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京都大学
1 Introduction

A regular space X is called rim-compact if there exists a base \( B \) for the open sets of \( X \) such that the boundary \( \text{Bd} U \) is compact for each \( U \) in \( B \).

In 1942 de Groot (cf. [1]) proved the following:

(*) A separable metrizable space \( X \) is rim-compact if and only if there is a metrizable compactification \( Y \) of \( X \) such that \( \text{ind} (Y \setminus X) \leq 0 \).

In an attempt to generalize (*), de Groot introduced two notions, the small inductive compactness degree \( \text{cmp} \) and the compactness definiency \( \text{def} \) (we will recall the definitions in Section 2 and Section 3 respectively). It is known that the inequality \( \text{cmp} X \leq \text{def} X \) holds for every separable metrizable space \( X \). The well known conjecture of de Groot (see for example [4]) was that the two invariants coincide in the class of separable metrizable spaces. As a way either to disprove or to support the conjecture de Groot and Nishiura [4] posed the following:

Question 1.1 Let \( Z_n = [0,1]^{n+1} \setminus (0,1)^n \times \{0\} \). Is it true that \( \text{cmp} Z_n \geq n \) for \( n \geq 3 \)?

In the quoted article, de Groot and Nishiura proved that \( \text{def} Z_n = n \) for every \( n \geq 1 \), and they also stated that \( \text{cmp} Z_i = i \) for \( i = 1,2 \).

In [9], R. Pol constructed a space \( P \subset R^4 \) such that \( \text{cmp} P = 1 < \text{def} P = 2 \). The space \( P \) is a modification of an example given by Luxemburg [7] of a compactum with noncoinciding transfinite inductive dimensions. After that, some other counterexamples to the de Groot’s conjecture were constructed by Hart (cf. [1]), Kimura [6], Levin and Segal [8]). However, Question 1.1 remained open (see also [10, Question 418] and [1, Problem 3, page 71]).

One of our main results is the following.

Theorem 1.1 Let \( n \leq 2^m - 1 \) for some integer \( m \). Then \( \text{cmp} Z_n \leq m + 1 \). In particular \( \text{cmp} Z_n < \text{def} Z_n \) for \( n \geq 5 \).

This is the answer to Question 1.1 for \( n \geq 5 \). Our paper is based on a construction of examples of compacta with noncoinciding transfinite inductive dimensions given in [2]. Our terminology follows [5] and [1].
2 Finite sum theorem for $\mathcal{P}$-ind

In this part, topological spaces are assumed to be regular $T_1$ and all classes of topological spaces considered are assumed to be nonempty and to contain any space homeomorphic with a closed subspace of one of their members. The letter $\mathcal{P}$ is used to denote such classes.

Recall the definition of the small inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-ind. Let $X$ be a space.

(i) $\mathcal{P}$-ind $X=-1$ iff $X \in \mathcal{P}$;

(ii) $\mathcal{P}$-ind $X \leq n$ ($\geq 0$) if each point in $X$ has arbitrarily small neighbourhoods $V$ with $\mathcal{P}$-ind $\text{Bd } V \leq n-1$.

(iii) $\mathcal{P}$-ind $X=n$ if $\mathcal{P}$-ind $X \leq n$ and $\mathcal{P}$-ind $X > n-1$;

(iv) $\mathcal{P}$-ind $X=\infty$ if $\mathcal{P}$-ind $X > n$ for $n = -1, 0, 1, ...$

It is clear that if $\mathcal{P} = \{\emptyset\}$ then $\mathcal{P}$-ind $X = \text{ind } X$. If $\mathcal{P}$ is the class of compact spaces then $\mathcal{P}$-ind $X = \text{cmp } X$.

The following is a list of properties of $\mathcal{P}$-ind we shall use in the paper.

1. If $A$ is closed in $X$ then $\mathcal{P}$-ind $A \leq \mathcal{P}$-ind $X$.

2. If $\mathcal{P}$-ind $X \leq n \geq 0$ and $U$ is open in $X$ then $\mathcal{P}$-ind $U \leq n$.

3. If $X = O_1 \cup O_2$, where $O_i$ is open in $X, i = 1, 2$, and $\max \{\mathcal{P}$-ind $O_1, \mathcal{P}$-ind $O_2\} \leq n \geq 0$. Then $\mathcal{P}$-ind $X \leq n$.

4. $\mathcal{P}$-ind $X \leq n \geq 0$ iff for each point $p$ and for each closed set $G$ of $X$ with $p \notin G$ there is a partition $S$ between $p$ and $G$ such that $\mathcal{P}$-ind $S \leq n-1$.

The following statement is contained implicitly in the proofs of [2, Theorem 3.9] and [3, Theorem 2].

Lemma 2.1. Let $X$ be a normal space such that $X = X_1 \cup X_2$, where $X_i$ is closed in $X$, and $A, B$ be two closed disjoint subsets of $X$ such that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose a partition $C_1$ in $X_1$ between the sets $A \cap X_1$ and $B \cap X_1$ such that $X_1 \setminus C_1 = U_1 \cup V_1$, where $U_1, V_1$ are open in $X_1$ and disjoint, and $A \cap X_1 \subset U_1, B \cap X_1 \subset V_1$. Choose also a partition $C_2$ in $X_2$ between the the sets $A \cap X_2$ and $((C_1 \cup V_1) \cup B) \cap X_2$ such that $X_2 \setminus C_2 = U_2 \cup V_2$, where $U_2, V_2$ are open in $X_2$ and disjoint, and $A \cap X_2 \subset U_2, (C_1 \cup V_1) \cup B) \cap X_2 \subset V_2$. Then the set $C = X \setminus (((U_1 \setminus X_2) \cup U_2) \cup (V_1 \cup (V_2 \setminus X_1)))$ is a partition in $X$ between the sets $A$ and $B$ such that $C \subset C_1 \cup C_2 \cup (X_1 \cap X_2)$.

Moreover, if $X$ is a regular $T_1$-space then the same statement is valid for a pair of closed subsets of $X$, where one of the sets is a point.
The following theorem and corollary are generalizations of [3, Theorem 2] and [2, Corollary 3.10 (a)] respectively.

**Theorem 2.1** Let \( X \) be a space such that \( X = X_1 \cup X_2 \), where \( X_i \) is closed in \( X \) and \( \mathcal{P}\text{-}\text{ind}\ X_i \leq n \geq 0 \) for every \( i = 1,2 \). Then \( \mathcal{P}\text{-}\text{ind}\ X \leq n + 1 \).

Moreover, if the space \( X \) is normal then for any closed subsets \( A \) and \( B \) of \( X \) there exists a partition \( C \) between \( A \) and \( B \) such that \( \mathcal{P}\text{-}\text{ind}\ C \leq n \).

**Corollary 2.1** Let \( X \) be a space and \( q \) be an integer. If \( X = \bigcup_{k=1}^{n+1} X_k \), where each \( X_k \) is closed in \( X \), \( 0 \leq n \leq 2^m - 1 \) for some integer \( m \) and \( \max\{\mathcal{P}\text{-}\text{ind}\ X_k\} \leq q \geq 0 \) then \( \mathcal{P}\text{-}\text{ind}\ X \leq q + m \).

For every normal space \( X \) one assigns the large inductive compactness degree \( \text{Cmp} \) as follows (cf. [1]).

(i) For \( n = -1 \) or 0, \( \text{Cmp} \ X = n \) iff \( \text{cmp} \ X = n \).

(ii) \( \text{Cmp} \ X \leq n \geq 1 \) if each pair of disjoint closed subsets \( A \) and \( B \) of \( X \) there exists a partition \( C \) such that \( \text{Cmp} \ C \leq n - 1 \).

(iii) \( \text{Cmp} \ X = n \) if \( \text{Cmp} \ X \leq n \) and \( \text{Cmp} \ X > n - 1 \).

(iv) \( \text{Cmp} \ X = \infty \) if \( \text{Cmp} \ X > n \) for every natural number \( n \).

It is clear that the following properties of \( \text{Cmp} \) are valid.

1. If \( A \) is closed in \( X \) then \( \text{Cmp} \ A \leq \text{Cmp} \ X \).

2. If \( X \) is a sum of closed subsets \( X_i, i = 1,2 \), then \( \text{Cmp} \ X = \max\{\text{Cmp} \ X_1, \text{Cmp} \ X_2\} \).

**Corollary 2.2** Let \( X \) be a normal space such that \( X = X_1 \cup X_2 \), where \( X_i \) is closed in \( X \) and \( \text{Cmp} \ X_i \leq 0 \) for every \( i \). Then \( \text{Cmp} \ X \leq 1 \). Moreover, if \( \text{Cmp} \ (X_1 \cap X_2) = -1 \) then \( \text{Cmp} \ X = 0 \); if \( \text{Cmp} \ X_1 = -1 \) then \( \text{Cmp} \ X = \text{Cmp} \ X_2 \).

Now we are ready to prove the following theorem.

**Theorem 2.2** Let \( X \) be a normal space such that \( X = X_1 \cup X_2 \), where \( X_i \) is closed for \( i = 1,2 \). Then \( \text{Cmp} \ X \leq \max\{\text{Cmp} \ X_1, \text{Cmp} \ X_2\} + \text{Cmp} \ (X_1 \cap X_2) + 1 \leq \text{Cmp} \ X_1 + \text{Cmp} \ X_2 + 1 \).

**Proof.** Put \( \text{Cmp} \ (X_1 \cap X_2) = k \) and \( \max\{\text{Cmp} \ X_1, \text{Cmp} \ X_2\} = m \). Observe that \( k \leq m \). Let \( k = -1 \). First we will prove the theorem for any \( m \geq -1 \) (\( k = -1 \)). By Corollary 2.2 the statement is valid for \( m = -1 \) and \( m = 0 \). Assume that our theorem is valid for \( m < p \geq 1 \). Put \( m = p \). Consider two disjoint closed subsets \( A \) and \( B \) of \( X \). We can suppose that \( A \cap X_i \neq \emptyset \) and \( B \cap X_i \neq \emptyset, i = 1,2 \). Choose partitions \( C_i, i = 1,2 \), as we
did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p - 1$. Denote $Y_1 = C_1 \cup C_2$ (recall that $C_1$ and $C_2$ are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\text{Cmp } (Y_1 \cap Y_2) = -1$, $\text{Cmp } Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p - 1$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \leq p - 1$. By inductive assumption, $\text{Cmp } Y \leq \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \leq -1 + (p - 1) + 1 = p - 1$. By Lemma 2.1 there is a partition $C$ between $A$ and $B$ in $X$ such that $C \subset Y$. Hence, $\text{Cmp } X \leq p = k + m + 1$.

Assume that our theorem is valid for any pair $(k,m): k < q \geq 0$ and $k \leq m$.

Put $k = q$. Consider the case $m = k = 0$. Then $\text{Cmp } X_i \leq 0$ for every $i = 1,2$, and by Corollary 2.2, $\text{Cmp } X \leq 1 = k + m + 1$. Let $k = m = q \geq 1$.

Consider two disjoint closed subsets $A$ and $B$ of $X$. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1,2$. Choose partitions $C_i, i = 1,2$, as we did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq q - 1$. Denote $Y_1 = C_1 \cup C_2$ (and $C_2$ are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\text{Cmp } Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq q - 1$, $\text{Cmp } (Y_1 \cap Y_2) \leq \min\{q, q - 1\} = q - 1 < q$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \leq q$. By inductive assumption, $\text{Cmp } Y \leq \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \leq q + (q - 1) + 1 = 2q$.

By Lemma 2.1 there is a partition $C$ between $A$ and $B$ in $X$ such that $C \subset Y$. Hence, $\text{Cmp } X \leq 2q + 1 = k + m + 1$.

Assume that our theorem is valid for any $m: k \leq m < p \geq 1$ (k=q). Put $m = p$. Consider two disjoint closed subsets $A$ and $B$ of $X$. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1,2$. Choose partitions $C_i, i = 1,2$, as we did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p - 1$. Denote $Y_1 = C_1 \cup C_2$ (and $C_2$ are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that $\text{Cmp } Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \leq p - 1$, $\text{Cmp } (Y_1 \cap Y_2) \leq \min\{q, p - 1\} = q$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \leq p - 1$. By inductive assumption, $\text{Cmp } Y \leq \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \leq q + (p - 1) + 1 = q + p$.

By Lemma 2.1 there is a partition $C$ between $A$ and $B$ in $X$ such that $C \subset Y$. Hence, $\text{Cmp } X \leq q + p + 1 = k + m + 1$.

**Corollary 2.3** Let $X$ be a normal space with $\text{Cmp } X = n \geq 1$. Then

(a) $X$ cannot be represented as a union of $n$ many closed subsets $R_1, R_2, \ldots, R_n$ with $\text{Cmp } R_i \leq 0$ for each $i$.

Furthermore, we suppose now that $X = \cup_{i=1}^{n+1} Z_i$, where each $Z_i$ is closed and $\text{Cmp } Z_i \leq 0$ for every $i = 1, \ldots, n + 1$, then we have

(b) $\text{Cmp } (Z_1 \cup \ldots \cup Z_{k+1}) = k$ for any $k$ with $0 \leq k \leq n$;

(c) $\text{Cmp } ((Z_1 \cup \ldots \cup Z_{k+1}) \cap (Z_{i+2} \cup \ldots \cup Z_{i+j+2})) = \min \{i,j\}$ for any nonnegative integers $i,j$ such that $i + j + 1 \leq n$.

**Remark.** The estimations from Corollary 2.2 and Theorem 2.2 can not be improved (see Corollary 3.3).
3 Spaces with cmp $\neq \text{def}$ (cmp $\neq \text{Cmp}$).

The deficiency $\text{def}$ is defined in the following way: For a separable metrizable space $X$,

$$\text{def } X = \min \{\text{ind } (Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$

In this section, the concept of $B$-special decomposition introduced in [2] essentially works. A decomposition $X = F \cup \bigcup_{i=1}^{\infty} E_i$ of a metric space $X$ into disjoint sets is called $B$-special if $E_i$ is clopen in $X$ and $\lim_{i \to \infty} \delta(E_i) = 0$, where $\delta(A)$ is the diameter of $A$.

The following proposition is easily obtained by use of [2, Lemma 2.3].

**Proposition 3.1** Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be a $B$-special decomposition of a metric space $X$ and $n \geq 0$ be an integer. If $\max\{\text{P-ind } F, \text{P-ind } E_i\} \leq n$ then $\text{P-ind } X \leq n$.

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all $i$ and $\lim_{i \to \infty} x_i = 0$. Put $C^n = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [x_{2i}, x_{2i-1}]) \subset I^{n+1}$.

**Theorem 3.1** (a) There are closed subsets $X_1, X_2, \ldots, X_{n+1}$ of $C^n$ such that $C^n = \bigcup_{k=1}^{n+1} X_k$ and $\text{cmp } X_k = 0$ for each $k = 1, 2, \ldots, n+1$.

(b) The equalities $\text{def } C^n = \text{Cmp } C^n = n (= \text{Comp } C^n)$ hold (see [1] for the definition of Comp).

(c) Let $m$ be an integer such that $0 \leq n \leq 2^m - 1$. Then we have $\text{cmp } C^n \leq m$. In particular $\text{cmp } C^n < \text{Cmp } C^n = \text{def } C^n$ for $n \geq 3$.

**Proof.** (a) For every $i$ choose finite systems $B^i_k, k = 1, \ldots, n+1$, consisting of disjoint compact subsets of $I^n$ with diameter $< \frac{1}{i}$ such that $I^n = \bigcup_{k=1}^{n+1} (\text{Bd } B^i_k)$. We put $X_k = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (B^i_k \times [x_{2i}, x_{2i-1}])$ for every $k = 1, \ldots, n+1$. Observe that the space $X_k$ admits a $B$-special decomposition into compact subsets and, by Proposition 3.1, $\text{cmp } X_k = 0$ for $k = 1, \ldots, n+1$.

(b) It is enough to prove that $\text{Comp } C^n \geq n$ i.e. there exist $n$ pairs $(F_1, G_1), \ldots, (F_n, G_n)$ of disjoint compact subsets of $C^n$ such that for any partitions $S_i$ between $F_i$ and $G_i$ in $X, i = 1, \ldots, n$, the intersection $S_1 \cap \ldots \cap S_n$ is not compact. (Recall that for every separable metrizable space $W$ we have $\text{Comp } W \leq \text{Cmp } W \leq \text{def } W$ (cf. [1]) and evidently $\text{def } C^n \leq n$.) For example such pairs are $((\{0\} \times I^n) \cap C^n, ((\{1\} \times I^n) \cap C^n), \ldots, ((I^{n-1} \times \{0\}) \times [0, 1]) \cap C^n, (I^{n-1} \times \{1\} \times [0, 1]) \cap C^n)$. Moreover, for any partition $C$ between $((\{0\} \times I^n) \cap C^n$ and $(\{1\} \times I^n) \cap C^n$ in $C^n$, $\text{Comp } C \geq n-1$.

(c) One can show (c) by applying Corollary 2.1 for cmp and the statement (a).

Now we are ready to show Theorem 1.1.

**Proof of Theorem 1.1.** Decompose the space $Z_n, n \geq 3$, into the union of two closed subsets $Z^1_n$ and $Z^2_n$ (each of them is homeomorphic to $C^n$), where $Z^1_n = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [1/(2i+1), 1/(2i)])$, $Z^2_n = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty} (I^n \times [1/(2i), 1/(2i-1)])$. 
Let $m$ be the integer such that $0 \leq n \leq 2^m - 1$. It follows from Theorem 3.1 (c) that $\text{cmp} Z^i_n \leq m$ for $i = 1, 2$. Thus, by Corollary 2.1, we have $\text{cmp} Z_n \leq m + 1$.

**Corollary 3.1** (a) For the space $C^2$ we have $\text{cmp} C^2 = \text{cmp} (C^2 \times [0, 1]) = 2$.

(b) $\text{cmp} C^3 = 2$.

The following question is discussed in [1, Problem 6, page 71].

**Question 3.1** For any $k$ and $m$ with $0 < k < m$, does there exist a separable metrizable space $X$ such that $\text{cmp} X = k$ and $\text{def} X = m$?

We shall partially answer the question as follows.

**Corollary 3.2** Let $m$ be an integer and $l(m) = \lfloor \log_2(m) \rfloor + 1$. Then for every $k$ with $m \geq k \geq l(m)$ there exists a separable metrizable space $X$ such that $\text{cmp} X = k$ and $\text{def} X = m$.

Let $C^n$ be the space defined above and $X_1, X_2, \ldots, X_{n+1}$ be closed subsets of $C^n$ described in Theorem 3.1. It follows from Theorem 3.1 (a) and Corollary 2.3 that $\text{Cmp} (X_1 \cup \ldots \cup X_{k+1}) = k$ for each $k$ with $0 \leq k \leq n$. However, we do not know the value of the deficiency of $X_1 \cup \ldots \cup X_{k+1}$. So we can ask the following.

**Question 3.2** Is it true that $\text{def} (X_1 \cup \ldots \cup X_{k+1}) = k$ for $1 \leq k < n$?

The question might be interesting when we consider a problem posed by Aarts and Nishiura [1, Problem 6, page 71]: Exhibit a separable metrizable space $X$ such that $\text{cmp} X < \text{Cmp} X < \text{def} X$. If the Question 3.1 would be answered negatively for example for the case of $n = 4$ and $k = 3$, then we have $\text{def} (X_1 \cup X_2 \cup X_3 \cup X_4) = 4$. We put $Y = X_1 \cup X_2 \cup X_3 \cup X_4$. Then, by the argument above, we have $\text{Cmp} Y = 3$. On the other hand, by Theorem 3.1 (a) and Corollary 2.1, it follows that $\text{cmp} Y \leq 2$. Hence $\text{cmp} Y < \text{Cmp} Y < \text{def} Y$. Even if the Question 3.1 would be answered positively, then one gets an interesting counterpart of Corollary 3.3 (see below) for $\text{def}$.

Now we will obtain a complement to Theorem 2.2 showing the exactness of the theorem's estimations.

**Corollary 3.3** For any integer $n \geq 1$ there exists a compact space $X_n (= C^n)$ with $\text{Cmp} X_n = n$ such that for any nonnegative integers $p, q$ with $p + q = n - 1$ there exist its closed subsets $X_n^{(p)}$ and $X_n^{(q)}$ such that $X_n = X_n^{(p)} \cup X_n^{(q)}$, $\text{Cmp} X_n^{(p)} = p$, $\text{Cmp} X_n^{(q)} = q$ and $\text{Cmp} (X_n^{(p)} \cap X_n^{(q)}) = \min \{p, q\}$. 
参考文献


