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Nonlinear Boundary Layers of the Boltzmann Equation

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1 Introduction and Main Result

We discuss the nonlinear half-space problem of the Boltzmann equation with the Dirichlet boundary condition at the boundary and with a given Maxwellian at infinity, which arises in the theory of the kinetic boundary layer, the analysis of the condensation-evaporation and so on [4], [12].

The linearized problem has been studied by many authors [2],[5],[6],[7], mainly in the context of the classical Milne and Kramers problems. Thus, boundary fluxes are specified as auxiliary conditions. In [8], an existence theorem was established for the nonlinear case with the specular boundary condition and the method of proof does not apply to other boundary condition including the Dirichlet condition. Recently, nonlinear existence and stability theorems have been established for the discrete velocity model of the Boltzmann equation [10], [11], [13]. In this paper, we present the first existence theorem on the full nonlinear problem. Our method provides also a new aspect of the linearized problem (Remark 1.5 and §3 below).

It should be noted that K. Aoki, Y. Sone and their group, (c.f. [1], [12]), made an extensive numerical computation on the nonlinear problem and have observed that the existence of solutions depends strongly on the choice of Maxwellians specified for the far field. Our result gives a partial proof of their numerical results (Remark 1.6).

Thus, we consider a gas filled in the half-space $\mathbb{R}^3_+$. Take the $x-$axis to be orthogonal to the boundary so that the boundary is the plane $x = 0$ and that the half-space extends for $x > 0$. Then, our problem is,

$$(1.1)\qquad \begin{cases} \xi_1 F_x = Q(F,F), & x \in (0,\infty),\ \xi \in \mathbb{R}^3, \\ F|_{x=0} = F_0(\xi), & \xi \in \mathbb{R}^3_+, \\ F \to M_\infty(\xi) \quad (x \to \infty), & \xi \in \mathbb{R}^3. \end{cases}$$

Here, $F = F(x,\xi)$ is the unknown which describes the mass density distribution of gas particles at position $x \in (0,\infty)$ with velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ where ξ_1 is the component along the x-axis. Q is the *collision operator* defined by a quadratic

$$Q(F,G)=\int_{\mathbb{R}^3\times S^2}\Big(F(\xi')G(\xi'_*)-F(\xi)G(\xi_*)\Big)\,q(\xi-\xi_*,\omega)\,d\xi_*d\omega, \tag{1.2}$$

with

$$\xi'=\xi-[(\xi-\xi_*)\cdot\omega]\,\omega,\qquad \xi'_*=\xi_*+[(\xi-\xi_*)\cdot\omega]\,\omega, \tag{1.3}$$

where "$\cdot$" is the inner product of $\mathbb{R}^3$. We restrict ourselves to the hard sphere gas for which the *collision kernel* q is given by

$$q(\zeta,\omega)=\sigma_0|\zeta\cdot\omega|, \tag{1.4}$$

where σ_0 is the surface area of the hard sphere. Here we shall recall two classical properties of Q which are needed later. See [3], [4] for details.

(i) $Q(F)=0$ if and only if F is a *Maxwellian*,

$$M[\rho,u,T](\xi)=\frac{\rho}{(2\pi T)^{3/2}}\exp\Big(-\frac{|\xi-u|^2}{2T}\Big), \tag{1.5}$$

which describes an equilibrium state of a gas with the mass density $\rho>0$, flow velocity $u=(u_1,u_2,u_3)\in\mathbb{R}^3$ and temperature $T>0$.

(ii) A function $\phi(\xi)$ is called a *collision invariant* of Q if

$$\langle\phi,Q(F)\rangle=0 \qquad \text{for all } F,$$

$\langle,\rangle$ being the inner product of $L^2(\mathbb{R}^3)$. Q has five collision invariants

$$1,\quad \xi_i\ (i=1,2,3),\quad |\xi|^2. \tag{1.6}$$

The second equation in (1.1) is the Dirichlet boundary condition. The Dirichlet data $F_0(\xi)$ can be assigned only for incoming particles, i.e. for $\xi_1>0$, but not for all $\xi\in\mathbb{R}^3$. Otherwise, the problem becomes over-determined and hence ill-posed, as seen from the estimates of solution derived in the next section.

In the third equation of (1.1), we specify a state $M_\infty(\xi)$ for all $\xi\in\mathbb{R}^3$ at $x=\infty$. Clearly, M_∞ cannot be specified arbitrarily but must be a zero of Q, and hence a Maxwellian. Thus, we must take

$$M_\infty=M[\rho_\infty,u_\infty,T_\infty](\xi),$$

and $\rho_\infty>0$, $u_\infty=(u_{\infty,1},u_{\infty,2},u_{\infty,3})\in\mathbb{R}^3$, and $T_\infty>0$ are the only quantities which we can control. By a shift of the variable ξ in the direction orthogonal to the x-axis, we can assume without loss of generality that $u_{\infty,2} = u_{\infty,3} = 0$, and then, the sound speed and Mach number of this equilibrium are given by

$$c_\infty = \sqrt{\frac{5}{3}T_\infty}, \qquad \mathrm{M}^\infty = \frac{u_{\infty,1}}{c_\infty}, \tag{1.7}$$

respectively, see [4]. We will see that the Mach number M^∞ provides significant changes on the solvability of our problem (1.1). Indeed, since our boundary condition at $x = \infty$ is specified for all ξ, it is over-determined, and as a consequence, (1.1) may not be solvable unconditionally. Actually, we will show that the number of solvability conditions changes with the Mach number M^∞. To state this precisely, set

$$n^+ = \begin{cases} 0, & \mathrm{M}^\infty < -1, \\ 1, & -1 < \mathrm{M}^\infty < 0, \\ 4, & 0 < \mathrm{M}^\infty < 1, \\ 5, & 1 < \mathrm{M}^\infty, \end{cases} \tag{1.8}$$

and introduce the weight function

$$\boldsymbol{W}_\beta(\xi) = (1 + |\xi|)^{-\beta}\Big(M[1, u_\infty, T_\infty](\xi)\Big)^{1/2}, \tag{1.9}$$

with $\beta \in \mathbb{R}$. Our main result is

Theorem 1.1 *Given $\rho_\infty > 0$, $u_{\infty,1} \in \mathbb{R}$, and $T_\infty > 0$, suppose $\mathrm{M}^\infty \neq 0, \pm 1$. Furthermore, let $\beta > 3/2$. Then, there exist positive numbers ϵ_0, σ, C_0, and a C^1 map*

$$\Psi : L^2(\mathbb{R}^3_+) \longrightarrow \mathbb{R}^{n+}, \quad \Psi(0) = 0, \tag{1.10}$$

and the following holds.

(i) *For any F_0 satisfying*

$$|F_0(\xi) - M_\infty(\xi)| \leq \epsilon_0 \boldsymbol{W}_\beta(\xi), \quad \xi \in \mathbb{R}^3_+, \tag{1.11}$$

and

$$\Psi(F_0 - M_\infty) = 0, \tag{1.12}$$

the problem (1.1) *has a unique solution F in the class*

$$|F(x,\xi) - M_\infty(\xi)| + |\xi_1 F_x(x,\xi)| \leq C_0 e^{-\sigma x} \boldsymbol{W}_\beta(\xi), \quad x \in (0,\infty), \xi \in \mathbb{R}^3. \tag{1.13}$$

(ii) *The set of F_0 satisfying* (1.11) *and* (1.12) *forms a (local) C^1 manifold of codimension n^+.*

Remark 1.2 The cases $\mathrm{M}^\infty = 0, \pm 1$ are not included in the theorem.

Remark 1.3 We put $\mathbb{R}^{n+} = \emptyset$ when $n^+ = 0$. Thus, the condition (1.12) is void for the case $\mathrm{M}^\infty < -1$.

Remark 1.4 Given a far field M_∞, (1.11) is a smallness condition on the deviation of F_0 from M_∞ whereas (1.12) gives restrictions on F_0 however small it may be. Thus, our theorem says that the problem (1.1) is solvable unconditionally for any F_0 sufficiently close to M_∞ if $\mathrm{M}^\infty < -1$, but otherwise not. A physical explanation of this is that if the far flow is supersonic and incoming to the boundary ($\mathrm{M}^\infty < -1$), then any phenomena near the boundary cannot affect the far field while if it is subsonic or outgoing, some of phenomena near boundary can propagate to infinity and affect the far field.

Remark 1.5 A similar theorem holds for the linearized problem of (1.1) at the far Maxwellian M_∞. In this case, the map Ψ becomes linear of deficiency n^+, that is, the set of admissible boundary data is just the orthogonal compliment of an n^+ dimensional (linear) subspace. This gives a new aspect of the linearized problem different from that in [2],[5],[6],[7]. See §3 below.

Remark 1.6 The numerical computation in [12] and the references therein deals with (1.1) with F_0 fixed to be the standard Maxwellian $M[1,0,1](\xi)$, and shows that the set of points $(\rho_\infty, u_{\infty,1}, T_\infty) \in \mathbb{R}^3$ which admit smooth solutions connecting F_0 and M_∞ is a union of a three-dimensional subdomain of the domain $\mathrm{M}^\infty < -1$ and a two-dimensional surface in $0 < \mathrm{M}^\infty < -1$ whereas no solutions exist for $\mathrm{M}^\infty > 0$. Our theorem agrees with this for the case $\mathrm{M}^\infty < 0$, but not for $\mathrm{M}^\infty > 0$. Probably $F_0 = M[1,0,1]$ may not satisfy the solvability condition (1.12) if $\mathrm{M}^\infty > 0$.

Remark 1.7 The stability of the stationary solutions obtained in Theorem 1.1 is an important issue. In our forthcoming paper, we will show their exponentially asymptotic stability for the case $\mathrm{M}^\infty < -1$.

2 Outline of the Proof

Our proof relies on the analysis of the corresponding linearized problem at M_∞. We will look for the solution of (1.1) in the form

$$F(x,\xi) = M_\infty(\xi) + \boldsymbol{W}_0(\xi) f(x,\xi), \tag{2.1}$$

where $\boldsymbol{W}_0$ is $\boldsymbol{W}_\beta$ of (1.9) with $\beta = 0$. Then, the problem (1.1) reduces to

$$\begin{cases} \xi_1 f_x - \boldsymbol{L} f &= \Gamma(f), \qquad x \in (0,\infty),\ \xi \in \mathbb{R}^3, \\ f|_{x=0} &= a_0(\xi), \qquad \xi \in \mathbb{R}^3_+, \\ f &\to 0\ (x \to \infty),\ \ \xi \in \mathbb{R}^3, \end{cases} \tag{2.2}$$

$$\begin{aligned} \boldsymbol{L}f &= \boldsymbol{W}_0^{-1}[Q(M_\infty, \boldsymbol{W}_0 f) + Q(\boldsymbol{W}_0 f, M_\infty)], \\ \Gamma(f) &= \boldsymbol{W}_0^{-1} Q(\boldsymbol{W}_0 f, \boldsymbol{W}_0 f), \\ a_0 &= \boldsymbol{W}_0^{-1}\left(F_0 - M_\infty\right). \end{aligned}$$

The operator $\boldsymbol{L}$ is linear while the remainder Γ is quadratic.

There are two ingredients in our proof. One is to add a "damping" term constructed as follows. Denote by N the space spanned by the collision invariants (1.6) weighted by $\boldsymbol{W}_0$,

$$N = \operatorname{span}\left\{\boldsymbol{W}_0(\xi),\ \boldsymbol{W}_0(\xi)\xi_i\ (i = 1,2,3),\ \boldsymbol{W}_0(\xi)|\xi|^2\right\}, \tag{2.3}$$

which we regard as a 5-dimensional subspace of $L^2(\mathbb{R}^3)$. Let $N^\perp$ be the orthogonal compliment of N and let

$$\boldsymbol{P}_0 : L^2(\mathbb{R}^3) \to N, \qquad \boldsymbol{P}_1 : L^2(\mathbb{R}^3) \to N^\perp,$$

be the orthogonal projections. Define the operator

$$A = \boldsymbol{P}_0 \xi_1 \boldsymbol{P}_0|_N. \tag{2.4}$$

A is a linear bounded self-adjoint operator on N and its eigenvalues are

$$\lambda_1 = u_{\infty,1} - c_\infty, \quad \lambda_i = u_{\infty,1}\ (i = 2,3,4), \quad \lambda_5 = u_{\infty,1} + c_\infty. \tag{2.5}$$

Notice that n^+ of (1.8) is the number of positive λ_i's and denote by $\boldsymbol{P}_0^+$ the eigenprojection for them. With this, we now modify (2.2) as

$$\begin{cases} \xi_1 f_x - \boldsymbol{L}f = & \Gamma(f) - \gamma \boldsymbol{P}_0^+ \xi_1 f, \quad x \in (0,\infty),\ \xi \in \mathbb{R}^3, \\ f|_{x=0} = & a_0(\xi), \qquad \xi \in \mathbb{R}^3_+, \\ f \to & 0\ (x \to \infty), \qquad \xi \in \mathbb{R}^3, \end{cases} \tag{2.6}$$

with a positive constant γ to be determined later. Note that for the case $\mathrm{M}^\infty < -1$, we have $n^+ = 0$ and hence $-\gamma \boldsymbol{P}_0^+ \xi_1 f = 0$, giving no modification to (2.2), but otherwise it has a good sign on the positive eigenspace $\boldsymbol{P}_0^+ N$.

Another ingredient is to introduce an exponential weight function in x, which is used to get a definitive estimate on the negative eigenspace $(1 - \boldsymbol{P}_0^+)N$. Thus, put

$$f = e^{-\sigma x} g, \tag{2.7}$$

with a constant $\sigma > 0$ to be determined later. Then, (2.6) becomes

$$\begin{cases} \xi_1 g_x - \sigma \xi_1 g - \boldsymbol{L}g = & h - \gamma \boldsymbol{P}_0^+ \xi_1 g, \quad x \in (0,\infty),\ \xi \in \mathbb{R}^3, \\ g|_{x=0} = & a_0(\xi), \qquad \xi \in \mathbb{R}^3_+, \\ g \to & 0\ (x \to \infty), \qquad \xi \in \mathbb{R}^3, \end{cases} \tag{2.8}$$

(2.9) $$h = e^{-\sigma x}\Gamma(g).$$

The new term $-\sigma\xi_1 g$ comes from the weight function in (2.7). Seemingly, this has not a good sign, but we can choose $\gamma, \sigma > 0$ so that the combination $-\sigma\xi_1 + \gamma \boldsymbol{P}_0^+\xi_1$ has a good sign on the space N if $\mathrm{M}^\infty \neq 0, \pm 1$.

If h is assumed a given function but not defined by (2.9), (2.8) is a linear problem. Using the good sign of the above mentioned linear combination, we can easily establish an L^2 energy estimate for this linear problem.

Proposition 2.1 *Any smooth solution g of the linear problem* (2.8) *satisfies*

(2.10) $$< |\xi_1| g^0, g^0 >_- + ||(1+|\xi|)^{1/2} g||^2 \leq C_0(< \xi_1 a_0, a_0 >_+ + ||h||^2),$$

where $g^0 = g|_{x=0}$ and C_0 is a positive constant independent of a_0 and h while $< \cdot, \cdot >_\pm$ and $||\cdot||$ are the inner products of $L^2(\mathbb{R}^3_\pm)$ and the norm of $L^2((0,\infty)\times\mathbb{R}^3)$, respectively.

This is enough to construct the solution. First, the same estimate can be derived for the adjoint problem to the linear problem (2.8), which then enable us, together with the Hahn-Banach theorem and Riesz representation theorm, to show the existence of weak L^2 solutions to the linear problem (2.8). Furthermore, taking suitable test functions, we can prove the "weak=strong" theorem, and thus get strong solutions satisfying the estimate (2.10).

Moreover, starting from this estimate and using the the bootstrap argument, we can get the L^∞ estimate, that is, (2.10) with all the L^2 norms replaced by L^∞ norms.

Now, the contraction argument allows us to construct L^∞ solutions of the nonlinear problem (2.8) with (2.9) for sufficently small boundary data a_0.

In the case $\mathrm{M}^\infty < -1$, this gives the solutions to (2.2) and hence to the original problem (1.1). For the case $\mathrm{M}^\infty > -1$, it is clear that if the solution g to (2.8) thus obtained satisfies

(2.11) $$\boldsymbol{P}_0^+\xi_1 g = 0,$$

it is also a solution of the original problem without the extra damping term. We can show that the condition (2.11) reduces to

(2.12) $$\boldsymbol{P}_0^+\xi_1 g|_{x=0} = 0.$$

Clearly, g and hence $g|_{x=0}$ as well is determined uniquely by the boundary data a_0 and so is the right hand side of (2.12). Put

(2.13) $$\Psi(a_0) = \boldsymbol{P}_0^+\xi_1 g|_{x=0}.$$

Identifying the space P_0^+N with $\mathbb{R}^{n^+}$, we can show that this is a C^1 map as

$$\Psi : L^2(\mathbb{R}^3_+, \xi_1 d\xi) \to \mathbb{R}^{n^+}, \tag{2.14}$$

with $\Psi(0) = 0$. Moreover, we can show, using the implicit function theorem, that the set of a_0's such that $\Psi(a_0) = 0$ is a C^1 manifold of codimention n^+, whence Theorem 1.1 follows. The detail will be given elsewhere.

3 A Remark on the Linearized Problem

The linearized problem of (1.1) at M_∞ is just (2.2) with the term $\Gamma(f)$ dropped;

$$\left\{\begin{array}{lll} \xi_1 f_x - \boldsymbol{L}f & = 0, & x \in (0,\infty),\ \xi \in \mathbb{R}^3, \\ f|_{x=0} & = a_0(\xi), & \xi \in \mathbb{R}^3_+, \\ f & \to 0\ (x \to \infty), & \xi \in \mathbb{R}^3, \end{array}\right. \tag{3.1}$$

This problem has been solved in [2],[5],[6],[7], but specifing some of boundary fluxes. In addition to this auxiliary condition, the solutions obtained there do not converge to 0 at $x = \infty$ but to an element of the space N of (2.3), and moreover, the proofs do not tell us how to determine the limit element.

Our argument in §2 applies also to this linearized problem and gives solutions which tend to 0 at $x = \infty$. We have only to solve (2.8) with $h = 0$ and to note that the map Ψ of (2.14) is linear for this case. Then, in virtue of the Riesz representation theorem, there exist $r_i \in L^2(\mathbb{R}^3_+, \xi_1 d\xi)$, $i = 1, 2, \cdots, n^+$ such that

$$\Psi(a_0) = (<\xi_1 r_1, a_0>_+, <\xi_1 r_2, a_0>_+, \cdots, <\xi_1 r_{n^+}, a_0>_+).$$

Put $R = \mathrm{span}\{r_1, r_2, \cdots, r_{n^+}\}$. Then, we conclude

Theorem 3.1 *For any $a_0 \in R^\perp$, the linearized problem* (3.1) *has a unique L^2 solution of the form $f = e^{-\sigma x} g$ with g satisfying the estimate* (2.10) *for $h = 0$.*

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