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Existence and asymptotic stability of stationary solution to the full compressible Navier–Stokes equations in the half space

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This talk is based on the work by Prof. Kawashima and myself. We shall divide it into two sections which are concerned, respectively, with the existence and asymptotic stability of the stationary solution to the full compressible Navier-Stokes equations in the half space. We consider the general constitutive equations. The theory on this subject is far from being complete. In fact, there is no any result on the other nonlinear waves except the stationary solution considered in this talk, or on the outflow problems.

1 Existence of stationary solution

1.1 Introduction

In this section, we investigate the existence of stationary solution to the full compressible Navier-Stokes equations in the half space. The one–dimensional motion of compressible viscous and heat conductive gas is described by the following system in the Eulerian coordinate

$$\rho_t + (\rho u)_x = 0, \quad x > 0, \quad t > 0, \tag{1.1}$$

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = (\mu u_x)_x, \tag{1.2}$$

$$\left(\rho\left(e + \frac{u^2}{2}\right)\right)_t + \left(\rho u\left(e + \frac{u^2}{2}\right) + pu\right)_x - (\mu u u_x)_x = (K\theta_x)_x. \tag{1.3}$$

We study the initial boundary value problem to the system (1.1)–(1.3) with the following initial data

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x), \quad \text{for all } x > 0, \quad \text{and} \quad \inf_{x>0} \rho_0(x), \theta_0(x) > 0, \tag{1.4}$$

the boundary condition at the infinity $x = \infty$

$$\lim_{x\to\infty} (\rho, u, \theta)(t, x) = (\rho_+, u_+, \theta_+), \quad (\rho_+, u_+, \theta_+ : \text{ constants for all } t > 0), \tag{1.5}$$

and also the boundary conditions at $x = 0$

$$u(t, 0) = u_b < 0, \quad \theta(t, 0) = \theta_b > 0 \quad \text{for all } t > 0. \tag{1.6}$$

The physical meaning of boundary conditions is that there exists constantly an outflow through the wall and the temperature is constant on the wall.

Here, $p = p(\rho, \theta)$, $e = e(\rho, \theta)$, $s = s(\rho, \theta)$. $\rho(> 0)$, u, p, θ and e are the density, the velocity of gas, the pressure, the absolute temperature and the internal energy, respectively. The coefficients $\mu, K(> 0)$ are assumed to be constants, and μ, K are the viscosity coefficient, heat conductivity respectively.

We shall make the assumptions on the thermodynamic quantities which are enumerated (A1)–(A3) below:

(A1) p, e, s are smooth functions of (ρ, θ), such that $p_\rho > 0, e_\theta > 0$.

(A2) The relationship for p and e. It follows from the first thermodynamic law, i.e.

$$de = \theta ds - pd\left(1/\rho\right) \tag{1.7}$$

that $\frac{1}{\rho^2}\left\{p - \theta\frac{\partial p}{\partial \theta}\right\} = \frac{\partial e}{\partial \rho}$. This relationship constrains possible laws for p and e.

(A3) The second law of thermodynamics admits only the function $e(v, s)$ that is convex in (v, s). □

Combining (1.7) with the above–mentioned three balance laws (1.1) – (1.3), we can define, up to a constant, a function s(the so-called entropy) that satisfies

$$(\rho s)_t + (\rho u s)_x = \left(\frac{K}{\theta}\theta_x\right)_x + \frac{1}{\theta}\left(\mu u_x^2 + \frac{K}{\theta}|\theta_x|^2\right) \leq \left(\frac{K}{\theta}\theta_x\right)_x \tag{1.8}$$

whence the second law of thermodynamics is satisfied automatically since we assume that $\mu, K > 0$.

In this section we are interested in the corresponding stationary problem which reads

$$(\tilde{\rho}\tilde{u})_x = 0, \quad x > 0, \tag{1.9}$$

$$\left(\tilde{\rho}\tilde{u}^2 + \tilde{p}\right)_x = (\mu\tilde{u}_x)_x, \tag{1.10}$$

$$\left(\tilde{\rho}\tilde{u}\left(\tilde{e} + \frac{\tilde{u}^2}{2}\right) + \tilde{p}\tilde{u}\right)_x - (\mu\tilde{u}\tilde{u}_x)_x = (K\tilde{\theta}_x)_x. \tag{1.11}$$

with the boundary condition at $x = 0$

$$(\tilde{u}, \tilde{\theta})(0) = (u_b, \theta_b) \tag{1.12}$$

and the boundary condition at infinity

$$\lim_{x\to\infty}(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) = (\rho_+, u_+, \theta_+). \tag{1.13}$$

Where $\tilde{p} = p(\tilde{\rho}, \tilde{\theta})$, $\tilde{e} = e(\tilde{\rho}, \tilde{\theta})$.

We are going to prove the existence of solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$ to the stationary problem (1.9) –(1.13). To this end, we firstly try to simplify the problem. In what follows, we still denote the functions $\tilde{\rho}, \tilde{u}, \tilde{\theta}, \cdots$ by $\rho, u, \theta, \cdots$ for the sake of simplicity. We integrate eq.s (1.9)–(1.11) with respect to x over (x, ∞), then (1.9)–(1.11) become

$$\rho(x)u(x) = \rho(0)u(0) = \rho_+u_+, \tag{1.14}$$

$$\rho u^2 + p(\rho, \theta) = \mu u_x + \rho_+u_+^2 + p_+. \tag{1.15}$$

$$\left(\rho\left(e + \frac{u^2}{2}\right) + p\right)u - \mu u u_x = K\theta_x + \left(\rho_+\left(e_+ + \frac{u_+^2}{2}\right) + p_+\right)u_+. \tag{1.16}$$

Where we have used the notations $p_+ = p(\rho_+, \theta_+)$, $e_+ = e(\rho_+, \theta_+), \cdots$.

Introducing

$$v = 1/\rho, \quad \hat{p} = p\,(1/v, \theta)\,, \quad \hat{e} = e\,(1/v, \theta)\,, \tag{1.17}$$

recalling (1.14), we arrive at

$$u = \frac{u_+}{v_+} v. \tag{1.18}$$

From the fact that $v(0) > 0$ and $u(0) = u_b < 0$ and Eq. (1.18), we find that u_+ must satisfy

$$u_+ = \frac{v_+}{v(0)} u(0) < 0. \tag{1.19}$$

Using (1.18) we can rewrite (1.15) and (1.16) as follows

$$v_x = f(v, \theta) := \frac{\gamma u_+}{v_+}(v - v_+) + \frac{\gamma v_+}{u_+}(\hat{p}(v, \theta) - \hat{p}_+), \tag{1.20}$$

$$\theta_x = g(v, \theta) := k \left(\frac{u_+}{v_+}(\hat{e}(v, \theta) - \hat{e}_+) - \frac{u_+^3}{2v_+^3}(v - v_+)^2 + \frac{u_+}{v_+}\hat{p}_+(v - v_+) \right), \tag{1.21}$$

where $\gamma = \mu^{-1}$ and $k = K^{-1}$. And the boundary conditions become

$$v(0) = \frac{v_+}{u_+} u_b, \quad \theta(0) = \theta_b, \quad \lim_{x \to \infty} v(x) = v_+, \quad \lim_{x \to \infty} \theta(x) = \theta_+. \tag{1.22}$$

If we denote

$$U = \begin{pmatrix} v \\ \theta \end{pmatrix} \quad F(U) = \begin{pmatrix} f(v, \theta) \\ g(v, \theta) \end{pmatrix}. \tag{1.23}$$

Then (1.20) and (1.21) can be rewritten as

$$U_x = F(U), \quad F(U_+) = 0. \tag{1.24}$$

Next we try to calculate the Jacobian of (1.24) at $x = \infty$:

$$J_+ = \begin{pmatrix} \gamma \frac{v_+}{u_+} \left((\frac{u_+}{v_+})^2 + \hat{p}_v^+ \right) & \gamma \frac{v_+}{u_+} \hat{p}_\theta^+ \\ k \frac{u_+}{v_+} \left(\hat{e}_v^+ + p^+ \right) & k \frac{u_+}{v_+} \hat{e}_\theta^+ \end{pmatrix}. \tag{1.25}$$

Here, $\hat{p}_v^+ = \hat{p}_v(v_+, \theta_+)$, $\hat{e}_v^+ = \hat{e}_v(v_+, \theta_+)$, $\cdots$. Assume that J_+ admits two distinct eigenvalues $\lambda_1 > \lambda_2$, then there exists a matrix P such that

$$P^{-1} J_+ P = \mathrm{diag}\{\lambda_1, \lambda_2\} =: \Lambda. \tag{1.26}$$

Let

$$Y := P^{-1}(U - U_+), \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{1.27}$$

Therefore, Eq. (1.24) can be tranformed to the following

$$\begin{aligned} Y_x &= \Lambda Y + P^{-1}(F(U) - J_+ U) =: \Lambda Y + H(Y), \\ & Y(0) = Y_0, \quad \lim_{x \to \infty} Y(x) = 0. \end{aligned} \tag{1.28}$$

Here, $H(Y) = \binom{h_1(Y)}{h_2(Y)}$.

We now can state the following lemma

Lemma 1.1 *Assume that* $\lambda_1 > 0 > \lambda_2$. *Then there exists a unique solution* $(y_1(x), y_2(x))$ *to the following problem*

$$y_1(x) = -\int_x^\infty e^{\lambda_1(x-s)} h_1(Y(s))ds, \quad y_2(x) = e^{\lambda_2 x} y_{20} + \int_0^x e^{\lambda_2(x-s)} h_2(Y(s))ds \tag{1.29}$$

□

We shall use this lemma when we deal with subsonic and transonic cases. Recalling the definition of sound speed,

$$C = C(\rho, s) := \sqrt{\partial p(\rho, s)/\partial \rho} = \sqrt{-v^2 \partial \tilde{p}(v, s)/\partial v}, \tag{1.30}$$

we then state our main result as following theorem

Theorem 1.1 *Suppose that* $u_b < 0$, $\theta_b, \theta_+, v_+ > 0$.
If $u_+ > 0$, *then there exists no stationary solution* $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *to the stationary problem. If* $u_+ < 0$, *then there exists a stationary solution and we can divide it into three cases:*

(i) Supersonic case: $C_+^2 < u_+^2$, *i.e. the Mach number at infinity* $M_+ > 1$. *Assume that for some small number* δ, *such that*

$$|u_b - u_+| + |\theta_b - \theta_+| \le \delta. \tag{1.31}$$

Then there exists a solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *to the stationary problem, such that*

$$\tilde{\rho} = 1/\tilde{v}, \quad \tilde{u} = \frac{u_+}{v_+}\tilde{v}, \tag{1.32}$$

and the estimates hold for some positive constant c

$$|\tilde{u}(x) - u_+| = \delta O(e^{-cx}), \quad |\tilde{\theta}(x) - \theta_+| = \delta O(e^{-cx}). \tag{1.33}$$

(ii) Subsonic case: $C_+^2 > u_+^2$. *Let*

$$Y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} =: P^{-1} \begin{pmatrix} \frac{v_+}{u_+}(u_b - u_+) \\ \theta_b - \theta_+ \end{pmatrix}.$$

Assume that (u_b, θ_b) *is chosen so that* Y_0 *satisfying*

$$y_{10} = -\int_0^\infty e^{-\lambda_1 s} h_1(Y(s))ds, \quad Y = Y(x; y_{20}). \tag{1.34}$$

Here Y *is the solution to the problem (1.29). Then there exists a solution* $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *to the stationary problem, such that* $|\tilde{u}(x) - u_+|$, $|\tilde{\theta}(x) - \theta_+| = \delta O(e^{-cx})$, *provided that* $|u_b - u_+| + |\theta_b - \theta_+| \le \delta$ *for some small constant* δ.

(iii) Transonic case: $C_+^2 = u_+^2$. *We can obtain similar conclusion as Case (ii), only the decay estimates are modified to* $|\tilde{u}(x) - u_+|$, $|\tilde{\theta}(x) - \theta_+| = \delta O(x^{-1})$.

We have used $C_+ = C(\rho_+, \theta_+)$ *to denote the sound speed at infinity.* □

Remark: On the curve (1.34), we only know that it can be written as $y_{10} = C_1 y_{20}^2 + C_2 y_{20}^3 + O(y_{20}^4)$. However, we do not know the signs of C_1, C_2. □

We now recall the references related to our subject. Concerening the one–dimensional case, we refer to Liu[13], Kawashima and Zhu[11, 12], Nishibata, Kawashima and Zhu[24], Matsumura and Nishihara[22], Huang, Matsumura and Shi[3], and so on.

The main difficulty of the proof of the existence of Theorem 1.1 is that the stationary problem is not a scalar equation, in fact it consists of three equations, and can be reduced to two independent equations. To prove the existence, we shall investiagte carefully the signs of the eigenvalues of the Jacobian matrix at the infinity state.

The remaining part of this section is as follows: in Subsection 1.2, we introduce some preliminaries which will be used frequently in our proof of the main theorem. Then making use of these lemmas we are able to prove in Subsection 1.3 our main results in this section.

1.2 Some preliminaries

To prove the existence of solution to (1.24), we shall investigate the signs of eigenvalues of J_+. We prepare the following simple lemmas.

Lemma 1.2 *Assume that a, b, c, d are real numbers. Then the matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.35}$$

i) has two negative eigenvalues if $a + d < 0$ *and* $\det A > 0$*;*
ii) two positive eigenvalues if $a + d > 0$ *and* $\det A > 0$*;*
iii) at least one zero eigenvalue if $\det A = 0$.

Next we shall use frequently the following thermodynamic relations to simplify the expressions later on. Throughout this section, we choose v, θ as the independent thermodynamic variables.

Lemma 1.3 *For the following thermodynamic quatities:* $s = \hat{s}(v, \theta)$, $p = \bar{p}(\rho, s) = \hat{p}(v, \theta)$, $e = e(v, \hat{s}(v, \theta)) = \hat{e}(v, \theta)$, *there hold*

$$\hat{e}_v = -p + \theta \hat{p}_\theta, \quad \hat{e}_\theta = \theta \hat{s}_\theta, \quad \hat{s}_v = \hat{p}_\theta. \tag{1.36}$$

Proof. From thermodynamics, one has $de = \theta ds - \bar{p} dv$. Moverover, it is easy to see that $de = \hat{e}_v dv + \hat{e}_\theta d\theta, \quad ds = \hat{s}_v dv + \hat{s}_\theta d\theta$. Then we have

$$\begin{cases} \hat{e}_v = -\bar{p} + \theta \hat{s}_v \\ \hat{e}_\theta = \theta \hat{s}_\theta. \end{cases} \tag{1.37}$$

On the other hand, it holds

$$0 = d^2 e = -dp \wedge dv + d\theta \wedge ds = (-\hat{p}_\theta + \hat{s}_v) d\theta \wedge dv, \tag{1.38}$$

thus we have

$$\hat{p}_\theta = \hat{s}_v. \tag{1.39}$$

Combination (1.39) with (1.37) yields $\hat{e}_v = -\bar{p} + \theta \hat{p}_\theta$. Q.E.D. □

Finally, we give the expression of the sound speed in the following lemma:

Lemma 1.4 *Let* $p = \bar{p}(\rho, s) = \tilde{p}(v, s), \quad s = \hat{s}(v, \theta)$. *Then we have*

$$\tilde{p}(v, s) = \tilde{p}(v, \hat{s}(v, \theta)) = \hat{p}(v, \theta)$$

and the sound speed function $C = C(v, \theta)$ *can be written as*

$$C = \sqrt{-v^2(\hat{p}_v - \theta \hat{p}_\theta^2 / \hat{e}_\theta)}. \tag{1.40}$$

Proof. For $p = \bar{p}(\rho, s)$, by the definition of sound speed we have

$$C = \sqrt{\partial \bar{p}(\rho, s)/\partial \rho} = \sqrt{-v^2 \partial \tilde{p}(v, s)/\partial v}.$$

Calculation yields $\hat{p}_v = \tilde{p}_v + \tilde{p}_s \hat{s}_v = \tilde{p}_v + \tilde{p}_s \hat{p}_\theta$ and $\hat{p}_\theta = \tilde{p}_s \hat{s}_\theta = 1/\theta \tilde{p}_s \hat{e}_\theta$. Thus

$$\tilde{p}_v = \hat{p}_v - \tilde{p}_s \hat{p}_\theta = \hat{p}_v - \theta \hat{p}_\theta^2 / \hat{e}_\theta.$$

Thus the proof of this lemma is complete. □

1.3 Proof of Theorem 1.1

After the preparation in the above subsection, we are now in a position to prove our main theorem. When we simplify the problem in Subsection 1.1, we have obtained that (1.19) should hold. That is $u_+ < 0$. We shall assume this condition is met. Otherwise, there exists no any stationary solution.

According to the Mach number, we divide the proof into several steps. To make use of Lemma 1.2 to the matrix J_+, we first calculate the values $a + d$ and $ad - bc$. Recalling Lemmas 1.4 and 1.3, we then have

$$a + d = \gamma \frac{v_+}{u_+} \left(\frac{u_+^2 - C_+^2}{v_+^2} + \theta_+ \frac{\hat{p}_\theta^{+\,2}}{\hat{e}_\theta^+} \right) + k \frac{u_+}{v_+} \hat{e}_\theta^+, \quad ad - bc = \hat{e}_\theta^+ \frac{u_+^2 - C_+^2}{v_+^2}. \tag{1.41}$$

Therefore, we can investigate the following cases:

Case i) Supersonic case, i.e. $u_+^2 > C_+^2$: then combining it with the fact that $u_+ < 0, v_+ > 0$, one has $a + d < 0, \quad ad - bc > 0$. Thus J_+ admits two negative eigenvalues $\lambda_1, \lambda_2 < 0$. (Consequently the case that $\lambda_1, \lambda_2 > 0$ is impossible since $a + d < 0$). Therefore we can conclude that there exists a unique solution to (1.24) provided that $u_b, u_+ < 0, \theta_b, \theta_+, v_+ > 0$ and $|u_b - u_+| + |\theta_b - \theta_+| \leq \delta$ for some small constant δ. The space-decay estimates are easy to get.

Case ii) Subsonic case, i.e. $u_+^2 < C_+^2$: For this case, we have $ad - bc < 0$. Thus J_+ has two eigenvalues such that $\lambda_2 < 0 < \lambda_1$.

The matrix P in (1.26) can be chosen as

$$P = \begin{pmatrix} \frac{2\gamma v_+ \hat{p}_\theta^+}{(B-A+\sqrt{\Delta})u_+} & \frac{2\gamma v_+ \hat{p}_\theta^+}{(B-A-\sqrt{\Delta})u_+} \\ 1 & 1 \end{pmatrix}, \tag{1.42}$$

with

$$A = \frac{\gamma v_+}{u_+}\left(\frac{u_+^2}{v_+^2} + \hat{p}_v^+\right), \quad B = \frac{\gamma u_+}{v_+}\hat{e}_\theta^+, \quad \Delta = (A-B)^2 + 4\gamma k\theta_+ \hat{p}_\theta^{+2}.$$

Then we can rewrite (1.24) as follows

$$\begin{aligned} Y_x &= \Lambda Y + H(Y), \\ Y(0) &= Y_0, \ \lim_{x\to\infty} Y(x) = 0. \end{aligned} \tag{1.43}$$

Where Y is defined in Subsection 1.1, and $H(Y)$ satisfies

$$PH(Y) = \begin{pmatrix} \frac{\gamma v_+}{u_+}[\hat{p}(v,\theta) - \hat{p}^+ - \hat{p}_v^+(v - v_+) - \hat{p}_\theta^+(\theta - \theta_+)] \\ \frac{k u_+}{v_+}\left[\hat{e}(v,\theta) - \hat{e}^+ - \hat{e}_v^+(v - v_+) - \hat{e}_\theta^+(\theta - \theta_+) - \frac{u_+^2}{2v_+^2}(v - v_+)^2\right] \end{pmatrix}, \tag{1.44}$$

and

$$|H(Y)| \le C(|y_1|^2 + |y_2|^2), \ \text{provided} \ \left|\frac{\partial^2}{\partial v^2}\hat{e}\right|, \dots \le C. \tag{1.45}$$

From (1.43), we have

$$\begin{cases} y_1(x) = e^{\lambda_1 x} y_{01} + \int_0^x e^{\lambda_1(x-s)} h_1(Y(s))ds = e^{\lambda_1 x}\left(y_{01} + \int_0^x e^{-\lambda_1 s} h_1(Y(s))ds\right), \\ y_2(x) = e^{\lambda_2 x} y_{02} + \int_0^x e^{\lambda_2(x-s)} h_2(Y(s))ds \end{cases} \tag{1.46}$$

Here, $\lim_{s\to\infty} Y(s) = 0$. We now consider the first equation in (1.46). Letting $x \to \infty$, recalling the fact that $\lambda_1 > 0$, we have

$$y_{01} = -\int_0^\infty e^{-\lambda_1 s} h_1(Y(s))ds. \tag{1.47}$$

Thus (1.46) is equivalent to

$$\begin{cases} y_1(x) = -\int_x^\infty e^{\lambda_1(x-s)} h_1(Y(s))ds, \\ y_2(x) = e^{\lambda_2 x} y_{02} + \int_0^x e^{\lambda_2(x-s)} h_2(Y(s))ds \end{cases} \tag{1.48}$$

To solve the equation (1.48), we define the function space

$$X := \{Y \in \mathcal{B}^0([0,\infty)); |Y(x)| \le \beta e^{-\alpha x}, \beta = 2|y_{02}|, \alpha > 0, x \ge 0\}$$

with $\alpha := \min\{\lambda_1, |\lambda_2|\}$ and suitably small data y_{02}. Then we can employ the contraction mapping theorem to prove the global existence of solution to (1.48).

In what follows, we want to obtain more information of the curve (1.47). We write

$$y_{10} = y_1(0) = -\int_0^\infty e^{-\lambda_1 s} h_1(Y(s; y_{02}))ds = C_1 y_{02}^2 + C_2 y_{02}^3 + \cdots. \tag{1.49}$$

We try to justify the signs of C_1, C_2, the coefficients of the terms y_{02}^2, y_{02}^3. It is easy to show that

$$\begin{cases} y_1(x) = a_1 y_{02}^2 + O(y_{02}^3), \\ y_2(x) = e^{\lambda_2 x} y_{02} + a_2 y_{02}^2 + O(y_{02}^3) \end{cases} \tag{1.50}$$

Here a_1, a_2 are functions in x. We write $h_1(Y)$ and $h_2(Y)$ in the following form

$$h_i(Y) = h_i^{11} y_1^2 + h_i^{12} y_1 y_2 + h_i^{22} y_2^2 + h_i^{03} y_2^3 + \cdots. \tag{1.51}$$

Here $i = 1, 2$. Making use of (1.48) we then have

$$a_1 = h_1^{22} e^{2\lambda_2 x}/(2\lambda_2 - \lambda_1), \quad a_2 = h_2^{22}\left(e^{2\lambda_2 x} - e^{\lambda_2 x}\right)/\lambda_2.$$

Therefore, C_1, C_2 can be expressed as

$$C_1 = \frac{h_1^{22}}{(2\lambda_2 - \lambda_1)}, \quad C_2 = \frac{h_1^{12} h_1^{22} - 2h_1^{22} h_2^{22} + h_1^{03}(2\lambda_2 - \lambda_1)}{(3\lambda_2 - \lambda_1)(2\lambda_2 - \lambda_1)}. \tag{1.52}$$

It remains to compute h_i^{kj}(They are so complicated that we can not justify the signs of C_1, C_2 till now!).
Case iii) Transonic case, i.e. $C_+^2 = u_+^2$. It is easy to deduce from (1.41) and $u_+ < 0$, $e_\theta > 0$ that for this case there hold $a + d < 0$ and $ad - bc = 0$, thus J_+ has one zero and one negative eigenvalues i.e. there holds $\lambda_2 < 0 = \lambda_1$.

Similar to the argument of Case ii), we can obtain the result with different space–decay estimates. We omit the details here. Q.E.D. □

2 Stability of stationary solution

2.1 Introduction

This section is devoted to asymptotic stability of stationary solution whose existence has been proved in Section 1. We simplify firstly the equations (1.1) –(1.3) and (1.8) to

$$\rho_t + (\rho u)_x = 0, \quad x > 0, \quad t > 0, \tag{2.1}$$

$$\rho(u_t + u u_x) + p_x = (\mu u_x)_x, \tag{2.2}$$

$$\rho\left(e_t + u e_x\right) + p u_x = (K\theta_x)_x + \mu u_x^2. \tag{2.3}$$

and the entropy equation

$$\rho\left(s_t + u s_x\right) = \theta^{-1}\left((K\theta_x)_x + \mu u_x^2\right). \tag{2.4}$$

The boundary and initial conditions are

$$u|_{x=0} = u_b, \quad \theta|_{x=0} = \theta_b, \tag{2.5}$$

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \tag{2.6}$$

And the corresponding stationary problem of (2.1)–(2.3), (2.5) and (2.6) are written as

$$(\tilde{\rho}\tilde{u})_x = 0, \quad x > 0, \tag{2.7}$$

$$\tilde{\rho}\tilde{u}\tilde{u}_x + \tilde{p}_x = (\mu\tilde{u}_x)_x. \tag{2.8}$$

$$\tilde{\rho}\tilde{u}\tilde{e}_x + \tilde{p}\tilde{u}_x = (K\tilde{\theta}_x)_x + \mu\tilde{u}_x^2. \tag{2.9}$$

and we need the following equation

$$\tilde{\rho}\tilde{u}\tilde{s}_x = \tilde{\theta}^{-1}\left((K\tilde{\theta}_x)_x + \mu\tilde{u}_x^2\right). \tag{2.10}$$

Where we have used $\tilde{p} = p(\tilde{\rho}, \tilde{\theta})$, $\tilde{e} = e(\tilde{\rho}, \tilde{\theta})$ and $\tilde{s} = s(\tilde{\rho}, \tilde{\theta})$.

Our main results in this section are

Theorem 2.1 *(The case $u_b < 0$) Suppose that $u_+ < 0$. Moreover, Case i) Assume that the infinity state is in Supersonic region, i.e. : $|u_+| > |C_+|$, or Case ii) Assume that the infinity state is in Subsonic region, i.e. : $|u_+| < |C_+|$. And we choose (u_b, θ_b) such that Y_0 satisfying*

$$y_{10} = -\int_0^\infty e^{-\lambda_1 s} h_1(Y(s))ds, \quad Y = Y(x; y_{20}). \tag{2.11}$$

With $Y_0 = \binom{y_{10}}{y_{20}} =: P^{-1}\begin{pmatrix} \frac{v_+}{u_+}(u_b - u_+) \\ \theta_b - \theta_+ \end{pmatrix}$. *Then asymptotic state is stationary solution denoted by $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$.*

Suppose furthermore that $\rho_0 \in \mathcal{B}^{1+\sigma}$, u_0, $\theta_0 \in \mathcal{B}^{2+\sigma}$ for some $\sigma \in (0,1)$, $\rho_0(x), \theta_0(x) > 0$ for all $x \in [0,1]$ and $(\rho_0 - \rho_+, u_0 - u_+, \theta_0 - \theta_+) \in H^1$, and that $\delta := |u_b - u_+| + |\theta_b - \theta_+|$, $\|(\rho_0 - \rho_+, u_0 - u_+, \theta_0 - \theta_+)\|_{H^1}$ are suitably small. And the compatibility condition $u_0(0) = u_b, \theta_0(0) = \theta_b$ are satisfied.

Then there exists a unique solution (ρ, u, θ) to (2.1)– (2.6) such that for any fixed $T > 0$

$\rho \in \mathcal{B}_T^{1+\sigma}, \quad u, \theta \in \mathcal{C}_T^{2+\sigma}$;
$\rho - \rho_+, u - u_+, \theta - \theta_+ \in C(\mathbb{R}^+; H^1)$;
$(\rho - \tilde{\rho})_x \in L^2(\mathbb{R}^+; L^2), \rho - \tilde{\rho} \in L^2(\mathbb{R}^+; L^\infty), (\rho - \tilde{\rho})_x(t, 0) \in L^2(\mathbb{R}^+)$;
$(u - \tilde{u})_x, (\theta - \tilde{\theta})_x \in L^2(\mathbb{R}^+; H^1)$.

And the a priori estimates hold

$$\begin{aligned} &\|(\rho - \rho_+, u - u_+, \theta - \theta_+)\|_{H^1}^2 + \int_0^t (\|(\rho - \tilde{\rho})_x\|^2 + \|(u - \tilde{u}, \theta - \tilde{\theta})_x\|_{H^1}^2)d\tau + \\ &\int_0^t (\|\rho - \tilde{\rho}\|_\infty^2 + |(\rho - \tilde{\rho})_x(\tau, 0)|^2 + |(\rho - \tilde{\rho})(\tau, 0)|^2)d\tau \\ &\le C\|(\rho_0 - \rho_+, u_0 - u_+, \theta_0 - \theta_+)\|_{H^1}^2 + C\delta^2. \end{aligned} \tag{2.12}$$

Moreover, we have

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}^+} |(\rho,u,\theta)(t,x)-(\tilde\rho,\tilde u,\tilde\theta)(x)| = 0.$$

Here, $C_+ := C(\rho_+,\theta_+)$, $(\tilde\rho,\tilde u,\tilde\theta)$ is the solution to the corresponding stationary problem of (2.1)-(2.6). □

As being pointed out at beginning, the theory of nonlinear waves for the initial boundary value problem of full compressible Navier-Stokes equations is far from being developed. There are only a few results. By far only the stationary solution is investigated. As for rarefaction waves, viscous shock waves etc., there is no result. Even the classification of asymptotic states remains open! There is no any result on the inflow problem of full compressible Navier-Stokes equations.

The main difficulties and our main ingredients in the proof of Theorem 2.1 are as follows: Since we consider the full compressible Navier-Stokes equations, the energy function becomes much more complicated than that of isentropic case. To derive the equation that the energy function satisfies, we shall frequently make use of the thermodynamic relations. Another one is the presence of boundary conditions and that we investigate the system in the eulerian coordinate, this will make it difficult when we try to justify the formal calculations for establishing the estimates for the derivatives of the unknown functions. Employing the technique in Kawashima and Nishida[10], we can overcome that difficulty.

The remains of this chapter is organized as follows: In Subsection 2.2, we reformulate the problem and restate our main theorem. We then introduce the energy function $\mathcal{E}$ in Subsection 2.3, and prove some properties of this function. The equation that $\mathcal{E}$ satisfies is also derived. After these preparations, we can obtain the Sobolev estimates in Subsection 2.4. Finally the large-time behavior is considered in Subsection 2.5.

2.2 Reformulation of the problem

We reformulate the problem and make it easy to be handled. Defining

$$\phi=\phi(t,x):=(\rho-\tilde\rho)(t,x),\quad \psi(t,x):=(u-\tilde u)(t,x),\quad \chi(t,x):=(\theta-\tilde\theta)(t,x) \tag{2.13}$$

Then we find that (ϕ,ψ) satisfy

$$\phi_t+(\psi+\tilde u)\phi_x+(\phi+\tilde\rho)\psi_x=f, \tag{2.14}$$

$$\psi_t+(\psi+\tilde u)\psi_x+\left(\frac{p_x}{\rho}-\frac{\tilde p_x}{\tilde\rho}\right)=\frac{\mu\psi_{xx}}{\phi+\tilde\rho}+g, \tag{2.15}$$

here, f,g are defined by

$$f:=-(\tilde\rho_x\psi+\tilde u_x\phi),\quad g:=\mu\tilde u_{xx}(1/\rho-1/\tilde\rho)-\psi\tilde u_x \tag{2.16}$$

and the estimates hold

$$|f|\le C(|\tilde\rho_x\psi|+|\tilde u_x\phi|),\quad |g|\le C(|\tilde u_{xx}\phi|+|\tilde u_x\psi|) \tag{2.17}$$

for suitably small δ, ϕ, ψ, χ.

The derivation of the equation of χ is somewhat complicated. We now choose ρ, θ as the two independent thermodynamic variables and write e as $e = e(\rho, \theta)$. Applying (2.1) and Lemma 1.3, then (2.3) is changed to the following

$$\rho e_\theta(\theta_t + u\theta_x) + \theta p_\theta u_x = K\theta_{xx} + \mu u_x^2. \tag{2.18}$$

In a similar way, we can obtain the corresponding stationary energy equation

$$\tilde{\rho}\tilde{u}\tilde{e}_\theta\tilde{\theta}_x + \tilde{\theta}\tilde{p}_\theta\tilde{u}_x = K\tilde{\theta}_{xx} + \mu\tilde{u}_x^2. \tag{2.19}$$

Whence combining (2.18) with (2.19) yields

$$\rho e_\theta(\chi_t + (\psi + \tilde{u})\chi_x) = K\chi_{xx} + h. \tag{2.20}$$

With h satisfying $h := \mu\psi_x^2 + 2\mu\psi_x\tilde{u}_x + (\tilde{\rho}\tilde{u}\tilde{e}_\theta - \rho u e_\theta)\tilde{\theta}_x + (\tilde{\theta}\tilde{p}_\theta - \theta p_\theta)\tilde{u}_x - \theta p_\theta \psi_x$, and the following estimate holds for suitably small δ, ϕ, ψ, χ

$$|h| \le C\left(\psi_x^2 + |\psi_x\tilde{u}_x| + |(\phi, \psi, \chi)\tilde{\theta}_x| + |(\phi, \chi)\tilde{u}_x| + |\psi_x|\right). \tag{2.21}$$

Finally the boundary and initial conditions become

$$\psi|_{x=0} = 0, \quad \chi|_{x=0} = 0, \quad \lim_{x\to\infty}(\phi, \psi, \chi)(x) = 0. \tag{2.22}$$

and

$$(\phi, \psi, \chi)(0, x) = (\rho_0, u_0, \theta_0)(x) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x). \tag{2.23}$$

Therefore, we can now restate our main results as follows

Theorem 2.2 *Assume that all the conditions in Theorem 2.1 are met. Then there exists a unique solution (ϕ, ψ, χ) to the problem (2.14), (2.15), (2.20)–(2.23) such that for any fixed $T > 0$*

$$\phi \in \mathcal{B}_T^{1+\sigma}, \quad \psi, \chi \in C_T^{2+\sigma},$$
$$\phi, \psi, \chi \in C(\mathbb{R}^+; H^1), \phi_x \in L^2(\mathbb{R}^+; L^2), \psi_x, \chi_x \in L^2(\mathbb{R}^+; H^1).$$

And the a priori estimates hold

$$\begin{aligned} &\|(\phi, \psi, \chi)(t)\|_{H^1} + \int_0^t \left(\|\phi_x(\tau)\|^2 + |(\phi, \phi_x)(s, 0)|^2 + \|(\psi, \chi)_x(\tau)\|_{H^1}^2\right) ds \\ \le\ & C\|(\phi_0, \psi_0, \chi_0)\|_{H^1}^2. \end{aligned} \tag{2.24}$$

Moreover, we have

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}^+} |(\phi, \psi, \chi)(t, x)| = 0.$$

□

Remark: Clearly Theorem 2.2 is equivalent to Theorem 2.1. So we prove only Theorem 2.2, and we use the standard continuation argument based on a local existence result and a priori estimates(i.e. Proposition 2.3) to prove Theorem 2.2.

2.3 Energy form

To establish the energy estimates, we introduce the energy form $\mathcal{E} = \mathcal{E}(v, u, s)$:

$$\rho\mathcal{E} := \rho\left(e + \frac{\psi^2}{2} - \tilde{e} + \tilde{p}\left(\frac{1}{\rho} - \frac{1}{\tilde{\rho}}\right) - \tilde{\theta}(s - \tilde{s})\right). \tag{2.25}$$

Here, $\tilde{e} = e(\tilde{v}, \tilde{s})$, $\tilde{p} = p(\tilde{v}, \tilde{s})$, $\tilde{\theta} = \theta(\tilde{v}, \tilde{s})$. Throughout this subsection we choose ρ, s as the two independent thermodynamic variables.

Lemma 2.1 *Assume that e, p are smooth functions of (ρ, s). Then there exist two positive constants k_1, k_2 such that*

$$\psi^2/2 + k_1(|\rho - \tilde{\rho}|^2 + |s - \tilde{s}|^2) \le \mathcal{E} \le \psi^2/2 + k_2(|\rho - \tilde{\rho}|^2 + |s - \tilde{s}|^2). \tag{2.26}$$

And $\mathcal{E}$ satisfies

$$\begin{aligned}
&(\rho\mathcal{E})_t + (\rho u\mathcal{E})_x + \mu\psi_x^2 + K\frac{\tilde{\theta}\chi_x^2}{\theta^2} + \left(\rho\psi^2 + p - \tilde{p} - \tilde{p}_\rho(\rho - \tilde{\rho}) - \tilde{p}_s(s - \tilde{s})\right)\tilde{u}_x \\
= \ & (\mu\frac{\psi_x^2}{2} + \frac{K\chi_x^2}{2\theta} - (p - \tilde{p})\psi)_x + \frac{K\tilde{\theta}_{xx} + \mu\tilde{u}_x^2}{\tilde{\theta}}(\theta - \tilde{\theta} - \tilde{\theta}_s(s - \tilde{s}) - \tilde{\theta}_\rho(\rho - \tilde{\rho})) \\
&+\mathcal{R}.
\end{aligned} \tag{2.27}$$

With

$$\begin{aligned}
\mathcal{R} \ := \ & -(\phi\psi + \phi\tilde{u} + \tilde{\rho}\psi)(\tilde{\theta}_s\tilde{s}_x + \tilde{p}_s\tilde{\rho}^{-2}\tilde{\rho}_x)(s - \tilde{s}) + \frac{K\chi\chi_x\tilde{\theta}_x}{\theta^2} \\
& - \left\{K\tilde{\theta}_{xx} + \mu\tilde{u}_x^2\right\}\frac{\chi^2}{\theta\tilde{\theta}} + \mu\frac{\chi}{\theta}\left(\psi_x^2 + 2\psi_x\tilde{u}_x\right) + \mu\tilde{u}_{xx}\frac{\tilde{\rho} - \rho}{\tilde{\rho}}\psi.
\end{aligned} \tag{2.28}$$

It can be estimated, provided that δ, $\|(\phi, \psi, \chi)\|_{H^1}$ are suitably small, as

$$\begin{aligned}
|\mathcal{R}| \ \le \ & C\left((|\phi| + |\psi|)(|\tilde{s}_x| + |\tilde{\rho}_x|)|s - \tilde{s}| + |\chi\chi_x\tilde{\theta}_x| + \left(|\tilde{\theta}_{xx}| + \tilde{u}_x^2\right)\chi^2\right) + \\
& +C\left(|\chi|\left(\psi_x^2 + |\psi_x\tilde{u}_x|\right) + |\tilde{u}_{xx}||\phi\psi|\right).
\end{aligned} \tag{2.29}$$

Here, we denote $\theta(\tilde{\rho}, \tilde{s})$, $p(\tilde{\rho}, \tilde{s})$, $\cdots$ by $\tilde{\theta}, \tilde{p}, \cdots$ respectively.

Proof. Let $v = 1/\rho$. For the proof of (2.26), we refer to Okada and Kawashima[25].

In what follows, we trun to verify (2.27). Making use of equations (2.2)– (2.4), we have

$$\begin{aligned}
&(\rho\mathcal{E})_t + (\rho u\mathcal{E})_x = (\rho_t + (\rho u)_x)\mathcal{E} + \rho(\mathcal{E}_t + u\mathcal{E}_x) \\
= \ & \left(1 - \frac{\tilde{\theta}}{\theta}\right)\left(K\theta_{xx} + \mu u_x^2\right) + \psi\mu\psi_{xx} + \mu\tilde{u}_{xx}\frac{\tilde{\rho} - \rho}{\tilde{\rho}}\psi - \rho\psi^2\tilde{u}_x - (p - \tilde{p})u_x \\
& -\rho\psi\left(\frac{p_x}{\rho} - \frac{\tilde{p}_x}{\tilde{\rho}}\right) - \rho u\left\{\tilde{e}_x - \tilde{p}_x(v - \tilde{v}) + \tilde{p}\tilde{v}_x + \tilde{\theta}_x(s - \tilde{s}) - \tilde{\theta}\tilde{s}_x\right\}.
\end{aligned} \tag{2.30}$$

Now we try to deal with right–hand side terms in (2.30) term by term. Firstly, we have

$$\left(1-\frac{\tilde\theta}{\theta}\right)\left(K\theta_{xx}+\mu u_x^2\right)=\tilde\theta^{-1}\left(K\tilde\theta_{xx}+\mu\tilde u_x^2\right)(\theta-\tilde\theta)+\left(\frac{K\chi\chi_x}{\theta}\right)_x$$
$$-\frac{\tilde\theta}{\theta}\frac{K\chi_x^2}{\theta}+\frac{K\chi\chi_x\tilde\theta_x}{\theta^2}-\left(K\tilde\theta_{xx}+\mu\tilde u_x^2\right)\frac{\chi^2}{\theta\tilde\theta}+\mu\frac{\chi}{\theta}\left(\psi_x^2+2\psi_x\tilde u_x\right), \qquad (2.31)$$

and $\mu\psi\psi_{xx}=(\mu\psi\psi_x)_x-\mu\psi_x^2$.

Secondly, invoking the relations $e_v=-p,\quad e_s=\theta$, we have $\tilde e_x=-\tilde p\tilde v_x+\tilde\theta\tilde s_x$. Thus

$$\rho u\left\{\tilde e_x-\tilde p_x(v-\tilde v)+\tilde p\tilde v_x+\tilde\theta_x(s-\tilde s)-\tilde\theta\tilde s_x\right\}=\rho u\left\{\tilde p_x\tilde v+\tilde\theta_x(s-\tilde s)\right\}-\tilde p_x u, \qquad (2.32)$$

Therefore, the following expression can be simplified.

$$-(p-\tilde p)u_x-\rho\psi\left(\frac{p_x}{\rho}-\frac{\tilde p_x}{\tilde\rho}\right)-\rho u\left\{\tilde e_x-\tilde p_x(v-\tilde v)+\tilde p\tilde v_x+\tilde\theta_x(s-\tilde s)-\tilde\theta\tilde s_x\right\}$$
$$=\ -((p-\tilde p)\psi)_x+\psi\left(\frac{\rho}{\tilde\rho}-1\right)\tilde p_x-(p-\tilde p)\tilde u_x+\tilde p_x u-\frac{\rho}{\tilde\rho}u\tilde p_x-$$
$$-\rho u\left(\frac{\tilde p_s\tilde\rho_x}{\tilde\rho^2}+\tilde\theta_s\tilde s_x\right)(s-\tilde s). \qquad (2.33)$$

Where we have made use of the expression $\theta_v=-p_s$. Next the terms except the first one in (2.33) are rewritten as following

$$\psi\left(\frac{\rho}{\tilde\rho}-1\right)\tilde p_x+\tilde p_x u-\frac{\rho}{\tilde\rho}u\tilde p_x=(1-\frac{\rho}{\tilde\rho})\tilde u\tilde p_x=\tilde p_\rho(\rho-\tilde\rho)\tilde u_x-\tilde\theta_v\tilde u\tilde s_x\frac{\tilde\rho-\rho}{\tilde\rho} \qquad (2.34)$$

and

$$-\rho u\tilde p_s\tilde\rho^{-2}\tilde\rho_x(s-\tilde s)=-(\phi\psi+\phi\tilde u+\tilde\rho\psi)\tilde p_s\tilde\rho^{-2}\tilde\rho_x(s-\tilde s)+\tilde p_s(s-\tilde s)\tilde u_x \qquad (2.35)$$

Finally we use $\theta_v=-\rho^2\theta_\rho$ to handle the following expression which is the sum of the final terms in (2.33) and (2.34)

$$-\tilde\theta_v\tilde u\tilde s_x\frac{\tilde\rho-\rho}{\tilde\rho}-\rho u\tilde\theta_s\tilde s_x(s-\tilde s)$$
$$=\ -(\phi\psi+\phi\tilde u+\tilde\rho\psi)\tilde\theta_s\tilde s_x(s-\tilde s)-\frac{K\tilde\theta_{xx}+\mu\tilde u_x^2}{\tilde\theta}\left(\tilde\theta_s(s-\tilde s)+\tilde\theta_\rho(\rho-\tilde\rho)\right)(2.36)$$

Here, equation (2.10) has been used. The final term in the above equation is not good. However, combining it with the first term in (2.31), we have

$$\tilde\theta^{-1}\left(K\tilde\theta_{xx}+\mu\tilde u_x^2\right)(\tilde\theta-\theta)-\tilde\theta^{-1}(K\tilde\theta_{xx}+\mu\tilde u_x^2)\left(\tilde\theta_s(s-\tilde s)+\tilde\theta_\rho(\rho-\tilde\rho)\right)$$
$$=\ \tilde\theta^{-1}\left(K\tilde\theta_{xx}+\mu\tilde u_x^2\right)\left(\theta-\tilde\theta-\tilde\theta_s(s-\tilde s)-\tilde\theta_\rho(\rho-\tilde\rho)\right). \qquad (2.37)$$

Therefore, combination of (2.30)–(2.37) yields (2.27). □

2.4 The energy estimates

In this subsection we use the energy function defined in the previous subsection to derive the energy estimates. We define

$$M(t)^2 := \int_0^t \left(\|\phi_x(\tau)\|^2 + \|(\psi_x,\chi_x)(\tau)\|_{H^1}^2 + |\phi(\tau,0)|^2 + |\phi_x(\tau,0)|^2\right) d\tau, \qquad (2.38)$$

and

$$N(t) := \sup_{0\le\tau\le t} \|(\phi,\psi,\chi)(\tau)\|_{H^1(\mathbb{R}^+)} \le E_0. \qquad (2.39)$$

and E_0 is suitably small so that $\rho \ge \frac{1}{2}\rho_-$ and $\theta \ge \frac{1}{2}\theta_b$.

This subsection is devoted to prove the following proposition

Proposition 2.3 *(A priori estimates) Let (ϕ,ψ,χ) be a solution to the problem (2.14), (2.15), (2.20)–(2.23) which satisfies*

$$\begin{aligned} &\phi \in C([0,T];H^1)\cap \mathcal{B}_T^{1+\sigma},\ \psi,\chi \in C([0,T];H^1)\cap \mathcal{C}_T^{2+\sigma};\\ &\inf_{Q_T} \rho(t,x),\ \theta(t,x) > 0. \end{aligned} \qquad (2.40)$$

for any fixed $T>0$. Then there exists a suitably small constant $\varepsilon_0 > 0$, such that if $N(t)+\delta \le \varepsilon_0$, then the following estimates hold

$$\|(\phi,\psi,\chi)\|_{H^1}^2 + \int_0^t (\|\phi_x\|^2 + |\phi,\phi_x|^2(\tau,0) + \|(\psi,\chi)_x\|_{H^1}^2) d\tau \le C\|(\phi_0,\psi_0,\chi_0)\|_{H^1}^2 \qquad (2.41)$$

for all $t\ge 0$. Here ε_0, C are independent of t,δ. □

To obtain the a priori estimates, we assume that (ϕ,ψ,χ) be a solution to the problem (2.14), (2.15), (2.20)–(2.23) which satisfies

$$\begin{aligned} &\phi \in C([0,T];H^1)\cap \mathcal{B}_T^{1+\sigma},\ \psi,\chi \in C([0,T];H^1)\cap \mathcal{C}_T^{2+\sigma};\\ &\inf_{Q_T} \rho(t,x),\ \theta(t,x) > 0. \end{aligned} \qquad (2.42)$$

for any fixed $T>0$.

Step 1. As a first step we state the first energy estimate

Lemma 2.2 *There exists a positive constant ε_1 such that if $N(t)+\delta\le\varepsilon_1$, then the following estimate holds for any $t\ge 0$*

$$\begin{aligned} &\|(\phi,\psi,\chi)\|^2 + \int_0^t \left\{\|(\psi,\chi)_x\|^2 + |\phi(\tau,0)|^2\right\} d\tau\\ \le\ & C(\|(\phi_0,\psi_0,\chi_0)\|^2 + (\delta+N(t))M(t)^2). \end{aligned} \qquad (2.43)$$

Here ε_1, C are independent of t,δ. □

Proof. Integrating Eq. (2.27) with respect to x over $(0,\infty)$, using the boundary conditions $\psi=0, \chi=0$, we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty \rho\mathcal{E}dx-\rho u\mathcal{E}|_{x=0}+\int_0^\infty\left(\mu\psi_x^2+K\frac{\tilde\theta\chi_x^2}{\theta^2}\right)dx\\ \le\ & C\int_0^\infty\Big\{(\psi^2+\phi^2+(s-\tilde s)^2)|\tilde u_x|+(|\phi|+|\psi|)(|\tilde s_x|+|\tilde\rho_x|)|s-\tilde s|+|\chi\chi_x\tilde\theta_x|+\\ &(\phi^2+\chi^2+(s-\tilde s)^2)\left(|\tilde\theta_{xx}|+\tilde u_x^2\right)+|\chi|\left(\psi_x^2+|\psi_x\tilde u_x|\right)+|\tilde u_{xx}||\phi\psi|\Big\}dx. \quad (2.44)\end{aligned}$$

We first consider the boundary term. It follows from Lemma 2.1, the equality

$$s-\tilde s=s_\rho(\bar\rho,\bar\theta)(\rho-\tilde\rho)+s_\theta(\bar\rho,\bar\theta)(\theta-\tilde\theta) \quad (2.45)$$

and the fact $u_b<0, \psi|_{x=0}=0, \chi|_{x=0}=0$, that $-\rho u\mathcal{E}|_{x=0}\ge C\phi(t,0)^2$.

Next, we deduce easily from $0<C^{-1}\le\theta,\tilde\theta\le C$ that

$$\int_0^\infty\left(\mu\psi_x^2+K\tilde\theta\chi_x^2/\theta^2\right)dx\ge C(\|\psi_x\|^2+\|\chi_x\|^2).$$

To handle the RHS term in (2.44), we apply the basic technique (see [9]), i.e., for any smooth real function f it holds $f(t,x)=f(t,0)+\int_0^x f_x(t,y)dy$. Thus

$$|f(t,x)|\le|f(t,0)|+\sqrt{x}\|f_x\|. \quad (2.46)$$

Therefore, making use of (2.45) and the decay estimates on the stationary solution $\tilde\rho,\tilde u,\tilde\theta$, we estimate the RHS term in (2.44) as

$$\begin{aligned}&\int_0^\infty(\psi^2+\phi^2+(s-\tilde s)^2)|\tilde u_x|\le C\delta\int_0^\infty\left(x\|(\phi,\psi_x,\chi)_x\|^2+\phi(t,0)^2\right)e^{-cx}dx\\ \le\ & \varepsilon\|\psi_x\|^2+\|\chi_x\|^2+\delta(\|\phi_x\|^2+\phi(t,0)^2). \quad (2.47)\end{aligned}$$

Using (2.45) and the Young inequality, one has

$$\begin{aligned}&\int_0^\infty\left((|\phi|+|\psi|)\left(|\tilde s_x|+|\tilde\rho_x|\right)|s-\tilde s|+|\tilde u_{xx}||\phi\psi|\right)dx\\ \le\ & C\int_0^\infty\left(|\phi|^2+|\psi|^2+|s-\tilde s|^2\right)\left(|\tilde s_x|+|\tilde\rho_x|+|\tilde u_{xx}|\right)dx. \quad (2.48)\end{aligned}$$

And for the term, $\int_0^\infty(\phi^2+\chi^2+(s-\tilde s)^2)\left(|\tilde\theta_{xx}|+\tilde u_x^2\right)dx$, invoking that the decay rate for $\tilde u_{xx}$ is better than that of $\tilde u_x$, we conclude that the above terms can be treated as in (2.47). Next, we have

$$\int_0^\infty\left(|\chi|\left(\psi_x^2+|\psi_x\tilde u_x|\right)+|\chi\chi_x\tilde\theta_x|\right)\le\varepsilon(\|(\psi,\chi)_x\|^2)+\int_0^\infty|\chi|^2(|\tilde\theta_x|^2+|\tilde u_x|^2). \quad (2.49)$$

So, this term can be also handled as in (2.47).

Combination of the above inequalities yields the RHS terms in (2.44) can be bounded by $\delta(\|\phi_x\|^2+|\phi(t,0)|^2)+\varepsilon\|(\psi,\chi)_x\|^2$. Thus taking δ,ε suitably small, applying Lemma 2.1 we prove this lemma. Q.E.D. □

Step 2. We now proceed to establish the second energy estimate i.e. to estimate the function ϕ_x in terms of $N(t)$ and $M(t)$.

Lemma 2.3 *There exists a suitably small positive constant $\varepsilon_2 \le \varepsilon_1$, such that if $N(t)+\delta \le \varepsilon_2$, then the following estimate holds for any $t \in [0,\infty)$*

$$\|\phi_x\|^2 + \int_0^t (\|\phi_x\|^2 + |\phi_x(\tau,0)|^2)d\tau \le C\left(\|(\phi_0,\psi_0,\chi_0)\|_{H^1}^2 + (N(t)+\delta)M(t)^2\right) \tag{2.50}$$

Here ε_2, C are independent of t,δ. □

Proof. We divide the proof of this lemma into two steps. Firstly, differentiating formally Eq. (2.14) with respect to x , we arrive at

$$\phi_{xt} + (\psi+\tilde{u})\phi_{xx} + (\phi+\tilde{\rho})\psi_{xx} = f_1, \tag{2.51}$$

$$f_1 := -2(\phi_x\psi_x + \phi_x\tilde{u}_x + \psi_x\tilde{\rho}_x) - \psi\tilde{\rho}_{xx} - \phi\tilde{u}_{xx}. \tag{2.52}$$

We shall transform the above equation of ϕ_x into that of $\phi_x/(\phi+\tilde{\rho})$. This will make the caculation simpler in the second step below. We have

$$\left(\frac{\phi_x}{\phi+\tilde{\rho}}\right)_t + (\psi+\tilde{u})\left(\frac{\phi_x}{\phi+\tilde{\rho}}\right)_x + \phi_{xx} = f_2, \tag{2.53}$$

with $f_2 := \frac{f_1}{\phi+\tilde{\rho}} - \frac{\phi_x(\psi+\tilde{u})_x}{\phi+\tilde{\rho}}$.

In what follows, we denote $\phi+\tilde{\rho}$ by ρ and $\psi+\tilde{u}$ by u in some places for simpilicity. Multiplying (2.53) by $\frac{\phi_x}{\rho}$ and integrating it with respect to x over $\mathbb{R}^+$, one has

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\phi_x}{\rho}\right\|^2 - \int_0^\infty \frac{u_x}{2}\left(\frac{\phi_x}{\rho}\right)^2 dx - \frac{u_b}{2}\left(\frac{\phi_x}{\rho}\right)^2(t,0) + \int_0^\infty \frac{\psi_{xx}\phi_x}{\rho}dx = \int_0^\infty \frac{f_2\phi_x}{\rho}dx. \tag{2.54}$$

Secondly, to remove $\psi_{xx}\phi_x$ in (2.54), we use (2.15), and multiply it by ϕ_x to get

$$\frac{d}{dt}(\psi,\phi_x) + ((\psi+\tilde{u})\psi_x,\phi_x) + \int_0^\infty (\psi_x\phi_t + \left(\frac{p_x}{\rho} - \frac{\tilde{p}_x}{\tilde{\rho}}\right)\phi_x) = \int_0^\infty (\mu\frac{\psi_{xx}}{\rho} + g)\phi_x. \tag{2.55}$$

We now proceed to treat the terms of (2.55). Firstly, it is easy to show that

$$\left|\int_0^\infty \psi\phi_x dx\right| \le \varepsilon\|\phi_x\|^2 + C\|\psi\|^2, \tag{2.56}$$

and

$$\left|\int_0^t ((\psi+\tilde{u})\psi_x,\phi_x)d\tau\right| \le \varepsilon\int_0^t \|\phi_x\|^2 d\tau + C\int_0^t \|\psi_x\|^2 d\tau. \tag{2.57}$$

Recalling the equation of ϕ, we obtain easily that $|\phi_t| \le C(|(\phi_x,\psi_x)| + |\tilde{\rho}_x\psi| + |\tilde{u}_x\phi|)$. Hence,

$$\left|\int_0^\infty \psi_x\phi_t dx\right| \le \varepsilon\int_0^t \left(\|\phi_x\|^2 + |\phi_x(\tau,0)|^2\right)d\tau + C\int_0^t \|\psi_x\|^2 d\tau. \tag{2.58}$$

For the term of p, if we write $p = p(\rho, \theta)$, by the mean value theorem one has

$$\frac{p_x}{\rho} - \frac{\tilde{p}_x}{\tilde{\rho}} = \frac{p_\rho \rho_x + p_\theta \theta_x}{\rho} - \frac{\tilde{p}_\rho \tilde{\rho}_x + \tilde{p}_\theta \tilde{\theta}_x}{\tilde{\rho}} = \frac{p_\rho}{\rho}\phi_x + O(\phi, \chi)(\tilde{\rho}_x, \tilde{\theta}_x) + O(\chi_x). \quad (2.59)$$

Combination of (2.54), (2.59) and (2.55), integrating it with respect to t yields

$$\begin{aligned} &\|\phi_x/\rho\|^2 - \varepsilon\|\phi_x\|^2 + \int_0^t \int_0^\infty \left(\|\phi_x\|^2 + \phi_x(\tau, 0)^2\right) d\tau \\ \leq\ & C\|\phi_{0x}\|^2 + C\int_0^\infty \psi_0 \phi_{0x} dx + \varepsilon \int_0^t \left(\|\phi_x\|^2 + \phi(\tau, 0)^2\right) d\tau + \\ & +\delta \int_0^t \|(\phi, \chi)_x\|^2 d\tau + C\|\psi\|^2 + C\int_0^t \|\psi_x\|^2 d\tau. \end{aligned} \quad (2.60)$$

Using the first energy estimate, taking ε suitably small we prove Lemma 2.3. □
Step 3. For the term $\|\psi_x\|^2$, we have

Lemma 2.4 *There exists a suitably small positive constant $\varepsilon_3 \leq \varepsilon_2$ such that if $N(t) + \delta \leq \varepsilon_3$, then the following estimate holds for any $t \geq 0$*

$$\|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \leq C\|(\phi_0, \psi_0, \chi_0)\|_{H^1}^2 + C(\delta + N(t))M(t)^2. \quad (2.61)$$

Here ε_3, C are independent of t, δ. □

Proof. To prove this lemma, we multiply eq. (2.15) by $-\psi_{xx}$, then integrating it with respect to t, x over $(0,t) \times (0, \infty)$, making use of (2.59), we have

$$\begin{aligned} &\frac{1}{2}\|\psi_x\|^2 + \int_0^t \int_0^\infty \frac{\mu \psi_{xx}^2}{\phi + \tilde{\rho}} \leq \frac{1}{2}\|\psi_{0x}\|^2 + \int_0^t \left(\varepsilon\|\psi_{xx}\|^2 + C\|(\phi, \psi, \chi)_x\|^2\right) d\tau + \\ & +C\int_0^t \int_0^\infty \left(\phi^2(\tilde{u}_{xx}^2 + \tilde{\theta}_x^2) + \psi^2(\tilde{\rho}_x^2 + \tilde{u}_x^2) + \chi^2(\tilde{\rho}_x^2 + \tilde{\theta}_x^2)\right) dx d\tau. \end{aligned} \quad (2.62)$$

Applying again the technique (2.46), we estimate the RHS terms of (2.62) as

$$RHS \leq \frac{1}{2}\|\psi_{0x}\|^2 + \varepsilon \int_0^t \|\psi_{xx}\|^2 + C\int_0^t \left(\|(\phi, \psi, \chi)_x\|^2 + \phi(\tau, 0)^2\right) d\tau. \, (2.63)$$

Recalling Lemmas 2.3, 2.2, taking ε suitably small, we have (2.61). Q.E.D. □
Step 4. Therefore, to complete the proof of Proposition 2.3, we need to prove following lemma on χ.

Lemma 2.5 *There exists a suitably small positive constant $\varepsilon_4 \leq \varepsilon_3$ such that if $N(t) + \delta \leq \varepsilon_4$, then the following estimate holds for any $t \geq 0$*

$$\|\chi_x(t)\|^2 + \int_0^t \|\chi_{xx}(\tau)\|^2 d\tau \leq C\|(\phi_0, \psi_0, \chi_0)\|_{H^1}^2 + C(\delta + N(t))M(t)^2. \quad (2.64)$$

Here ε_4, C are independent of t, δ. □

Proof. Multiplying (2.20) by $\frac{-\chi_{xx}}{\rho e_\theta}$ and integrating it with respect to x yields

$$\frac{1}{2}\frac{d}{dt}\|\chi_x\|^2 + ((\psi+\tilde{u})\chi_x, -\chi_{xx}) + K\int_0^\infty \frac{\chi_{xx}^2}{\rho e_\theta}dx = \int_0^\infty \frac{-h\chi_{xx}}{\rho e_\theta}dx. \qquad (2.65)$$

We now turn to handle terms in the above equation. Firstly, it follows from $e_\theta > 0$ and the fact $0 < C^{-1} \le \rho, \theta \le C$, that $0 < C \le \rho e_\theta \le C' < \infty$. Whence

$$K\int_0^\infty \frac{\chi_{xx}^2}{\rho e_\theta}dx \ge C\|\chi_{xx}\|^2. \qquad (2.66)$$

Similar to Step 3, we estiamte easily the second term as

$$|((\psi+\tilde{u})\chi_x), -\chi_{xx})| \le \varepsilon\|\chi_{xx}\|^2 + C\|\chi_x\|^2. \qquad (2.67)$$

Using the estimate (2.21), for the right–hand side term in (2.65), one has

$$\left|\int_0^\infty \frac{-h\chi_{xx}}{\rho e_\theta}\right| \le C\int_0^\infty \left(\psi_x^2 + |\psi_x\tilde{u}_x| + |(\phi,\chi)\tilde{u}_x| + |\psi_x| + |(\phi,\psi,\chi)\tilde{\theta}_x|\right)|\chi_{xx}|dx$$

$$\le \varepsilon\|\chi_{xx}\|^2 + C\|\psi_x\|^2 + (\delta+N)\|(\phi,\psi,\chi)_x\|^2 + \delta|\phi(t,0)|^2 + C\int_0^\infty \psi_x^2|\chi_{xx}|. \qquad (2.68)$$

It remains to handle the following term by using the Hölder inequality

$$\int_0^t\int_0^\infty \psi_x^2|\chi_{xx}|dx \le C\int_0^t \|\psi_x\|_\infty\|\psi_x\|\|\chi_{xx}\|dx \le C\int_0^t \|\psi_x\|_{H^1}\|\psi_x\|\|\chi_{xx}\|dx$$

$$\le C\left(\int_0^t \|\psi_x\|\|\psi_x\|_{H^1}^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|\psi_x\|\|\chi_{xx}\|^2 d\tau\right)^{\frac{1}{2}} \le CN(t)M(t)^2. \qquad (2.69)$$

Thus, using Lemmas 2.4, 2.2, 2.3, taking ε suitably small, we get (2.64). Q.E.D. □

Completion of the proof of Proposition 2.3: Combination of Lemmas 2.2–2.5, taking $N(t)+\delta \le \varepsilon_0 := \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}(=\varepsilon_4$, since we choose them such that $\varepsilon_1 \le \varepsilon_2 \le \varepsilon_3 \le \varepsilon_4)$. Therefore, if $N(t)+\delta \le \varepsilon_0$, then the following estimate holds

$$N^2(t) + M^2(t) \le C\|(\phi_0,\psi_0,\chi_0)\|_{H^1}^2 + C\delta^2 + C(N(t)+\delta)M(t)^2. \qquad (2.70)$$

If we take $\varepsilon_0 < 1$, using the Young inequality one has

$$N^2(t) + M^2(t) \le C\left(\|(\phi_0,\psi_0,\chi_0)\|_{H^1}^2 + \delta^2\right).$$

Which implies the results of Proposition 2.3 by the definition of N, M.Q.E.D. □

2.5 Large time behavior

In this subsection, we shall consider large time behavior of the solution to the full compressible Navier-Stokes equations. To this end, we first show that

$$\|\phi_x(t)\|, \|\psi_x(t)\|, \|\chi_x(t)\| \to 0. \qquad (2.71)$$

In fact, if this holds, recalling that $\|(\phi,\psi,\chi)\|_{H^1} \le C$, by interpolation we have

$$\|\phi(t)\|_\infty \;\le\; C\|\phi(t)\|^{\frac{1}{2}}\|\phi_x(t)\|^{\frac{1}{2}} \to 0, \quad \text{as } t \to \infty. \tag{2.72}$$

Similarly, we have $\|\psi(t)\|_\infty, \|\chi(t)\|_\infty \to 0$ as $t \to \infty$.

So, it remains to show (2.71). We define

$$P(t) = \int \phi_x^2/\rho^2(t,x)dx, \quad U(t) = \int \psi_x^2(t,x)dx, \quad X(t) = \int \chi_x^2(t,x)dx.$$

It follows from the first energy estimates that $\int_0^\infty (P(s)+U(s)+X(s))\,ds \le C$.

Step 1. We try to prove further that $\int_0^\infty |\frac{d}{ds}P(s)|ds \le C$. Recalling Eq.s (2.54) and (2.55), we have

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\phi_x}{\rho}(t)\right\|^2 - \int_0^\infty \frac{u_x}{2}\left(\frac{\phi_x}{\rho}\right)^2 dx - \frac{u_b}{2}\left(\frac{\phi_x}{\rho}\right)^2(t,0) + \int_0^\infty \frac{\psi_{xx}\phi_x}{\rho} = \int_0^\infty \frac{f_2\phi_x}{\rho}. \tag{2.73}$$

Which combined with the estimates in Proposition 2.3, we can obtain $\int_0^\infty \left|\frac{d}{dt}\|\frac{\phi_x}{\rho}(t)\|^2\right| dt \le C$. That is

$$\int_0^\infty |\frac{d}{ds}P(s)|ds \le C.$$

Recalling the fact $\int_0^\infty P(s)ds \le C$, one has $P(t) \to 0$, thus $\|\phi_x(t)\| \to 0$ as $t \to \infty$.

Step 2. In a similar way, we can show that

$$\int_0^\infty (|\frac{d}{ds}U(s)| + |\frac{d}{ds}X(s)|)ds \le C.$$

Thus $\|(\psi,\chi)_x(t)\| \to 0$ as $t \to \infty$, and (2.71) is proved. Q.E.D. □

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