+1+\alpha}{2} \right] \right\},
\]

respectively. By \( C^{l+\alpha}(D) \), we define the space of functions \( f(x), x \in D \), with the norm

\[
|f|_{D}^{l+\alpha} \equiv \sum_{|m| \leq l} |D_{x}^{m}f|_{D}^{(0)} + (f|_{D}^{(l+\alpha)}, \quad |f|_{D}^{(0)} \equiv \sup_{x \in D} |f(x)|,
\]
$$(f)_{D}^{(\gamma,\alpha)} \equiv \sum_{|m|=l} \langle D_{x}^{m} f \rangle_{D}^{(\gamma,\alpha)} \equiv \sup_{x,y \in D, \ m=|l|} \frac{|D_{x}^{m} f(x) - D_{y}^{m} f(y)|}{|x-y|^{\alpha}}.$$ 

We also need the following seminorm:

$$\langle(f)\rangle_{D_{T}}^{(\gamma,\alpha)} \equiv \sup_{r, t \in (0, T), \ r \neq t} \frac{\langle f(x, t) - f(x, \tau) \rangle_{D}^{(\gamma)}}{|t - \tau|^{\frac{1+\alpha}{2}}}.$$ 

where $\alpha, \gamma \in (0,1)$. Furthermore, by $\mathcal{H}^{\gamma,\alpha}$, $X_{\sigma,T}^{\gamma,\alpha}$ and $\mathcal{X}_{\sigma,T}^{\gamma,\alpha}$ we mean function spaces $C^{\gamma,\alpha}((0, \infty) \times \Omega^{(1)}) \times C^{\gamma,\alpha}((0, \infty) \times \Omega^{(1)}) \times C^{\gamma,\alpha}((0, \infty) \times \Omega^{(2)})$, $C^{\gamma,\alpha}((0, \infty) \times \Omega^{(1)}) \times C^{\gamma,\alpha}((0, \infty) \times \Omega^{(1)}) \times C^{\gamma,\alpha}((0, \infty) \times \Omega^{(2)})$, $\Sigma \in C^{\gamma,\alpha}$

and the inequalities

$$\kappa_{0} < \kappa^{(1)}(\theta_{\sigma,0}) < \kappa_{0}^{-1}, \quad \sum_{i=1,2} \left| (-1)^{i-1} \kappa^{(i)}(\theta_{\sigma,0}) \nabla \theta_{\sigma,0} \cdot n \right|_{\Gamma} > a_{0},$$

$$|\rho - \rho_{0}| \leq b_{0}, \quad \sum_{i=1,2} \left| (-1)^{i-1} \kappa^{(i)}(\theta_{\sigma,0}) \nabla \theta_{\sigma,0} \cdot n \right|_{\Gamma}^{(0)} \leq b_{0}$$

hold for some positive constants $\kappa_{0} \leq 1$, $a_{0}$ and $b_{0} < 1/(4C_{3})$, $C_{3}$ in (4.1), where $\tau$ is a tangential vector to $\Gamma$. Moreover we assume that the compatibility conditions up to order 1 hold. Then problem $(P_{\sigma})$ has a unique solution $(\psi, \nabla p, \theta_{\sigma,0}, \theta_{\sigma,1}, \theta_{\sigma,2}, \Gamma_{t}) \in \mathcal{X}_{\sigma,T_{0}}^{\alpha}$ for some $T_{0} > 0$ which is independent of $\sigma$.

Furthermore, let $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(1)}, \theta_{\sigma,1}^{(1)}, \theta_{\sigma,2}^{(1)}, \Gamma_{\sigma,t})$ be a set of solutions of problem $(P_{\sigma})$ in the space $\mathcal{X}_{\sigma,T_{0}}^{2,\alpha}$, $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(2)}, \theta_{\sigma,1}^{(2)}, \theta_{\sigma,2}^{(2)}, \Gamma_{\sigma,t})$ be a solution of problem $(P)$ in the space $\mathcal{X}_{\sigma,T}^{2,\alpha}$ and $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(1)}, \theta_{\sigma,1}^{(1)}, \theta_{\sigma,2}^{(1)}, \Gamma_{\sigma,t})$ converge to $(\psi_{0}, \theta_{0}^{(1)}, \theta_{0}^{(2)}, \Gamma_{t})$ in the space $\mathcal{H}^{\alpha}$ as $\sigma$ tends to 0, then $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(1)}, \theta_{\sigma,1}^{(1)}, \theta_{\sigma,2}^{(1)}, \Gamma_{\sigma,t})$ converges to $(\psi, \nabla p, \theta_{1}^{(1)}, \theta_{2}^{(1)}, \Gamma_{t})$ in the space $\mathcal{X}_{\sigma,T}^{2,\alpha}$ as $\sigma$ tends to 0 on some interval $[0, T]$ which is independent of $\sigma$. 

THEOREM 1.1. Assume that

$$\Gamma \equiv \Gamma_{0} \in C^{5+\alpha}, \quad \Sigma \in C^{4+\alpha},$$

$$f \in C^{1+\alpha}(0, \infty), \quad \kappa^{(i)} \in C^{3+\alpha}(0, \infty), \quad \psi_{\sigma,0} \in C^{3+\alpha}((0, \infty),$$

$$\theta_{\sigma,0} \in C^{4+\alpha}((0, \infty), \quad \theta_{1} \in C^{4+\alpha, 2a}((R^{3} \times (0, T)), \quad \theta_{2} \in C^{4+\alpha, 2a}((R^{3} \times (0, T)),$$

and the inequalities

$$\kappa_{0} < \kappa^{(i)}(\theta_{\sigma,0}) < \kappa_{0}^{-1}, \quad \sum_{i=1,2} \left| (-1)^{i-1} \kappa^{(i)}(\theta_{\sigma,0}) \nabla \theta_{\sigma,0} \cdot n \right|_{\Gamma} > a_{0},$$

$$|\rho - \rho_{0}| \leq b_{0}, \quad \sum_{i=1,2} \left| (-1)^{i-1} \kappa^{(i)}(\theta_{\sigma,0}) \nabla \theta_{\sigma,0} \cdot n \right|_{\Gamma}^{(0)} \leq b_{0}$$

hold for some positive constants $\kappa_{0} \leq 1$, $a_{0}$ and $b_{0} < 1/(4C_{3})$, $C_{3}$ in (4.1), where $\tau$ is a tangential vector to $\Gamma$. Moreover we assume that the compatibility conditions up to order 1 hold. Then problem $(P_{\sigma})$ has a unique solution $(\psi, \nabla p, \theta_{\sigma,0}, \theta_{\sigma,1}, \theta_{\sigma,2}, \Gamma_{t}) \in \mathcal{X}_{\sigma,T_{0}}^{\alpha}$ for some $T_{0} > 0$ which is independent of $\sigma$. 

Furthermore, let $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(1)}, \theta_{\sigma,1}^{(1)}, \theta_{\sigma,2}^{(1)}, \Gamma_{\sigma,t})$ be a set of solutions of problem $(P_{\sigma})$ in the space $\mathcal{X}_{\sigma,T_{0}}^{2,\alpha}$, $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(2)}, \theta_{\sigma,1}^{(2)}, \theta_{\sigma,2}^{(2)}, \Gamma_{\sigma,t})$ be a solution of problem $(P)$ in the space $\mathcal{X}_{\sigma,T}^{2,\alpha}$ and $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(1)}, \theta_{\sigma,1}^{(1)}, \theta_{\sigma,2}^{(1)}, \Gamma_{\sigma,t})$ converge to $(\psi_{0}, \theta_{0}^{(1)}, \theta_{0}^{(2)}, \Gamma_{t})$ in the space $\mathcal{H}^{\alpha}$ as $\sigma$ tends to 0, then $(\psi_{\sigma}, \nabla p_{\sigma}, \theta_{\sigma,0}^{(1)}, \theta_{\sigma,1}^{(1)}, \theta_{\sigma,2}^{(1)}, \Gamma_{\sigma,t})$ converges to $(\psi, \nabla p, \theta_{1}^{(1)}, \theta_{2}^{(1)}, \Gamma_{t})$ in the space $\mathcal{X}_{\sigma,T}^{2,\alpha}$ as $\sigma$ tends to 0 on some interval $[0, T]$ which is independent of $\sigma$. 

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2. Reduction of the problem. Let $\mathcal{M}$ be a 2-dimensional manifold which is isometric to $\Gamma$, $\omega = (\omega_1, \omega_2)$ be a generic point on $\mathcal{M}$ and

\[ X_0 : \mathcal{M} \rightarrow \Gamma \]

be a $C^{5+\alpha}$-diffeomorphism. We define a mapping $X$ from $\mathcal{M} \times [-\gamma_0, \gamma_0]$ to a neighborhood $N_0$ of $\Gamma$ in the form

\[ X(\omega, \lambda) = X_0(\omega) + \mathbf{n}(X_0(\omega))\lambda \]

where $\mathbf{n}(X_0(\omega))$ is a unit normal to $\Gamma$ at $X_0(\omega)$ directing into $\Omega^{(1)}$. Here a positive number $\gamma_0$ is assumed to be chosen so small that the mapping $X$ is regular and one-to-one. Let $(\omega(x), \lambda(x))$ be the inverse mapping of $X$, and introduce the following notation:

\[ \phi^{(i)}(\omega, \lambda) = \nabla_x \omega_i(x)|_{x=X(\omega, \lambda)}, \quad i = 1, 2, \]

\[ \phi^{(3)}(\omega, \lambda) = \nabla_x \lambda(x)|_{x=X(\omega, \lambda)}, \]

\[ M^{(k)} = \left( \frac{\partial^2}{\partial x_i \partial x_j} \omega_k(x) \right)_{i,j=1,2,3}|_{x=X(\omega, \lambda)}, \quad k = 1, 2, \]

\[ M^{(3)} = \left( \frac{\partial^2}{\partial x_i \partial x_j} \lambda(x) \right)_{i,j=1,2,3}|_{x=X(\omega, \lambda)}. \]

Now, for some $T > 0$, let us assume that the interface $\Gamma_t$, $t \in [0, T]$, is represented by $X_0(\omega(x)) + \mathbf{n}(X_0(\omega(x)))d(\omega(x), t)$ with some function $d(\omega, t)$ satisfying $d(\omega, 0) = 0$. Then $\bigcup_{0 \leq t \leq T} (\Gamma_t \times \{t\})$ can be represented as

\[ \{(x, t) \in N_0 \times [0, T] | \Phi_d(x, t) \equiv \lambda(x) - d(\omega(x), t) = 0\}. \]

Accordingly, the Stefan condition (1.7) can be written as

\[ \rho_e \frac{\partial \Phi_d}{\partial t} - \kappa^{(1)}(\theta^{(1)})(\nabla \Phi_d \cdot \nabla \theta^{(1)}) + \kappa^{(2)}(\theta^{(2)})(\nabla \Phi_d \cdot \nabla \theta^{(2)}) = 0 \]

and the twice mean curvature of $\Gamma_t$ as

\[ H(\omega, t) = \frac{1}{|\nabla_x \Phi_d|} \left( \sum_{i,j=1,2} a_{ij}(\omega, d, \nabla \omega) \frac{\partial^2 d}{\partial \omega_i \partial \omega_j} + b(\omega, d, \nabla \omega) \right) \]

with

\[ a_{ij}(\omega, d, p_1, p_2) = (\phi^{(i)} \cdot \phi^{(j)}) - \left[ \sum_{k,l=1,2} (p_k \phi^{(k)} \cdot \phi^{(i)})(p_l \phi^{(l)} \cdot \phi^{(j)}) \right] \times \left( 1 + \left[ \sum_{k=1,2} p_k \phi^{(k)} \right]^2 \right)^{-1}, \]
\[
b(\omega, d, p_1, p_2) = \sum_{k=1,2} p_k \text{Tr}(M^{(k)}) - \text{Tr}(M^{(3)}) - \left[ \sum_{k,l=1,2} p_k p_l \phi^{(k)} \phi^{(l)} - \sum_{k,l,m=1,2} p_k p_l p_m \phi^{(k)} \phi^{(l)} \right] = \left( 1 + \left| \sum_{k=1,2} p_k \phi^{(k)} \right|^2 \right)^{-1}
\]

where \( p_k = \partial d / \partial \omega_k, k = 1, 2 \) (see [2]). Here we denote by \((\mathbf{a} \cdot \mathbf{b}), \text{Tr}(A)\) and \(a^T\) the scalar product of the vectors \(\mathbf{a}\) and \(\mathbf{b}\) in \(\mathbb{R}^3\), the trace of the matrix \(A\) and the transposed vector of \(\mathbf{a}\), respectively.

Next we introduce a transformation \(e_d\) (see [4]). Let \(X_T\) and \(Y_T\) be two coordinates \((x_1, x_2, x_3, t)\) and \((y_1, y_2, y_3, t)\) in \(\mathbb{R}^3 \times [0,T]\) such that \(x = X(\omega, \lambda), y = X(\omega, \eta)\). Then the mapping \(e_d : Y_T \rightarrow X_T\) is defined by

\[
e_d(X(\omega, \eta), t) = \begin{cases} 
(X(\omega, \eta + \chi(\eta) d(\omega, t)), t) & \text{if } (x, t) \in N_0 \times [0,T], \\
(X(\omega, \lambda), t) & \text{if } (x, t) \in N_0' \times [0,T],
\end{cases}
\]

where \(\chi(\lambda) \in C^\infty(-\infty, +\infty)\) is a cut-off function satisfying

\[
\chi(\lambda) = \begin{cases} 
1 & \text{for } |\lambda| \leq \frac{\gamma_0}{4}, \\
0 & \text{for } |\lambda| \geq \frac{3\gamma_0}{4}, \\
|\chi'(\lambda)| \leq \frac{4}{3\gamma_0}
\end{cases}
\]

It is obvious that \(Q_T^{(1)} = \Omega^{(1)} \times (0, T), Q_T^{(2)} = \Omega^{(2)} \times (0, T)\) and \(\Gamma_T = \Gamma \times (0, T)\) are transformed onto \(U_{0 \leq t \leq T}(\Omega^{(1)}_t \times \{t\}), U_{0 \leq t \leq T}(\Omega^{(2)}_t \times \{t\})\) and \(U_{0 \leq t \leq T}(\Gamma_t \times \{t\})\), respectively by \(e_d\). By denoting simply the transformed functions \(\theta^{(1)} \circ e_d, \theta^{(2)} \circ e_d, \nabla v, \nabla p, \phi\) and \(\phi\), respectively, problem \((P_\sigma)\) can be rewritten in the fixed domain \(Q_T^{(1)} \cup Q_T^{(2)}\) of the variables \((y, t)\).

\[
\begin{cases}
\frac{\partial v}{\partial t} + (h_d \cdot \nabla)v + (v \cdot \nabla d)v - \nu \nabla^2 v + \nabla d p = f(\theta^{(1)}) & \text{in } Q_T^{(1)}, \\
\nabla_d \cdot v = 0 & \text{in } Q_T^{(1)}, \\
v|_{t=0} = v_{\sigma,0} & \text{on } \overline{\Omega}^{(1)}, \\
v = 0 & \text{on } \Sigma_T, \\
\begin{cases} 
v \cdot n_d = \left(1 - \frac{p_e}{\rho}\right) V, \\
2\nu \Pi_d D_d(v) n_d = \Pi_d [v(v - V n_d)^*] n_d & \text{on } \Gamma_T,
\end{cases}
\end{cases}
\]

(2.1)
\[
\begin{align*}
\frac{\partial \theta^{(1)}}{\partial t} + (h_d \cdot \nabla) \theta^{(1)} + (v \cdot \nabla_d) \theta^{(1)} - \frac{1}{\rho C_p^{(1)}} \nabla_d \cdot (\kappa^{(1)}(\theta^{(1)})\nabla_d \theta^{(1)}) &= \frac{2\nu}{C_p^{(1)}} \mathrm{D}_{d}(v) \quad \text{in } Q_T^{(1)}, \\
\frac{\partial \theta^{(2)}}{\partial t} + (h_d \cdot \nabla) \theta^{(2)} - \frac{1}{\rho C_p^{(2)}} \nabla_d \cdot (\kappa^{(2)}(\theta^{(2)})\nabla_d \theta^{(2)}) &= 0 \quad \text{in } Q_T^{(2)},
\end{align*}
\]

(2.2)
\[
\begin{align*}
\theta^{(1)}|_{t=0} &= \theta_{\sigma,0}^{(1)} \quad \text{on } \overline{\Omega}^{(1)}, \\
\theta^{(2)}|_{t=0} &= \theta_{\sigma,0}^{(2)} \quad \text{on } \overline{\Omega}^{(2)}, \\
\theta^{(1)} &= \theta^{(2)} = \theta_1 \left(1 - \frac{\sigma}{l} \right) H \quad \text{on } \Sigma_T,
\end{align*}
\]

Here we set
\[
\begin{align*}
\nabla_d &= (E_d^*)^{-1} \nabla, \\
h_d &= \frac{\partial y}{\partial t} e_d, \\
n_d &= \frac{\nabla_d \eta}{|\nabla_d \eta|}, \\
\mathrm{D}_{d}(v) &= \mathrm{D}(v) \circ e_d, \\
\Pi_{d}g &= \Pi g \circ e_d, \\
\end{align*}
\]

and \(E_d = (a_{ij})\) is the Jacobian matrix of the mapping from \(y\) to \(x\), \(a_{ij}\) is the \(ij\)-component of \((E_d^*)^{-1}\) and \(E_d^*\) is the transposed matrix of \(E_d\).

Extensions \(\hat{\theta}_{\sigma}^{(1)} \in C^{4+\alpha,\frac{4+\alpha}{2}}(Q_T^{(1)}), \hat{\theta}_{\sigma}^{(2)} \in C^{4+\alpha,\frac{4+\alpha}{2}}(Q_T^{(2)}), \hat{d}_{\sigma} \in C^{5+\alpha,\frac{5+\alpha}{2}}(\mathcal{M} \times [0,T]), \hat{v}_{\sigma} \in C^{3+\alpha,\frac{3+\alpha}{2}}(Q_T^{(1)}), \nabla \hat{p}_{\sigma} \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T^{(1)})\) of the initial data can be constructed to satisfy the conditions:

\[
\begin{align*}
\hat{\theta}_{\sigma}^{(1)}(y,0) &= \theta_{\sigma,0}^{(1)}(y), \\
\frac{\partial \hat{\theta}_{\sigma}^{(1)}}{\partial t}(y,0) &= \theta_{\sigma,1}^{(1)}(y), \\
\hat{\theta}_{\sigma}^{(2)}(y,0) &= \theta_{\sigma,0}^{(2)}(y), \\
\frac{\partial \hat{\theta}_{\sigma}^{(2)}}{\partial t}(y,0) &= \theta_{\sigma,1}^{(2)}(y), \\
\hat{d}_{\sigma}(\omega,0) &= 0, \\
\frac{\partial \hat{d}_{\sigma}}{\partial t}(\omega,0) &= d_{\sigma,0}^{(1)}(\omega), \\
\frac{\partial^2 \hat{d}_{\sigma}}{\partial t^2}(\omega,0) &= d_{\sigma,0}^{(2)}(\omega), \\
\hat{\nu}_{\sigma}(y,0) &= \nu_{\sigma,0}(y), \\
\frac{\partial \hat{\nu}_{\sigma}}{\partial t}(y,0) &= \nu_{\sigma,0}(y), \\
\nabla \hat{v}_{\sigma}(y,0) &= -\nu_{\sigma,0}(y) - (\nu_{\sigma,0}(y) \cdot \nabla) \nu_{\sigma,0}(y) + \nu \nabla^2 \nu_{\sigma,0}(y) + f(\theta_{\sigma,0}(y))
\end{align*}
\]

and the inequality:

\[
\sum_{i=1,2} |\hat{\theta}_{\sigma}^{(i)}|_{Q_T^{(i)}}^{(4+\alpha,\frac{4+\alpha}{2})} + |\hat{d}_{\sigma}|_{\Gamma_T}^{(5+\alpha,\frac{5+\alpha}{2})} + |\nabla \hat{p}_{\sigma}|_{Q_T^{(i)}}^{(1+\alpha,\frac{1+\alpha}{2})} + |\hat{v}_{\sigma}|_{Q_T^{(i)}}^{(3+\alpha,\frac{3+\alpha}{2})}
\leq c \left( \sum_{i=1,2} |\theta_{\sigma,0}^{(i)}|_{Q_T^{(i)}}^{(4+\alpha)} + |\nu_{\sigma,0}|_{Q_T^{(i)}}^{(2+\alpha)} \right),
\]

with a constant \(c\) being bounded as \(T \rightarrow 0\). Here the functions \(\theta_{\sigma,0}^{(1)}, \theta_{\sigma,0}^{(2)}, d_{\sigma,0}^{(1)}, d_{\sigma,0}^{(2)}, \nu_{\sigma,0}^{(1)}\) are defined from the compatibility conditions between the equations and the data.
in (2.1) and (2.1) (see, [7]). Furthermore, the inequality

\begin{equation}
\sum_{i=1,2} |\hat{\theta}_{\sigma}^{(i)} - \hat{\theta}^{(i)}|_{Q_{T}^{(i)}}^{(4+\alpha, \frac{4\alpha}{2})} + |\hat{d}_{\sigma} - \hat{d}|_{\Gamma_{T}}^{(5+\alpha,)} + |\nabla \hat{p}_{\sigma} - \nabla \hat{p}|_{Q_{T}^{(1)}}^{(1+\alpha, \frac{1+a}{2})} + |\hat{v}_{\sigma} - \hat{v}|_{Q_{T}^{(1)}}^{(3+\alpha, \frac{3+a}{2})}
\leq c \left( \sum_{i=1,2} |\theta_{\sigma,0}^{(i)} - \theta_{0}^{(i)}|_{\Omega t}^{(4+\alpha)} + |v_{\sigma,0} - v_{0}|_{\Omega(1)}^{(3+\alpha)} \right),
\end{equation}

obviously holds, where \((\hat{v}, \nabla \hat{p}, \hat{\theta}^{(1)}, \hat{\theta}^{(2)})\) is an extension corresponding to problem \((P)\). Then by introducing the new unknown functions \(w^{(i)} \equiv \theta^{(i)} - \hat{\theta}_{\sigma}^{(i)} - \chi(\nabla \eta \cdot \nabla \hat{\theta}_{\sigma}) \delta\), \(i = 1, 2\), \(\delta \equiv d - \hat{d}_{\sigma}\), \(u \equiv v - \hat{v}_{\sigma}\) and \(\nabla q \equiv \nabla p - \nabla \hat{p}_{\sigma}\), problem (2.1)-(2.2) can be written in the equivalent form

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + \nabla q = F_{1}(u, \nabla q, w^{(1)}, \delta) \text{ in } Q_{T}^{(1)}, \\
\nabla \cdot u = F_{2}(u, \delta) \text{ in } Q_{T}^{(1)}, \\
u \cdot n = \mathcal{F}_{3}(u, \delta) \text{ on } \Sigma_{T}, \\
u \cdot n = \mathcal{F}_{4}(u, \delta) \text{ on } \Gamma_{T},
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\frac{\partial w^{(1)}}{\partial t} - \frac{1}{\rho C_{p}^{(1)}} \nabla \cdot \left( \kappa^{(1)}(\hat{\theta}_{\sigma}) \nabla w^{(1)} \right) - \frac{4\nu}{C_{p}^{(1)}} \mathrm{D}(\hat{u}) : \mathrm{D}(\hat{\theta}) = G_{1}(u, w^{(1)}, \delta) \text{ in } Q_{T}^{(1)}, \\
\frac{\partial w^{(2)}}{\partial t} - \frac{1}{\rho C_{p}^{(2)}} \nabla \cdot \left( \kappa^{(2)}(\hat{\theta}_{\sigma}) \nabla w^{(2)} \right) = G_{2}(w^{(2)}, \delta) \text{ in } Q_{T}^{(2)}, \\
w^{(1)}|_{t=0} = 0 \text{ on } \bar{\Omega}^{(1)}, \\
w^{(2)}|_{t=0} = 0 \text{ on } \bar{\Omega}^{(2)}, \\
w^{(1)} = \theta_{2} - \hat{\theta}_{\sigma}^{(1)} \text{ on } \Sigma_{T},
\end{cases}
\end{equation}

where by \(\mathcal{F}_{i}, i = 1, \cdots, 4\), in (2.5) and \(\mathcal{G}_{i}, i = 1, \cdots, 5\), in (2.6) we mean nonlinear terms derived by the above linearization. The explicit representations of \(\mathcal{F}_{i}, i = 1, \cdots, 4\), and \(\mathcal{G}_{i}, i = 1, \cdots, 3\), have the same form given in [7], hence we omit them here. The representations of \(\mathcal{G}_{4}\) and \(\mathcal{G}_{5}\) will be given in section 4.
3. The linear problems. In this section we are concerned with the linear problems.

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + \nabla q &= F_1, \quad \nabla \cdot u = F_2 \quad \text{in}\ Q_T^{(1)}, \\
u \mathbf{n} &= F_3, \quad 2\nu \Pi \mathbf{D}(u) = F_4 \quad \text{on}\ \Gamma_T, \\
\frac{\partial w^{(1)}}{\partial t} &= \frac{1}{\rho_c C_p^{(1)}} \nabla \cdot \left( \kappa^{(1)}(\sigma^{(1)}) \nabla w^{(1)} \right) \\
&\quad - \frac{4\nu}{C_p^{(1)}} \mathbf{D}(u) \colon \mathbf{D}(\sigma) = G_1 \quad \text{in}\ Q_T^{(1)}, \\
\frac{\partial w^{(2)}}{\partial t} &= \frac{1}{\rho_c C_p^{(2)}} \nabla \cdot \left( \kappa^{(2)}(\sigma^{(2)}) \nabla w^{(2)} \right) = G_2 \quad \text{in}\ Q_T^{(2)}, \\
w^{(1)}|_{t=0} &= 0 \quad \text{on}\ \Omega^{(1)}, \\
w^{(2)}|_{t=0} &= 0 \quad \text{on}\ \Omega^{(2)}, \\
w^{(1)} &= H_2 \quad \text{on}\ \Sigma_T, \\
w^{(2)} &= H_2 \quad \text{on}\ \Sigma_T, \\
\frac{\partial \delta}{\partial t} + \frac{1}{\rho'_l} \kappa^{(1)}(\sigma^{(1)})(\nabla \eta \cdot \nabla w^{(1)}) \\
&\quad - \frac{1}{\rho'_c} \kappa^{(2)}(\sigma^{(2)})(\nabla \eta \cdot \nabla w^{(2)}) = G_3, \\
\frac{\partial w^{(1)}}{\partial \mathbf{n}} \delta - \frac{\sigma^{(1)}}{l} \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^2 \delta}{\partial \omega_i \partial \omega_j} = G_4, \\
\frac{\partial w^{(2)}}{\partial \mathbf{n}} \delta - \frac{\sigma^{(2)}}{l} \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^2 \delta}{\partial \omega_i \partial \omega_j} = G_5 \quad \text{on}\ \Gamma_T
\end{align*}
\]

We treat the above problem separately, that is,

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + \nabla q &= F_1, \quad \nabla \cdot u = F_2 \quad \text{in}\ Q_T^{(1)}, \\
u \mathbf{n} &= F_3, \quad \Pi \mathbf{D}(u) = F_4 \quad \text{on}\ \Gamma_T
\end{align*}
\]
For problem (3.2) we have already obtained the following theorem (see [6]).

**THEOREM 3.1.** Let us assume that

\[ \begin{align*}
F_1 \in C_0^{\alpha, \frac{\alpha}{2}}(Q_T^{(1)}), & \quad F_2 \in C_0^{1+\alpha, \frac{1+\alpha}{2}}(Q_T^{(1)}), & \quad H_1 \in C_0^{2+\alpha, \frac{2+\alpha}{2}}(\Sigma_T), \\
F_3 \in C_0^{2+\alpha, \frac{2+\alpha}{2}}(\Gamma_T), & \quad F_4 \in C_0^{1+\alpha, \frac{1+\alpha}{2}}(\Gamma_T).
\end{align*} \]

and there exist a vector field \( r \in C_0^{\alpha, \frac{\alpha}{2}}(Q_T^{(1)}) \) and a tensor \( R \) satisfying

\[ \frac{\partial F_2}{\partial t} - \nabla \cdot F_1 = \nabla \cdot r, \quad r = \nabla \cdot R, \quad \langle \langle R \rangle \rangle^{(1+\alpha, \gamma)}_{Q_T^{(1)}} < \infty, \]

in the sense of distribution and

\[ \int Q_T F_2 dx = - \int_{\Gamma_T} F_3 d\Gamma - \int \Sigma G_1 \cdot nd\Sigma. \]

Then problem (3.2) has a unique solution \( u \in C_0^{2+\alpha, \frac{2+\alpha}{2}}(Q_T^{(1)}), \nabla q \in C_0^{\alpha, \frac{\alpha}{2}}(Q_T^{(1)}) \) which satisfies

\[ \begin{align*}
|u|^{(2+\alpha, \frac{2+\alpha}{2})}_{Q_T^{(1)}} + |\nabla q|^{(\alpha, \frac{\alpha}{2})}_{Q_T^{(1)}} & \leq C \left( |F_1|^{(\alpha, \frac{\alpha}{2})}_{Q_T^{(1)}} + |F_2|^{(1+\alpha, \frac{1+\alpha}{2})}_{Q_T^{(1)}} + |H_1|^{(2+\alpha, \frac{2+\alpha}{2})}_{\Sigma_T} \\
& + |F_3|^{(2+\alpha, \frac{2+\alpha}{2})}_{\Gamma_T} + |F_4|^{(1+\alpha, \frac{1+\alpha}{2})}_{\Gamma_T} + |r|^{(\alpha, \frac{\alpha}{2})}_{Q_T^{(1)}} + \langle \langle R \rangle \rangle^{(1+\alpha, \gamma)}_{Q_T^{(1)}} \right),
\end{align*} \]

where a constant \( C \) does not depend on \( F_j, j = 1, \ldots, 4, H_1 \), and remains bounded as
For problem (3.3), we first consider the model problem in the half space:

\[
\begin{cases}
\frac{\partial w_1}{\partial t} - a_1 \Delta w_1 = g_1 & \text{in } D_\infty^3, \\
\frac{\partial w_2}{\partial t} - a_2 \Delta w_2 = g_2 & \text{in } \tilde{D}_\infty^3, \\
w_1|_{t=0} = 0 & \text{on } R_+^3, \\
w_2|_{t=0} = 0 & \text{on } R_-^3, \\
\frac{\partial \delta}{\partial t} + d_1 \frac{\partial w_1}{\partial z_3} - d_2 \frac{\partial w_2}{\partial z_3} = g_3 & \text{on } R_\infty^2, \\
w_1 + b_1 \delta - c \sigma \Delta_{z'} \delta = g_4 & \text{on } R_\infty^2, \\
w_2 + b_2 \delta - c \sigma \Delta_{z'} \delta = g_5 & \text{on } R_\infty^2.
\end{cases}
\]  

(3.4)

Here \(D_\infty^3 \equiv \{(z_1, z_2, z_3, t) \in \mathbb{R}^4 | z_3 > 0, t > 0\}\), \(\tilde{D}_\infty^3 \equiv \{(z_1, z_2, z_3, t) \in \mathbb{R}^4 | z_3 < 0, t > 0\}\), \(R_\infty^2 \equiv \{(z_1, z_2, t) \in \mathbb{R}^3 | t > 0\}\), \(R_+^3 \equiv \{(z_1, z_2, z_3) \in \mathbb{R}^3 | z_3 > 0\}\), and \(R_-^3 \equiv \{(z_1, z_2, z_3) \in \mathbb{R}^3 | z_3 < 0\}\), and \(a_1, a_2, b_1, b_2, c, d_1, d_2\) are positive constants. We begin with the derivation of an estimate of a solution to the following homogeneous problem:

\[
\begin{cases}
\frac{\partial w_1'}{\partial t} - a_1 \Delta w_1' = 0 & \text{in } D_\infty^3, \\
\frac{\partial w_2'}{\partial t} - a_2 \Delta w_2' = 0 & \text{in } \tilde{D}_\infty^3, \\
w_1'|_{t=0} = 0 & \text{on } R_+^3, \\
w_2'|_{t=0} = 0 & \text{on } R_-^3, \\
\frac{\partial \delta}{\partial t} + d_1 \frac{\partial w_1'}{\partial z_3} - d_2 \frac{\partial w_2'}{\partial z_3} = g_3 & \text{on } R_\infty^2, \\
w_1' + b_1 \delta - c \sigma \Delta_{z'} \delta = 0 & \text{on } R_\infty^2, \\
w_2' + b_2 \delta - c \sigma \Delta_{z'} \delta = 0 & \text{on } R_\infty^2.
\end{cases}
\]  

(3.5)

Making use of the Fourier transformation with respect to \(z' = (z_1, z_2)\) and the Laplace transformation with respect to \(t\):

\[
FL\{f\}(\xi', z_3, s) \equiv \tilde{f}(\xi', z_3, s) \equiv \int_0^\infty e^{-st} dt \int_{\mathbb{R}^2} e^{-i \xi' \cdot z'} f(z', z_3, t) dz',
\]

we have a representation of a solution of the transformed problem of (3.5) as follows:

\[
\tilde{w}_1' = -(b_1 + c \sigma |\xi'|^2) \tilde{\delta} \exp \left[- \left( \frac{s + a_1 |\xi'|^2}{a_1} \right)^{1/2} z_3 \right],
\]  

(3.6)

\[
\tilde{w}_2' = -(b_2 + c \sigma |\xi'|^2) \tilde{\delta} \exp \left[ \left( \frac{s + a_2 |\xi'|^2}{a_2} \right)^{1/2} z_3 \right]
\]  

(3.7)

and

\[
\tilde{\delta} = \frac{\tilde{g}_3}{s + \sum_{i=1,2} d_i (b_i + c \sigma |\xi'|^2) \left( \frac{s + a_i |\xi'|^2}{a_i} \right)^{1/2}}.
\]  

(3.8)

The following theorem in [3] makes it possible to estimate these transformed functions in Hölder norms.
Theorem 3.2. Suppose that a function $f(x, t)$ belongs to $C^{\alpha, a}_{0} (\mathbb{R}^{2}_{\infty})$ for some $\alpha > 0$ and a symbol $\tilde{K}(\xi, s)$, satisfies the condition:

$$
\Gamma_{h}^{\nu_{j}}(\tilde{K}) \equiv \int_{0}^{\infty} \frac{dz_{0}}{z_{0}^{3/2}} \int_{0}^{\infty} \frac{dz_{1}}{z_{1}^{3/2}} \int_{0}^{\infty} \frac{dz_{2}}{z_{2}^{3/2}} \times ||\Delta_{0}(z_{0})\Delta_{1}(z_{1})\Delta_{2}(z_{2})[\nu_{j}^{\nu_{j}} \Phi(\eta, \eta_{0})\tilde{K}(\eta_{h}, s_{h})]||_{L_{2}(\mathbb{R}^{3}_{L\cdot D0})} \leq Ch^{l}
$$

for sufficiently large $\nu_{j}$, $j = 0, 1, 2$, where $\eta_{h} = \eta/h$, $s_{h} = a + i\xi_{0}/h^{2}$, (if $j = 0$), $1$ (if $j = 1, 2$) and $C$ is a positive constant independent of $h$.

Then the convolution $u = K * f$ satisfies the inequality

$$
\langle u_{A} \rangle_{\mathbb{R}^{2}_{\infty}}^{(l + \alpha, )} \leq C(\langle f_{A} \rangle_{\mathbb{R}^{2}_{\infty}}^{(\alpha^{l})} \pm_{2} |\underline{\propto} ,)
$$

where the notation $f_{A}$ means the function defined as $f_{A} \equiv F^{-1}[(FL)f]$, and $C$ is a positive constant independent of $f_{A}$.

Here $\Delta_{i}(z_{i})$ are finite differences of step size $Z_{j}$ in the variable $\eta_{i}$ and $\Phi(x, t) = \phi(x_{1})\phi(x_{2})\phi(t)$, where $\phi(x) = \sum_{k=1}^{N} \frac{(-1)^{k+1}N!}{k!(N-k)!} \frac{1}{k} \omega\left(\frac{x}{k}\right)$, $N$ is a sufficiently large positive integer and $\omega$ is a function belongs to $C^{\infty}(\mathbb{R})$ satisfying $\text{supp}\omega \subset [0, 1], \omega \geq 0$ and $\int \omega(x)dx = 1$.

Introduce the notation

$$
R(\xi', s) \equiv s + \sum_{i=1, 2} d_{i}(b_{i} + c\sigma|\xi'|^{2}) \left(\frac{s + a_{i}|\xi'|^{2}}{a_{i}}\right)^{1/2},
$$

$$
P(\xi', s) \equiv s + \sum_{i=1, 2} d_{i}b_{i} \left(\frac{s + a_{i}|\xi'|^{2}}{a_{i}}\right)^{1/2}, \quad r_{i}(\xi', s) \equiv \left(\frac{s + a_{i}|\xi'|^{2}}{a_{i}}\right)^{1/2} (i = 1, 2),
$$

then considering $\text{arg}r_{1}, \text{arg}r_{2} \in (-\pi/4, \pi/4)$, we have

$$
|R(\xi', s)| \geq C|P(\xi', s)|,
$$

where $C$ is an arbitraly positive constant satisfying $0 < C < (2 - 2^{1/2})^{1/2}/2$. This inequality plays essential role to derive uniform estimates of $R(\xi', s)$ with respect to $\sigma \geq 0$. Indeed, by the calculation given in [7] with this inequality, it is easily seen that $R(\xi', s)$ satisfies the following lemma.

**Lemma 3.3.** The symbol $R(\xi', s)$ satisfies

$$
\Gamma_{h}^{\nu_{j}}(R(\xi', s)) \leq Ch, \quad j = 0, 1, 2,
$$

where $C$ is a positive constant independent of $h$ and $\sigma \geq 0$, and $\nu_{j}$'s are sufficiently large positive constants.

Then firstly we have the estimate

$$
|\delta|_{\mathbb{R}^{2}_{h}}^{(3\alpha, 3\alpha/2)} \leq C|g_{3}|_{\mathbb{R}^{2}_{h}}^{(2\alpha, 2\alpha/2)}.
$$

Furthermore the following lemma obviously holds because of the homogeneity of the symbol $r_{i}$.  

**Lemma 3.4.** The symbols $r_{i}$ $(i=1, 2)$ satisfy

$$
\Gamma_{h}^{\nu_{j}}(r_{i}) \leq Ch, \quad j = 0, 1, 2,
$$
where $C$ is a positive constant independent of $h$, and $\nu'_j$s are sufficiently large positive constants.

Then the relation

$$
\left( s + c\sigma|\xi'|^2 \sum_{j=1,2} d_j r_j \right) \tilde{\delta} = - \sum_{j=1,2} d_j b_j r_j \tilde{\delta} + \tilde{g}_3
$$

implies the estimate

$$
\frac{\partial \delta}{\partial t} \bigg|_{R_T^2}^{(2+\alpha, \frac{2+\alpha}{2})} + \sigma |\Delta_{z'} \delta|_{R_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} \leq C \left( |\delta|_{R_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} + |g_3|_{R_T^2}^{(2+\alpha, \frac{2+\alpha}{2})} \right)
$$

Hence we have

$$
|\delta|_{\sigma, R_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} \leq C |g_3|_{R_T^2}^{(2+\alpha, \frac{2+\alpha}{2})},
$$

where $C'$s are positive constants which is independent of $\sigma$. Since $w'_i$, $i=1,2$, in (3.5) can be considered as solutions of the Dirichlet problem of heat equations, we have also the estimate:

$$
|w'_i|_{D_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} + |w'_j|_{\tilde{D}_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} \leq C \left( b_1 + b_2 \right) |\delta|_{R_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} + c\sigma |\delta|_{\sigma, R_T^2}^{(3+\alpha, \frac{3+\alpha}{2})} \leq C |g_3|_{R_T^2}^{(2+\alpha, \frac{2+\alpha}{2})}.
$$

The solution of the non-homogeneous problem (3.4) is given by adding the above $w'_i$ to a solution of the problem

$$
\begin{align*}
\frac{\partial w'_1}{\partial t} - a_1 \Delta w'_1 &= g_1 & \text{in } D_\infty^3, \\
\frac{\partial w'_2}{\partial t} - a_2 \Delta w'_2 &= g_2 & \text{in } \tilde{D}_\infty^3, \\
w'_1|_{t=0} &= 0 & \text{on } R_\infty^3, \\
w'_2|_{t=0} &= 0 & \text{on } \tilde{R}_\infty^3, \\
w'_1 &= g_4 & \text{on } R_\infty^2, \\
w'_2 &= g_5 & \text{on } \tilde{R}_\infty^2.
\end{align*}
$$

Hence we have the following theorem:

**Theorem 3.5.** Suppose that

$$
g_1 \in C^{1+\alpha, \frac{1+\alpha}{2}}_0 (D_\infty^3), \quad g_2 \in C^{1+\alpha, \frac{1+\alpha}{2}}_0 (\tilde{D}_\infty^3),
$$

$$
g_3 \in C^{2+\alpha, \frac{2+\alpha}{2}}_0 (R_\infty^2), \quad g_4 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0 (R_\infty^2), \quad g_5 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0 (R_\infty^2),
$$

then problem (3.4) has a unique solution

$$
w'_1 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0 (D_\infty^3), \quad w'_2 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0 (\tilde{D}_\infty^3), \quad \delta \in C^{3+\alpha, \frac{3+\alpha}{2}}_{\sigma, 0} (R_\infty^2).$$
satisfying
\[
|w_1|^{(3+\alpha, \frac{3+\alpha}{2})}_{D_T^3} + |w_2|^{(3+\alpha, \frac{3+\alpha}{2})}_{D_T^2} + |\delta|^{(3+\alpha, \frac{3+\alpha}{2})}_{\sigma,R_T^2} \\
\leq C \left(|g_1|^{(1+\alpha, \frac{1+\alpha}{2})}_{D_T^3} + |g_2|^{(1+\alpha, \frac{1+\alpha}{2})}_{D_T^2} \\
+ |g_3|^{(2+\alpha, \frac{2+\alpha}{2})}_{R_T^2} + |g_4|^{(3+\alpha, \frac{3+\alpha}{2})}_{R_T^2} + |g_5|^{(3+\alpha, \frac{3+\alpha}{2})}_{R_T^2}\right),
\]
where \(C\) is a constant independent of \(\sigma\), \(g_i\), \(i = 1, \ldots, 5\), and remains bounded as \(T \to 0\).

On the basis of this theorem, we can solve problem (3.3) by the method of regularizer (see [8], [7], [5]).

**Theorem 3.6.** Let us assume that
\[
G_1 \in C^{1+\alpha, \frac{1+\alpha}{2}}_0(Q_T^{(1)}) \quad G_2 \in C^{1+\alpha, \frac{1+\alpha}{2}}_0(Q_T^{(2)}) \quad H_2 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0(\Sigma_T),
\]
\[
G_3 \in C^{2+\alpha, \frac{3+\alpha}{2}}_0(\Gamma_T) \quad G_4 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0(\Gamma_T) \quad G_5 \in C^{3+\alpha, \frac{3+\alpha}{2}}_0(\Gamma_T).
\]
Then problem (3.3) has a unique solution \(w^{(i)} \in C^{3+\alpha, \frac{3+\alpha}{2}}_0(Q_T^{(i)})\), \(i = 1, 2\), \(\delta \in C^{3+\alpha, \frac{3+\alpha}{2}}(\Gamma_T)\) which satisfies
\[
\sum_{i=1,2} |w^{(i)}|^{(3+\alpha, \frac{3+\alpha}{2})}_{Q_T^{(i)}} + |\delta|^{(3+\alpha, \frac{3+\alpha}{2})}_{\sigma,\Gamma_T} \leq C \left(|G_1|^{(1+\alpha, \frac{1+\alpha}{2})}_{Q_T^{(1)}} + |G_2|^{(1+\alpha, \frac{1+\alpha}{2})}_{Q_T^{(2)}} \\
+ |H_2|^{(2+\alpha, \frac{2+\alpha}{2})}_{\Sigma_T} + |G_3|^{(2+\alpha, \frac{2+\alpha}{2})}_{\Gamma_T} + |G_4|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T} + |G_5|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T}\right)
\]
where a constant \(C\) does not depend on \(\sigma\), \(G_j\), \(j = 1, \ldots, 5\), \(H_2\), and remains bounded as \(T \to 0\).

**4. Proof of Theorem 1.1.**

**Lemma 4.1.** Let \(0 < \alpha < 1\). The following inequalities hold for any \(u_1, u_2 \in C^{2+\alpha, \frac{2+\alpha}{2}}_0(Q_T^{(1)})\), \(\nabla q_1, \nabla q_2 \in C^{\alpha, \frac{\alpha}{2}}_0(Q_T^{(1)})\), \(\delta_1, \delta_2 \in C^{3+\alpha, \frac{3+\alpha}{2}}(\Gamma_T), w_1^{(1)}, w_2^{(1)} \in C^{3+\alpha, \frac{3+\alpha}{2}}_0(Q_T^{(1)})\), \(w_1^{(2)}, w_2^{(2)} \in C^{3+\alpha, \frac{3+\alpha}{2}}_0(Q_T^{(2)})\).

\[
|\mathcal{F}_1(u_1, \nabla q_1, w_1^{(1)}, \delta_1) - \mathcal{F}_1(u_2, \nabla q_2, w_2^{(1)}, \delta_2)|_{Q_T^{(1)}}^{(\alpha, \frac{\alpha}{2})} \\
\leq c(T) \left(|u_1 - u_2|^{(2+\alpha, \frac{2+\alpha}{2})}_{Q_T^{(1)}} + |\nabla q_1 - \nabla q_2|^{(\alpha, \frac{\alpha}{2})}_{Q_T^{(1)}} \\
+ |\delta_1 - \delta_2|^{(3+\alpha, \frac{3+\alpha}{2})}_{\sigma,\Gamma_T} + |w_1^{(1)} - w_2^{(1)}|^{(3+\alpha, \frac{3+\alpha}{2})}_{Q_T^{(1)}}\right),
\]
\[
|\mathcal{F}_2(u_1, \delta_1) - \mathcal{F}_2(u_2, \delta_2)|_{Q_T^{(1)}}^{(1+\alpha, \frac{1+\alpha}{2})} \\
\leq c(T) \left(|u_1 - u_2|^{(2+\alpha, \frac{2+\alpha}{2})}_{Q_T^{(1)}} + |\delta_1 - \delta_2|^{(3+\alpha, \frac{3+\alpha}{2})}_{\sigma,\Gamma_T}\right),
\]
\[ |\mathcal{F}_3(u_1, \delta_1) - \mathcal{F}_3(u_2, \delta_2)|_{1 \Gamma_T}^{(2+\alpha, \frac{3+\alpha}{2})} \leq (c(T) + \left| 1 - \frac{\rho_e}{\rho} \right|) \times \left( |u_1 - u_2|_{Q_T^{(1)}}^{(2+\alpha, \frac{3+\alpha}{2})} + |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})} \right), \]

\[ |\mathcal{F}_4(u_1, \delta_1) - \mathcal{F}_4(u_2, \delta_2)|_{1 \Gamma_T}^{(1+\alpha, \frac{1+\alpha}{2})} \leq c(T) \left( |u_1 - u_2|_{Q_T^{(1)}}^{(2+\alpha, \frac{3+\alpha}{2})} + |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})} \right), \]

\[ |\mathcal{G}_1(u_1, w_{1}^{(1)}, \delta_1) - \mathcal{G}_1(u_2, w_{2}^{(1)}, \delta_2)|_{Q_T^{(1)}}^{(1+\alpha, \frac{1+\alpha}{2})} \leq c(T) \left( |u_1 - u_2|_{Q_T^{(1)}}^{(2+\alpha, \frac{3+\alpha}{2})} + |w_1 - w_2|_{Q_T^{(1)}}^{(3+\alpha, \frac{3+\alpha}{2})} + |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})} \right), \]

\[ |\mathcal{G}_2(w_{1}^{(2)}, \delta_1) - \mathcal{G}_2(w_{2}^{(2)}, \delta_2)|_{Q_T^{(2)}}^{(1+\alpha, \frac{1+\alpha}{2})} \leq c(T) \left( |w_1 - w_2|_{Q_T^{(2)}}^{(3+\alpha, \frac{3+\alpha}{2})} + |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})} \right), \]

\[ |\mathcal{G}_3(w_{1}^{(1)}, w_{1}^{(2)}, \delta_1) - \mathcal{G}_3(w_{2}^{(1)}, w_{2}^{(2)}, \delta_2)|_{1 \Gamma_T}^{(2+\alpha, \frac{3+\alpha}{2})} \leq \left( c(T) + \left| \kappa(\theta_0) - \kappa(\theta_0') \right| \nabla \theta_1 \left( 1 - \frac{\sigma}{l} H_0 \right) \cdot \tau \right)_{\Gamma_T^{(0)}} \times \left( |w_1^{(1)} - w_2^{(1)}|_{Q_T^{(1)}}^{(3+\alpha, \frac{3+\alpha}{2})} + |w_1^{(2)} - w_2^{(2)}|_{Q_T^{(2)}}^{(3+\alpha, \frac{3+\alpha}{2})} + |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})} \right), \]

\[ |\mathcal{G}_4(\delta_1) - \mathcal{G}_4(\delta_2)|_{1 \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})} \leq c(\epsilon, T) |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})}, \]

\[ |\mathcal{G}_5(\delta_1) - \mathcal{G}_5(\delta_2)|_{1 \Gamma_T^{(3+\alpha, \frac{3+\alpha}{2})}} \leq c(\epsilon, T) |\delta_1 - \delta_2|_{\sigma, \Gamma_T}^{(3+\alpha, \frac{3+\alpha}{2})}, \]

where \( c(T) \) is a positive constant depending on \( T, u_1, u_2, q_1, q_2, w_{1}^{(1)}, w_{1}^{(2)}, w_{2}^{(1)}, w_{2}^{(2)}, \delta_1, \delta_2 \) which converges to 0 as \( T \to 0 \) and \( c(\epsilon, T) \) is a positive constant depending not
only on $T$ and the above functions but also on $\epsilon$ which will appear below, and be taken arbitrarily small for suitably chosen $\epsilon$ and $T$.

Proof. Terms $F_i$ ($i = 1, \cdots, 7$) and $G_i$ ($i = 1, \cdots, 3$) can be treated by the same way as in [7], hence here we give the estimates only for $G_i$ ($i = 4, 5$).

Set

$$
G_4(\delta) = -\hat{\theta}_{\sigma}^{(1)} + \theta_1 - \frac{\theta_1 \sigma}{l} \frac{\partial^2 \delta}{\partial \omega \partial \omega_j} 
$$

$$
+ \frac{\theta_1 \sigma}{l} \frac{1}{|\nabla_{\sigma + \delta}\eta|} \sum_{i,j=1,2} a_{ij}(\omega, \sigma_\delta + \delta, \nabla_{\sigma + \delta}(\sigma_\delta + \delta)) \frac{\partial^2 (\sigma_\delta + \delta)}{\partial \omega \partial \omega_j} 
$$

$$
+ \frac{\theta_1 \sigma}{l} \frac{1}{|\nabla_{\sigma + \delta}\eta|} b(\omega, \sigma_\delta + \delta, \nabla_{\sigma + \delta}(\sigma_\delta + \delta)) 
$$

$$
\equiv -\hat{\theta}_{\sigma}^{(1)} + \theta_1 - \frac{\theta_1 \sigma}{l} (A(\delta) + B(\delta)),
$$

where

$$
A(\delta) \equiv -\frac{1}{|\nabla_{\sigma + \delta}\eta|} \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^2 \delta}{\partial \omega \partial \omega_j} 
$$

$$
+ \frac{1}{|\nabla_{\sigma + \delta}\eta|} \sum_{i,j=1,2} a_{ij}(\omega, \sigma_\delta + \delta, \nabla_{\sigma + \delta}(\sigma_\delta + \delta)) \frac{\partial^2 (\sigma_\delta + \delta)}{\partial \omega \partial \omega_j} 
$$

and

$$
B(\delta) \equiv -\frac{1}{|\nabla_{\sigma + \delta}\eta|} b(\omega, \sigma_\delta + \delta, \nabla_{\sigma + \delta}(\sigma_\delta + \delta)).
$$

Firstly we rewrite $A(\delta)$ as follows:

$$
A(\delta) = \left( \frac{1}{|\nabla_{\sigma + \delta}\eta|} - \frac{1}{|\nabla_{\sigma + \delta}\eta|} \right) \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^2 \delta}{\partial \omega \partial \omega_j} 
$$

$$
+ \frac{1}{|\nabla_{\sigma + \delta}\eta|} \sum_{i,j} (a_{ij}(\omega, 0, 0, 0) 
$$

$$
- a_{ij}(\omega, \sigma_\delta + \delta, \nabla_{\sigma + \delta}(\sigma_\delta + \delta)) \frac{\partial^2 \delta}{\partial \omega \partial \omega_j} 
$$

$$
+ \frac{1}{|\nabla_{\sigma + \delta}\eta|} \sum_{i,j=1,2} a_{ij}(\omega, \sigma_\delta + \delta, \nabla_{\sigma + \delta}(\sigma_\delta + \delta)) \frac{\partial^2 \sigma_\delta}{\partial \omega \partial \omega_j}.
$$

For example, the first term is evaluated as follows:

$$
\left( \left( \frac{1}{|\nabla_{\sigma + \delta_1}\eta|} - \frac{1}{|\nabla_{\sigma + \delta_2}\eta|} \right) \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^2 \delta_1}{\partial \omega \partial \omega_j} 
$$

$$
+ \left( \frac{1}{|\nabla_{\sigma + \delta_1}\eta|} - \frac{1}{|\nabla_{\sigma + \delta_2}\eta|} \right) \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^2 (\delta_1 - \delta_2)}{\partial \omega \partial \omega_j} \right)_{\Gamma T}^{(3+\alpha,2)}.
$$
\[
\leq \frac{1}{|\nabla_{d_{\sigma}+\delta_{1}}\eta|} - \frac{1}{|\nabla_{d_{\sigma}+\delta_{2}}\eta|} \left[ \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^{2} \delta_{1}}{\partial \omega_{i} \partial \omega_{j}} \right]^{(0)}_{\Gamma_T}
+ \frac{1}{|\nabla_{d_{\sigma}+\delta_{1}}\eta|} - \frac{1}{|\nabla_{d_{\sigma}+\delta_{2}}\eta|} \left[ \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^{2} \delta_{1}}{\partial \omega_{i} \partial \omega_{j}} \right]^{(0)}_{\Gamma_T}
+ \frac{1}{|\nabla\eta|} - \frac{1}{|\nabla_{d_{\sigma}+\delta_{2}}\eta|} \left[ \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^{2} (\delta_{1} - \delta_{2})}{\partial \omega_{i} \partial \omega_{j}} \right]^{(0)}_{\Gamma_T}
+ \frac{1}{|\nabla\eta|} - \frac{1}{|\nabla_{d_{\sigma}+\delta_{2}}\eta|} \left[ \sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0) \frac{\partial^{2} (\delta_{1} - \delta_{2})}{\partial \omega_{i} \partial \omega_{j}} \right]^{(0)}_{\Gamma_T}
\leq c(T) |\Delta_{\Gamma}(\delta_{1} - \delta_{2})|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T},
\]

where by $\Delta_{\Gamma}$ we denote the operator $\sum_{i,j} a_{ij}(\omega, 0, 0, 0) \frac{\partial^{2}}{\partial \omega_{i} \partial \omega_{j}}$. Taking into account the definition of $a_{ij}$, the second and the last terms are also estimated by $|\Delta_{\Gamma}(\delta_{1} - \delta_{2})|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T}$. Furthermore, considering the following estimate

\[
\left| b \left( \omega, d_{\sigma} + \delta_{1}, \nabla_{d_{\sigma}+\delta_{1}}(d_{\sigma} + \delta_{1}) \right) \right|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T}
- \left| b \left( \omega, d_{\sigma} + \delta_{2}, \nabla_{d_{\sigma}+\delta_{2}}(d_{\sigma} + \delta_{2}) \right) \right|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T}
\leq C_{1} |\nabla(\delta_{1} - \delta_{2})|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T}
\leq C_{1} \left( \epsilon |\Delta_{\Gamma}(\delta_{1} - \delta_{2})|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T} + C(\epsilon) |\nabla(\delta_{1} - \delta_{2})|^{(0)}_{\Gamma_T} \right),
\]

$B(\delta_{1}) - B(\delta_{2})$ is obviously estimated in the form

\[
|B(\delta_{1}) - B(\delta_{2})|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T} \leq C_{2} (\epsilon + c(T)C(\epsilon)) |\Delta_{\Gamma}(\delta_{1} - \delta_{2})|^{(3+\alpha, \frac{3+\alpha}{2})}_{\Gamma_T},
\]

where $C_{1}$ and $C_{2}$ are positive constants independent of $\sigma$. \(\square\)

Now, on the basis of this lemma, we construct a solution of problem (2.5)-(2.6) in the function space:

\[
X^{k+\alpha}_{\sigma,T} = \left\{ (u, \nabla q, w^{(1)}, w^{(2)}, \delta) \in X^{k+\alpha}_{\sigma,T} \left| \right. \begin{array}{l}
||u^{(1)}, w^{(2)}, \delta, \nabla q||_{X^{k+\alpha}_{\sigma,T}} = ||u^{(k+2+\alpha, \frac{k+3\alpha}{2})}_{Q_{T}^{(1)}} + |\nabla q|^{(k+\alpha, \frac{k+\alpha}{2})}_{Q_{T}^{(1)}} + |w^{(1)}|^{(k+3+\alpha, \frac{k+3\alpha}{2})}_{Q_{T}^{(1)}} + |\delta|^{(k+3+\alpha, \frac{k+3\alpha}{2})}_{\Gamma_T} \\
+ M = |F_1(0,0,0,0)|^{(k+\alpha, \frac{k+\alpha}{2})}_{Q_{T}^{(1)}} + |F_2(0,0)|^{(k+1+\alpha, \frac{k+1\alpha}{2})}_{Q_{T}^{(1)}} + |F_1(0,0,0,0)|^{(k+2+\alpha, \frac{k+3\alpha}{2})}_{\Gamma_T} + |F_4(0,0,0,0)|^{(k+4+\alpha, \frac{1\alpha}{2})}_{\Gamma_T} \leq \end{array} \right\}
\]

where

\[
M = \left\{ F_1(0,0,0,0)|^{(k+\alpha, \frac{k+\alpha}{2})}_{Q_{T}^{(1)}} + |F_2(0,0)|^{(k+1+\alpha, \frac{k+1\alpha}{2})}_{Q_{T}^{(1)}} + |F_1(0,0,0,0)|^{(k+2+\alpha, \frac{k+3\alpha}{2})}_{\Gamma_T} + |F_4(0,0,0,0)|^{(k+4+\alpha, \frac{1\alpha}{2})}_{\Gamma_T} \}
\]
and $K$ and $T$ are positive constants to be determined later. Choose $(u, \nabla q, w^{(1)}, w^{(2)}, \delta) \in X_{\sigma,T}^{\alpha}$ arbitrarily, and $(\hat{u}, \nabla \hat{q}, \hat{w}^{(1)}, \hat{w}^{(2)}, \hat{\delta})$ be a solution of problem (3.1) with $(F_{1}, F_{2}, 0, H_{1}, F_{3}, F_{4}, G_{1}, G_{2}, 0, 0, H_{2}, G_{3}, G_{4}, G_{5})$ replaced by $(F_{1}, F_{2}, 0, -\hat{v}_{\sigma}, F_{3}, F_{4}, G_{1}, G_{2}, 0, 0, \theta_{2}-\theta_{\sigma}^{(1)}, G_{3}, G_{4}, G_{5})$. Let $P$ be a mapping corresponds $(u, \nabla q, w^{(1)}, w^{(2)}, \delta)$ to $(\hat{u}, \nabla \hat{q}, \hat{w}^{(1)}, \hat{w}^{(2)}, \hat{\delta})$. Then theorem 3.1 and theorem 3.6 garantee that $P$ maps $X_{\sigma,T}^{\alpha}$ into itself. Actually, it is shown as follows.

$$
||P(u, \nabla q, w^{(1)}, w^{(2)}, \delta)||_{X_{\sigma,T}^{\alpha}} \equiv ||(\hat{u}, \nabla \hat{q}, \hat{w}^{(1)}, \hat{w}^{(2)}, \hat{\delta})||_{X_{\sigma,T}^{\alpha}}
$$

$$
\leq ||P(u, \nabla q, w^{(1)}, w^{(2)}, \delta) - P(0,0,0,0,0)||_{X_{\sigma,T}^{\alpha}}
$$

$$
+ ||P(0,0,0,0,0)||_{X_{\sigma,T}^{\alpha}}
$$

$$
\leq C_{3} \left[ \epsilon + c(T)(C(\epsilon) + 1) + \left| 1 - \frac{\rho_{s}}{\rho} \right| + \left| \left( \kappa^{(1)}(\theta_{0}^{(1)}) - \kappa^{(2)}(\theta_{0}^{(2)}) \right) \nabla(\theta_{1}H_{0}) \cdot \tau \right|_{\Gamma}^{(0)} \right]
$$

$$
+ \left| \left( \kappa^{(1)}(\theta_{0}^{(1)}) - \kappa^{(2)}(\theta_{0}^{(2)}) \right) \nabla(\theta_{1}H_{0}) \cdot \tau \right|_{\Gamma}^{(0)} \right] KM
$$

$$
\equiv (L(\epsilon, T)K + 1) M,
$$

where $H_{0}$ is the twice mean curvature of $\Gamma$ and $C_{3}$ is a positive constant independent of $\sigma$. Considering the smallness assumptions for $|\rho - \rho_{e}|$ and $|\kappa^{(1)}(\theta_{0}^{(1)}) - \kappa^{(2)}(\theta_{0}^{(2)})|$ given in theorem 1.1, we can take $\sigma$ satisfying

$$
C_{3} \frac{\sigma}{l} \left| \left( \kappa^{(1)}(\theta_{0}^{(1)}) - \kappa^{(2)}(\theta_{0}^{(2)}) \right) \nabla(\theta_{1}H_{0}) \cdot \tau \right|_{\Gamma}^{(0)}
$$

$$
\leq \frac{1}{2} - C_{3} \left( \left| 1 - \frac{\rho_{s}}{\rho} \right| + \left| \left( \kappa^{(1)}(\theta_{0}^{(1)}) - \kappa^{(2)}(\theta_{0}^{(2)}) \right) \nabla(\theta_{1}H_{0}) \cdot \tau \right|_{\Gamma}^{(0)} \right).
$$

Hence, for some $\epsilon_{0}$ satisfying $C_{3}\epsilon_{0} < 1/2$, there exist $T_{0} > 0$ independent of $\sigma \in (0, \sigma^{*})$ such as $L(\epsilon_{0}, T_{0}) < 1$. Here by $\sigma^{*}$ we denote the upperbound of $\sigma$ satisfying (4.1). Then taking $K > 0$ larger than $1/(1 - L(\epsilon_{0}, T_{0}))$, we have $||P(u_{\sigma}, \nabla q_{\sigma}, w_{\sigma}^{(1)}, w_{\sigma}^{(2)}, \delta_{\sigma})||_{X_{\sigma,T_{0}}^{\alpha}} \leq KM$.

Contractiveness of mapping $P$ also follows from $L(\epsilon_{0}, T_{0}) < 1$. Hence the contractive mapping theorem yields a unique solution of the problem.

Moreover, the convergence of the solution of problem $(P_{\sigma})$ can be proved as follows. Let $(u_{\sigma_{1}}, \nabla q_{\sigma_{1}}, w_{\sigma_{1}}^{(1)}, w_{\sigma_{1}}^{(2)}, \delta_{\sigma_{1}})$ in $X_{\sigma_{1},T_{0}}^{2+\alpha}$ and $(u_{\sigma_{2}}, \nabla q_{\sigma_{2}}, w_{\sigma_{2}}^{(1)}, w_{\sigma_{2}}^{(2)}, \delta_{\sigma_{2}})$ in $X_{\sigma_{2},T_{0}}^{2+\alpha}$ be solutions of problems $(P_{\sigma_{1}})$ and $(P_{\sigma_{2}})$, respectively. Then we have

$$
||(u_{\sigma_{1}}, \nabla q_{\sigma_{1}}, w_{\sigma_{1}}^{(1)}, w_{\sigma_{1}}^{(2)}, \delta_{\sigma_{1}}) - (u_{\sigma_{2}}, \nabla q_{\sigma_{2}}, w_{\sigma_{2}}^{(1)}, w_{\sigma_{2}}^{(2)}, \delta_{\sigma_{2}})||_{X_{T_{0}}^{\alpha}}
$$

$$
\leq C_{4}||(v_{\sigma_{1},0}, \theta_{\sigma_{1},0}^{(1)}, \theta_{\sigma_{1},0}^{(2)}) - (v_{\sigma_{2},0}, \theta_{\sigma_{2},0}^{(1)}, \theta_{\sigma_{2},0}^{(2)})||_{H^{\alpha}}
$$
\[ C_{5} \left( c(T_{0}) + \left| 1 - \frac{\rho_{c}}{\rho} \right| + \left| \kappa^{(1)}(\theta_{0}^{(1)}) - \kappa^{(2)}(\theta_{0}^{(2)}) \right| (\nabla \theta_{1}) \cdot \tau \right|_{\Gamma}^{(0)} \]
\[ \times \left| (u_{\sigma_{1}}, \nabla q_{\sigma_{1}}, w_{\sigma_{1}}^{(1)}, w_{\sigma_{1}}^{(2)}, \delta_{\sigma_{1}}) - (u_{\sigma_{2}}, \nabla q_{\sigma_{2}}, w_{\sigma_{2}}^{(1)}, w_{\sigma_{2}}^{(2)}, \delta_{\sigma_{2}}) \right|_{X_{T_{o}}^{2+\alpha}} + \left| (v_{\sigma_{1},0}, \theta_{\sigma_{1},0}^{(1)}, \theta_{\sigma_{2},0}^{(2)}, \delta_{\sigma_{2}}) \right|_{H^{1+\alpha}} \]
\[ \equiv M(T_{0}) \left| (u_{\sigma_{1}}, \nabla q_{\sigma_{1}}, w_{\sigma_{1}}^{(1)}, w_{\sigma_{1}}^{(2)}, \delta_{\sigma_{1}}) - (u_{\sigma_{2}}, \nabla q_{\sigma_{2}}, w_{\sigma_{2}}^{(1)}, w_{\sigma_{2}}^{(2)}, \delta_{\sigma_{2}}) \right|_{X_{T_{o}}^{2+\alpha}} + \sigma_{1} \sigma_{2} \left| K^{\prime} M \right|_{1 - M(T_{0})} \]

where \( C_{i}, i = 4, 5, 6 \), are positive constants independent of \( \sigma \), and \( K \) is a positive constant satisfying \( K^{\prime} > K \). Noting that \( M(T_{0}) < L(\epsilon_{0}, T_{0}) < 1 \), we have

\[ \left| (u_{\sigma_{1}}, \nabla q_{\sigma_{1}}, w_{\sigma_{1}}^{(1)}, w_{\sigma_{1}}^{(2)}, \delta_{\sigma_{1}}) - (u_{\sigma_{2}}, \nabla q_{\sigma_{2}}, w_{\sigma_{2}}^{(1)}, w_{\sigma_{2}}^{(2)}, \delta_{\sigma_{2}}) \right|_{X_{T_{o}}^{2+\alpha}} \leq \sigma_{1} \sigma_{2} \left| K^{\prime} M \right|_{1 - M(T_{0})} \rightarrow 0 \quad (\sigma_{1}, \sigma_{2} \rightarrow 0). \]

Thus \( \{(u_{\sigma}, \nabla q_{\sigma}, w_{\sigma}^{(1)}, w_{\sigma}^{(2)}, \delta_{\sigma})\} \) is a Cauchy sequence in \( X_{T_{o}}^{2+\alpha} \) as \( \sigma \rightarrow 0 \). Hence the proof of theorem 1.1 is completed.

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