NON-TRIVIAL LIMIT LAWS IN TOPOLOGICAL GROUPS: HOW SIMPLE CAN THEY BE?

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ABSTRACT. A limit law is a map \( f : (D, \leq) \to F(X) \) from a directed set \((D, \leq)\) to a free group \( F(X) \) over some set \( X \). A topological group \( G \) satisfies limit law \( f \) (we also say that \( f \) holds in \( G \)) provided that for every group homomorphism \( \pi : F(X) \to G \) from \( F(X) \) to \( G \) and each open set \( U \) containing the identity element \( e_G \) of \( G \) there exists some \( d \in D \) such that \( \pi(f(c)) \in U \) for all \( c \geq d \). For a group \( G \) a limit law \( f : (D, \leq) \to F(X) \) is called \( G \)-algebraic provided that there exists \( d \in D \) such that

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\pi(f(c)) = e_G \quad \text{whenever} \quad c \geq d \quad \text{and} \quad \pi : F(X) \to G \quad \text{is a group homomorphism. A limit law that is not} \quad G \quad \text{-algebraic is called essentially} \quad G \text{-topological. Main result: If a a Hausdorff group} \quad G \quad \text{satisfies some essentially} \quad G \text{-topological limit law} \quad f : (D, \leq) \to F(X) \quad \text{such that} \quad (D, \leq) \quad \text{is either a linearly ordered set or a countable partially ordered set, then} \quad G \quad \text{also satisfies some essentially} \quad G \text{-topological limit law} \quad f' : (N, \leq) \to F(X) \quad \text{having the usual set of integers} \quad (N, \leq) \quad \text{as its domain. It follows that if} \quad G \quad \text{is one of the three classical locally compact groups,} \quad \mathbb{Z} \quad \text{(integers)}, \quad \mathbb{R} \quad \text{(reals)} \quad \text{or} \quad T \quad \text{(unit circle), then every limit law with a linearly ordered domain that holds in} \quad G \quad \text{is} \quad G \text{-algebraic.}

As usual, the symbol \( F(X) \) denotes the free group over a set \( X \). If \( G \) is a group, then \( e_G \) denotes the identity element of \( G \). The identity element of \( F(X) \) will be simply denoted by \( e \).

A partially ordered set (or shortly, poset) is a pair \((D, \leq)\) consisting of a set \( D \) together with a relation \( \leq \) which is:

(i) reflexive, i.e. \( d \leq d \) for each \( d \in D \), and

(ii) transitive, i.e. \( d_0 \leq d_1 \) and \( d_1 \leq d_2 \) implies \( d_0 \leq d_2 \).

A partially ordered set \((D, \leq)\) is directed provided that for every pair \( d_0, d_1 \in D \) of elements of \( D \) there exists \( d \in D \) such that \( d_0 \leq d \) and \( d_1 \leq d \).

Limit laws were studied extensively in [1, 3] and recently studied extensively in [4]. A limit law is a map \( f : (D, \leq) \to F(X) \) from a directed set \((D, \leq)\) to a free group \( F(X) \) over some set \( X \). We say that a limit law \( f : (D, \leq) \to F(X) \) holds in a topological group \( G \), or that \( G \) satisfies law \( f \), provided that for every group homomorphism \( \pi : F(X) \to G \) from \( F(X) \) to \( G \) the directed set \( \{\pi(f(d)) : d \in D\} \) converges to the identity element \( e_G \) of \( G \); that is, for every open set \( U \) containing \( e_G \) there exists \( d \in D \) such that \( \pi(f(c)) \in U \) for all \( c \geq d \).

Let \( G \) be a group. A limit law \( f : (D, \leq) \to F(X) \) will be called \( G \)-algebraic provided that there exists some \( d \in D \) such that \( \pi(f(c)) = e_G \) whenever \( c \geq d \) and \( \pi : F(X) \to G \) is a group homomorphism. If \( G \) is a topological group, then a \( G \)-algebraic limit law automatically holds in \( G \) for an obvious algebraic reason, thereby justifying its name.

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Note that the topology of the group $G$ plays absolutely no role in "deciding" whether a $G$-algebraic law holds in $G$ or not; everything is determined by the algebraic structure of $G$. Therefore, from a topological point of view, $G$-algebraic laws are trivial and not particularly interesting. A limit law that is not $G$-algebraic will be called essentially $G$-topological. Contrary to $G$-algebraic laws, the topology of the group $G$ plays a crucial role in (really!) deciding whether an essentially $G$-topological law holds in $G$ or not; this both explains the choice of our terminology and indicates that essentially $G$-topological laws are of special interest from the topological point of view.

It appears to be natural to adopt the (luck of) complexity of a partially ordered set $(D,\leq)$ as a measure of "simplicity" of a limit law $f : (D,\leq) \to F(X)$. The main purpose of this article is to demonstrate that the three classical locally compact groups, the group $\mathbb{Z}$ of integer numbers, the group $\mathbb{R}$ of real numbers and the unit circle group $\mathbb{T}$, do not satisfy any "simple" essentially $G$-topological law (see Corollary 14).

**Lemma 1.** Let $f : (D,\leq) \to F(X)$ be a limit law that holds in a Hausdorff topological group $G$. If $(D,\leq)$ has a biggest element (in particular, if the poset $(D,\leq)$ is finite), then $f$ is $G$-algebraic.

**Proof.** Let $a$ be a biggest element of $(D,\leq)$. First suppose that there exists a group homomorphism $\pi : F(X) \to G$ such that $\pi(f(a)) \neq e_G$. Then $U = G \setminus \{\pi(f(a))\}$ is an open neighbourhood of $e_G$ by Hausdorffness of $G$. Since $f$ holds in $G$, there exists $c \in D$ with $\pi(f(d)) \in U$ for all $d \geq c$. Since $a$ is the biggest element of $(D,\leq)$, it follows that $\pi(f(a)) \in U = G \setminus \{\pi(f(a))\}$, a contradiction. Therefore $\pi(f(a)) = e_G$ for every group homomorphism $\pi : F(X) \to G$. Since $a$ is the biggest element of $(D,\leq)$, we also have that $\pi(f(d)) = e_G$ whenever $d \in D$, $d \geq a$ and $\pi : F(X) \to G$ is a group homomorphism. This means that $f$ is $G$-algebraic.

A subset $C$ of a directed set $(D,\leq)$ is called cofinal in $(D,\leq)$ if for every $d \in D$ there exists $c \in C$ with $d \leq c$.

Let $f$ and $g$ be limit laws. We will write $f \Leftarrow g$ provided that $f$ holds in every topological group in which $g$ holds.

**Lemma 2.** If $f : (D,\leq) \to F(X)$ is a limit law and $C$ is a cofinal subset of $(D,\leq)$, then the restriction $f|_C : C \to F(X)$ of $f$ to $C$ is a limit law and $f|_C \Leftarrow f$.

**Proof.** Being a cofinal subset of a directed set $(D,\leq)$, the partially ordered set $(C,\leq)$ is also directed, and so $f|_C$ is a limit law. Let $G$ be a topological group in which $f$ holds. We are going to prove that $f|_C$ also holds in $G$. Indeed, let $\pi : F(X) \to G$ be a homomorphism from $F(X)$ to $G$. Let $U$ be an open subset of $G$ which contains the identity element $e_G$. Since $f$ holds in $G$, there exists some $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$. Since $C$ is cofinal in $(D,\leq)$, one can find $c_0 \in C$ with $c_0 \geq d$. Clearly, $\pi(f(c)) \in U$ for all $c \geq c_0$.

A sequential law is a limit law $f : (\mathbb{N},\leq) \to F(X)$ with the set $(\mathbb{N},\leq)$ of natural numbers as its directed set. A countable law is a limit law $f : (D,\leq) \to F(X)$ whose domain $(D,\leq)$ is a countable directed set.

In view of Lemma 1, the cardinality of the domain of an essentially $G$-topological law must be infinite, and thus countable laws are potentially the simplest possible essentially $G$-topological laws. This explains why our first theorem considers such laws.

**Theorem 3.** Let $G$ be a Hausdorff group. If $G$ satisfies some essentially $G$-topological countable law, then it also satisfies some essentially $G$-topological sequential law.
Proof. Let $G$ be a Hausdorff group and let \( f : (D, \leq) \to F(X) \) be an essentially $G$-topological countable limit law that holds in $G$. Let $D = \{d_n : n \in \mathbb{N}\}$ be an enumeration of $D$. According to Lemma 1 the poset $(D, \leq)$ does not have the biggest element. Using this fact, directness of $(D, \leq)$ and the fact that $f$ is essentially $G$-topological we can easily choose, by induction on $n$, an element $c_n \in D$ and a group homomorphism $\pi_n : F(X) \to G$ such that $d_n \leq c_n$, $c_{n-1} < c_n$ and $\pi_n(f(c_n)) \neq e_G$. By our construction, $C = \{c_n : n \in \mathbb{N}\}$ is cofinal in $(D, \leq)$ and therefore $f|_C \preceq f$ by Lemma 2. Since $f$ holds in $G$, so does $f|_C$. By our construction, $(C, \leq)$ is order isomorphic to $(\mathbb{N}, \leq)$ and $\pi_n(f(c)) \neq e_G$ for all $c \in C$. Thus $f|_C$ is an essentially $G$-topological sequential law. \( \Box \)

Recall that a cardinal $\tau$ is called singular provided that there exists a cardinal $\kappa < \tau$ and a transfinite sequence $\{\tau_\beta : \beta < \kappa\}$ of cardinals such that $\sup\{\tau_\beta : \beta < \kappa\} = \tau$ and $\tau_\beta < \tau$ for each $\beta < \kappa$. A cardinal is regular if it is not singular.

If $X$ is a set, $G$ is a group and $\varphi : X \to G$ is a map, then $\hat{\varphi} : F(X) \to G$ will denote the (unique) extension of $\varphi$ over $F(X)$ that is a group homomorphism. If $y \in F(X)$ and $y \neq e$, then $\text{supp}$ denotes the smallest subset $Y$ of $X$ such that $y$ belongs to the subgroup of $F(X)$ generated by $Y$. Note that $\text{supp}$ is always finite.

Our next lemma establishes an algebraic fact about free groups that is perhaps of some independent interest.

**Lemma 4.** If $X$ is a set and $Z$ is a subset of $F(X)$ of uncountable regular cardinality, then there exist $Y \subseteq Z$, $y^* \in Y$ and a map $\varphi : X \to X$ such that $|Y| = |Z|$ and $\hat{\varphi}(y) = y^*$ for all $y \in Y$.

**Proof.** Without loss of generality we will assume that $z \neq e$ for each $z \in Z$. Note that $\{\text{supp} z : z \in Z\}$ is a family of non-empty finite subsets of $X$, so by the $\Delta$-system Lemma (see, for example, [2, Ch. II, Theorem 1.6]) there exists a finite (possibly empty) set $T \subseteq X$ and $Z' \subseteq Z$ such that $|Z'| = |Z|$ and $\text{supp} z \cap \text{supp} z' = T$ whenever $z, z' \in Z'$ and $z \neq z'$. For each $n \in \mathbb{N} \setminus \{0\}$ define $Z'_n = \{z \in Z' : |\text{supp} z| = n\}$ and note that $Z' = \bigcup\{Z'_n : n \in \mathbb{N} \setminus \{0\}\}$. Since $|Z'| = |Z|$ is an uncountable regular cardinal, it follows that $|Z'_n| = |Z'|$ for some $n \in \mathbb{N} \setminus \{0\}$. Pick arbitrarily $z^* \in Z'_n$. For each $z \in Z'_n \setminus \{z^*\}$ choose a bijection $h_z : \text{supp} z \to \text{supp} z^*$ such that $h_z(t) = t$ for all $t \in T$, and let $\hat{h}_z : F(\text{supp} z) \to F(\text{supp} z^*)$ be the natural homomorphic extension of $h_z$ over $F(\text{supp} z)$. Since the set $F(\text{supp} z^*)$ is at most countable, and $|Z'_n \setminus \{z^*\}| = |Z'_n| = |Z'| = |Z|$ is an uncountable regular cardinal, there exist $g \in F(\text{supp} z^*)$ and $Y \subseteq Z'_n$ such that $|Y| = |Z'_n|$ and $\hat{h}_y(y) = g$ for all $y \in Y$. Pick $y^* \in Y$ arbitrarily. For each $y \in Y$ define the map $f_y : \text{supp} y \to \text{supp} y^*$ by $f_y = h_{y^*}^{-1} \circ h_y$ and note that the restriction of $f_y$ to $T$ is the identity map of $T$. This allows us to define the map $\varphi : X \to X$ by $\varphi(x) = f_y(x)$ if $x \in \text{supp} y$ for some $y \in Y$ and $\varphi(x) = x$ if $x \in X \setminus \text{supp} y \in Y$. Finally, by our construction

$$\hat{\varphi}(y) = \hat{f}_y(y) = \hat{h}_{y^*}^{-1}(\hat{h}_y(y)) = \hat{h}_{y^*}^{-1}(g) = y^*$$

for each $y \in Y$. \( \Box \)

Another potential candidate for a "simple" limit law is the law with a linearly ordered domain. Recall that a poset $(D, \leq)$ is linearly ordered provided that for every pair $d, d'$ of elements of $D$ either $d \leq d'$ or $d' \leq d$ holds. A linearly ordered law is a limit law $f : (D, \leq) \to F(X)$ whose domain $(D, \leq)$ is a linearly ordered set. Sequential laws are particular types of linearly ordered laws.
Theorem 5. If a Hausdorff topological group $G$ satisfies some essentially $G$-topological linearly ordered law, then $G$ also satisfies some essentially $G$-topological sequential law.

Proof. The proof of this theorem will be split into a sequence of claims.

Let $G$ be a Hausdorff topological group and $f : (D, \leq) \to F(X)$ be an essentially $G$-topological linearly ordered law which holds in $G$. Let $\tau$ be the smallest cardinality of a cofinal subset of $(D, \leq)$. Choose a cofinal subset $E = \{d_\alpha : \alpha < \tau\}$ of $(D, \leq)$ of cardinality $\tau$.

Claim 6. If $C \subseteq D$ and $|C| < \tau$, then there exists $d \in D$ such that $c < d$ for all $c \in C$.

Proof. Since $\tau$ is a minimal cardinality of a cofinal subset of $(D, \leq)$, the set $C$ cannot be cofinal in $(D, \leq)$. Therefore there exists some $d \in D$ such that for all $c \in C$ the inequality $d \leq c$ does not hold. It is precisely here where we use the fact that $(D, \leq)$ is a linearly ordered set to conclude that $c < d$ for all $c \in C$. \hfill \Box

By transfinite recursion we will choose points $\{c_\alpha : \alpha < \tau\} \subseteq D$ and a family $\{\pi_\alpha : \alpha < \tau\}$ of group homomorphisms from $F(X)$ to $G$ in such way that, for every $\alpha < \tau$, one has $d_\alpha < c_\alpha$, $\pi_\alpha(f(c_\alpha)) \neq e_G$ and $c_\beta < c_\alpha$ for $\beta < \alpha$. Assume that $\alpha < \tau$ and that points $\{c_\beta : \beta < \alpha\} \subseteq D$ and group homomorphisms $\{\pi_\beta : \beta < \alpha\}$ from $F(X)$ to $G$ have already been chosen. From Claim 6 it follows that there exists $d \in D$ such that $c_\beta < d$ for all $\beta < \alpha$. Since $(D, \leq)$ is directed, $d_\alpha \leq d'$ and $d \leq d'$ for some $d' \in D$. Now use the fact that $f$ is essentially $G$-topological to pick $c_\alpha \in D$ and a group homomorphism $\pi_\alpha : F(X) \to G$ such that $d' \leq c_\alpha$ and $\pi_\alpha(f(c_\alpha)) \neq e_G$. Clearly $c_\alpha$ has all necessary properties.

Claim 7. $\beta < \alpha < \tau$ implies $c_\beta < c_\alpha$.

Proof. This was guaranteed as part of our inductive construction. \hfill \Box

Claim 8. $C = \{c_\alpha : \alpha < \tau\}$ is a cofinal subset of $(D, \leq)$.

Proof. $E = \{d_\alpha : \alpha < \tau\}$ is cofinal in $(D, \leq)$ and $d_\alpha \leq c_\alpha$ for all $\alpha < \tau$ implies that $C$ is also cofinal in $(D, \leq)$. \hfill \Box

Claim 9. If $\Gamma$ is a cofinal subset of $\tau$, then $\{c_\gamma : \gamma \in \Gamma\}$ is cofinal in $(D, \leq)$.

Proof. Suppose that $\Gamma$ is cofinal in $\tau$. Let $d \in D$. From Claim 8 it follows that $d \leq c_\beta$ for some $\beta < \tau$. Cofinality of $\Gamma$ in $\tau$ yields $\gamma \in \Gamma$ such that $\beta < \gamma$. Now $d \leq c_\beta \leq c_\gamma$ by Claim 7. \hfill \Box

Claim 10. $\tau$ is infinite.

Proof. If $\tau$ is finite, then $(D, \leq)$ must have a biggest element $a$, and then $f$ will be $G$-algebraic by Lemma 1. \hfill \Box

Claim 11. $\tau$ is a regular cardinal.

Proof. Assume the contrary, i.e. that $\tau$ is singular. Then there exists a cardinal $\kappa < \tau$ and a transfinite sequence $\{\tau_\beta : \beta < \kappa\}$ of cardinals such that $\sup\{\tau_\beta : \beta < \kappa\} = \tau$ and $\tau_\beta < \tau$ for each $\beta < \kappa$. For each $\beta < \kappa$ applying $\tau_\beta < \tau$ and Claim 6 to the set $C_\beta = \{d_\alpha : \alpha < \tau_\beta\}$ one can find $b_\beta \in D$ such that $d_\alpha < b_\beta$ for $\alpha < \tau_\beta$. We now claim that the set $\{b_\beta : \beta < \kappa\}$ is cofinal in $(D, \leq)$, thereby contradicting minimality of $\tau$. Indeed, let $d \in D$. Since $E$ is cofinal in $(D, \leq)$, one has $d \leq d_\alpha$ for some $\alpha < \tau$. Since $\sup\{\tau_\beta : \beta < \kappa\} = \tau$, there exists $\beta < \kappa$ with $\alpha < \tau_\beta$. It remains only to note that $d \leq d_\alpha < b_\beta$. \hfill \Box
Claim 12. $\tau$ is countable.

Proof. Assume the contrary. Then $\tau$ is an uncountable regular cardinal by Claims 10 and 11. We can now apply Lemma 4 to the set $Z = \{ f(c_\alpha) : \alpha < \tau \}$ to find a subset $\Gamma \subseteq \tau$, an ordinal $\gamma^* \in \Gamma$ and a map $\varphi : X \to X$ such that $|\Gamma| = \tau$ and $\varphi(f(c_\gamma)) = f(c_{\gamma^*})$. Recall now that the group homomorphism $\pi_{\gamma^*} : F(X) \to G$ satisfies $g = \pi_{\gamma^*}(f(c_{\gamma^*})) \neq e_G$. Since $G$ is Hausdorff, $U = G \setminus \{g\}$ is an open neighbourhood of $e_G$ in $G$. Define a group homomorphism $\pi : F(X) \to G$ via $\pi = \pi_{\gamma^*} \circ \varphi$. Then $\pi(f(c_\gamma)) = \pi_{\gamma^*}(\varphi(f(c_\gamma))) = \pi_{\gamma^*}(f(c_{\gamma^*})) = g$ for $\gamma \in \Gamma$, and therefore

$$\pi(f(c_\gamma)) \notin U \text{ for each } \gamma \in \Gamma.$$ 

Since $|\Gamma| = \tau$, $\Gamma$ is cofinal in $\tau$, and so the set $\{c_\gamma : \gamma \in \Gamma\}$ is cofinal in $(D, \leq)$ by Claim 9. Cofinality of $\{c_\gamma : \gamma \in \Gamma\}$ in $(D, \leq)$ and (1) imply that $f$ does not hold in $G$, a contradiction. \hfill \square

By Claim 8 $C = \{c_\alpha : \alpha < \tau = \omega\}$ is a cofinal subset of $(D, \leq)$, and so the restriction $h = f|_C$ of $f$ to $C$ is a limit law such that $h \Leftarrow f$ (see Lemma 2). Since $f$ holds in $G$, so does $h$. From the choice of homomorphisms $\pi_\alpha$ it follows that $h$ is essentially $G$-topological. Claim 7 implies that $(C, \leq)$ is order isomorphic to $(\mathbb{N}, \leq)$, i.e. that $h$ is a sequential law. \hfill \square

From Theorems 3 and 5 we immediately get the following

Corollary 13. For a Hausdorff group $G$ the following conditions are equivalent:

(i) $G$ satisfies some essentially $G$-topological linearly ordered law,

(ii) $G$ satisfies some essentially $G$-topological countable law,

(iii) $G$ satisfies some essentially $G$-topological sequential law.

If $G$ is either the group $\mathbb{Z}$ of integer numbers, the group $\mathbb{R}$ of real numbers or the unit circle group $\mathbb{T}$, then each sequential law that holds in $G$ is $G$-algebraic [1]. From this result and Corollary 13 we obtain

Corollary 14. Let $G$ be one of the groups $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{T}$. Then all countable or linearly ordered laws that hold in $G$ are $G$-algebraic.

If a locally compact Abelian group $G$ satisfies some essentially $G$-topological sequential law, then $G$ is totally disconnected [1]. From this and Corollary 13 we get our last

Corollary 15. If a locally compact Abelian group $G$ satisfies either some essentially $G$-topological linearly ordered law or some essentially $G$-topological countable law, then $G$ is totally disconnected.

References


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