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HOMOTOPY TYPES OF DIFFEOMORPHISM GROUPS
OF NONCOMPACT 2-MANIFOLDS

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1. INTRODUCTION

This is a report on the study of topological properties of the diffeomorphism groups of noncompact smooth 2-manifolds endowed with the compact-open $C^\infty$-topology [18].

When $M$ is a compact smooth 2-manifold, the diffeomorphism group $\mathcal{D}(M)$ with the compact-open $C^\infty$-topology is a smooth Fréchet manifold [6, Section I.4], and the homotopy type of the identity component $\mathcal{D}(M)_0$ has been classified by S. Smale [15], C. J. Earle and J. Eel [4], et. al. In the $C^0$-category, for any compact 2-manifold $M$, the homeomorphism group $\mathcal{H}(M)$ with the compact-open topology is a topological Fréchet manifold [3, 11, 19], and the homotopy type of the identity component $\mathcal{H}(M)_0$ has been classified by M. E. Hamstrom [7].

Recently we have shown that $\mathcal{H}(M)_0$ is a topological Fréchet-manifold even if $M$ is a noncompact connected 2-manifold, and have classified its homotopy type [17]. The argument in [17] is based on the following ingredients: (i) the ANR-property and the contractibility of $\mathcal{H}(M)_0$ for compact $M$, (ii) the bundle theorem connecting the homeomorphism group $\mathcal{H}(M)_0$ and the embedding spaces of submanifolds into $M$ [16, Corollary 1.1], and (iii) a result on the relative isotopies of 2-manifolds [17, Theorem 3.1]. The same strategy based on the $C^\infty$-versions of these results implies a corresponding conclusion for the diffeomorphism groups of noncompact smooth 2-manifolds.

Suppose $M$ is a smooth 2-manifold and $X$ is a closed subset of $M$. We denote by $\mathcal{D}_X(M)$ the group of $C^\infty$-diffeomorphisms $h$ of $M$ onto itself with $h|_X = id_X$, endowed with the compact-open $C^\infty$-topology [9, Ch.2, Section 1], and by $\mathcal{D}_X(M)_0$ the identity connected component of $\mathcal{D}_X(M)$.

The following is our main result:

**Theorem 1.1.** Suppose $M$ is a noncompact connected smooth 2-manifold without boundary.
(1) $\mathcal{D}(M)_0$ is a topological $\ell_2$-manifold.
(2) (i) $\mathcal{D}(M)_0 \cong S^1$ if $M$ is a plane, an open Möbius band or an open annulus.
(ii) $\mathcal{D}(M)_0 \cong \ast$ in all other cases.

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Any separable infinite-dimensional Fréchet space is homeomorphic to the Hilbert space \( \ell_2 \equiv \{(x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty\} \). A topological \( \ell_2 \)-manifold is a separable metrizable space which is locally homeomorphic to \( \ell_2 \). Topological types of \( \ell_2 \)-manifolds are classified by their homotopy types. Theorem 1.1 implies the following conclusion:

**Corollary 1.1.**

(i) \( \mathcal{D}(M)_0 \cong S^1 \times \ell_2 \) if \( M = a \) plane, an open Möbius band or an open annulus.

(ii) \( \mathcal{D}(M)_0 \cong \ell_2 \) in all other cases.

For the subgroup of diffeomorphisms with compact supports, we have the following results:

Let \( \mathcal{D}(M)_0^c \) denote the subgroup of \( \mathcal{D}(M)_0 \) consisting of \( h \in \mathcal{D}(M) \) which admits a \( C^\infty \)-isotopy with a compact support, \( h_t : M \to M \) such that \( h_0 = \text{id}_M \) and \( h_1 = h \).

We say that a subspace \( A \) of a space \( X \) has the homotopy negligible (h.n.) complement in \( X \) if there exists a homotopy \( \varphi_t : X \to X \) such that \( \varphi_0 = \text{id}_X \) and \( \varphi_t(X) \subset A \) \( (0 < t \leq 1) \). In this case, the inclusion \( A \subset X \) is a homotopy equivalence, and \( X \) is an ANR iff \( A \) is an ANR.

**Theorem 1.2.** Suppose \( M \) is a noncompact connected smooth 2-manifold without boundary. Then \( \mathcal{D}(M)_0^c \) has the h.n. complement in \( \mathcal{D}(M)_0 \)

**Corollary 1.2.**

(1) \( \mathcal{D}(M)_0^c \) is an ANR.

(2) The inclusion \( \mathcal{D}(M)_0^c \subset \mathcal{D}(M)_0 \) is a homotopy equivalence.

Section 2 contains fundamental facts on diffeomorphism groups of 2-manifolds and \( \ell_2 \)-manifolds. Section 3 contains a sketch of proofs of Theorems 1.1 and 1.2.

## 2. Fundamental Properties of Diffeomorphism Groups

In this preliminary section we list fundamental facts on diffeomorphism groups of 2-manifolds (general properties, bundle theorem, homotopy type, relative isotopies, etc) and basic facts on ANR's and \( \ell_2 \)-manifolds. Throughout the paper all spaces are separable and metrizable and maps are continuous.

### 2.1. General property of diffeomorphism groups.

Suppose \( M \) is a smooth \( n \)-manifold possibly with boundary and \( X \) is a closed subset of \( M \).

**Lemma 2.1.** (c.f. [9, Ch 2., Section 4], etc)

\( \mathcal{D}_X(M) \) is a topological group, which is separable, completely metrizable, infinite-dimensional and not locally compact.
When $N$ is a smooth submanifold of $M$, the symbol $\mathcal{E}_X(N, M)$ denotes the space of $C^\infty$-embeddings $f : N \hookrightarrow M$ with $f|_X = id_X$ with the compact-open $C^\infty$-topology, and $\mathcal{E}_X(N, M)_0$ denotes the connected component of the inclusion $i_N : N \subset M$ in $\mathcal{E}_X(N, M)$.

**Lemma 2.2.** (i) Suppose $M$ is a smooth manifold without boundary, $N$ is a compact smooth submanifold of $M$ and $X$ is a closed subset of $N$. Then $\mathcal{E}_X(N, M)$ is a Fréchet manifold. (ii) Suppose $M$ is a compact smooth $n$-manifold and $X$ is a closed subset of $M$ with $\partial M \subset X$ or $\partial M \cap X = \emptyset$. Then $D_X(M)$ is a Fréchet manifold.

In Lemma 2.2 $\mathcal{E}_X(N, M)_0$ and $D_X(M)_0$ are path-connected. Thus any $h \in D_X(M)_0$ can be joined with $id_M$ by a path $h_t \ (t \in [0, 1])$ in $D_X(M)_0$.

**2.2. Bundle theorems.**

The bundle theorem asserts that the natural restriction maps from diffeomorphism groups to embedding spaces are principal bundles [2, 12]. This has been used to study the homotopy types of diffeomorphism groups. This theorem also plays an essential role in our argument.

Suppose $M$ is a smooth $m$-manifold without boundary, $N$ is a compact smooth $n$-submanifold of $M$ and $X$ is a closed subset of $N$.

**Case 1:** $n < m$ [2, 12]

Let $U$ be any open neighborhood of $N$ in $M$.

**Theorem 2.1.** For any $f \in \mathcal{E}_X(N, U)$ there exist a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{E}_X(N, U)$ and a map $\varphi : \mathcal{U} \rightarrow D_{X|(M \setminus U)}(M)_0$ such that $\varphi(g)f = g \ (g \in \mathcal{U})$ and $\varphi(f) = id_M$.

**Corollary 2.1.** The restriction map $\pi : D_{X|(M \setminus U)}(M)_0 \rightarrow \mathcal{E}_X(N, U)_0, \pi(h) = h|_N$, is a principal bundle with fiber $D_{X|(M \setminus U)}(M)_0 \cap D_N(M)$.

**Case 2:** $n = m$

In this case we have a weaker conclusion: Suppose $N'$ is a compact smooth $n$-submanifold of $M$ obtained from $N$ by attaching a closed collar $\partial N \times [0, 1]$ to $\partial N$. Let $U$ be any open neighborhood of $N'$ in $M$. We can apply Theorem 2.1 to $\partial N'$ to obtain the following result:

**Theorem 2.2.** For any $f \in \mathcal{E}_X(N', U)$ there exist a neighborhood $\mathcal{U}'$ of $f$ in $\mathcal{E}_X(N', U)$ and a map $\varphi : \mathcal{U}' \rightarrow D_{X|(M \setminus U)}(M)_0$ such that $\varphi(g)f|_N = g|_N \ (g \in \mathcal{U}')$ and $\varphi(f) = id_M$.

For the sake of simplicity, we set $D_0 = D_{X|(M \setminus U)}(M)_0, \mathcal{E}_0 = \mathcal{E}_X(N, U)_0, \mathcal{E}_0' = \mathcal{E}_X(N', U)_0$.

Consider the restriction map $p : \mathcal{E}_0' \rightarrow \mathcal{E}_0, p(f) = f|_N$ and $\pi : D_0 \rightarrow \mathcal{E}_0, \pi(h) = h|_N$. We have the pullback diagram:
\[
p^*(D_0) \xrightarrow{p_*} D_0 \\
\pi_* \downarrow \quad \downarrow \pi \\
E'_0 \xrightarrow{p} E_0,
\]

where \( p^*D_0 = \{(f, h) \in E'_0 \times D_0 \mid f|_N = h|_N \} \), \( p_*(f, h) = h \) and \( \pi_*(f, h) = f \). The map \( p_* \) admits a natural right inverse \( q : D_0 \to p^*D_0 \), \( q(h) = (h|_N, h) \). The group \( D_0 \cap D_N(M) \) acts on \( p^*D_0 \) by \( (f, h)g = (f, hg) \) \((g \in D_0 \cap D_N(M))\).

**Corollary 2.2.**

1. \( \pi_* : p^*(D_0) \to E'_0 \) is a principal bundle with fiber \( D_0 \cap D_N(M) \).
2. \( p_* : p^*(D_0) \to D_0 \) is a homotopy equivalence with the homotopy inverse \( q : D_0 \to p^*(D_0) \).
3. \( p : E'_0 \to E_0 \) is a homotopy equivalence if \( X \subset \text{int} N \).

The statements (2) and (3) exhibit a close relation between the restriction map \( \pi \) and the pullback \( \pi_* \).

2.3. Diffeomorphism groups of 2-manifolds.

Next we recall fundamental facts on diffeomorphism groups of compact 2-manifolds. The following theorem shows that \( D_X(M)_0 \simeq \ast \) except a few cases. The symbols \( S^1 \), \( S^2 \), \( T \), \( P \), \( K \), \( D \), \( A \) and \( M \) denote the 1-sphere, 2-sphere, torus, projective plane, Klein bottle, disk, annulus and Möbius band respectively.

**Theorem 2.3.** ([4, 15] etc.) Suppose \( M \) is a compact connected smooth 2-manifold. Then the homotopy type of \( D(M)_0 \) is classified as follows:

<table>
<thead>
<tr>
<th>( M )</th>
<th>( D(M)_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S^2, P )</td>
<td>( SO(3) )</td>
</tr>
<tr>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( K, D, A, M )</td>
<td>( S^1 )</td>
</tr>
<tr>
<td><strong>all other cases</strong></td>
<td>( \ast )</td>
</tr>
</tbody>
</table>

- \( D_0(D) \simeq \ast \), \( D_0(M) \simeq \ast \).
- If \( X \) is a disjoint union of a compact smooth 2-submanifold and finitely many smooth circles and points in \( M \) and \( \partial M \subset X \), then \( D_X(M)_0 \simeq \ast \).

For 2-manifolds there is no difference among the conditions: homotopic, \( C^0 \)-isotopic, \( C^\infty \)-isotopic and joinable by a path in the diffeomorphism group. By [4] and a \( C^\infty \)-analogue of [5] we have

**Proposition 2.1.** Suppose \( M \) is a compact smooth 2-manifold.

1. Suppose \( N \) is a closed collar of \( \partial M \). If \( h \in D_N(M) \) is homotopic to \( \text{id}_M \) rel \( N \), then \( h \) is \( C^\infty \)-isotopic to \( \text{id}_M \) rel \( N \).

2. Suppose \( N \) is a compact smooth 2-submanifold of \( M \) with \( \partial M \subset N \). For \( h \in D_N(M) \), the
following conditions are equivalent:
(a) \( h \) is \( C^0 \)-isotopic to \( \text{id}_M \) rel \( N \).
(b) \( h \) is \( C^\infty \)-isotopic to \( \text{id}_M \) rel \( N \).
(c) \( h \in \mathcal{D}_N(M)_0 \).

In Corollaries 2.1 and 2.2 we have a principal bundle with fiber \( \mathcal{G} = \mathcal{D}_X(M)_0 \cap \mathcal{D}_N(M) \). The next theorem gives us a sufficient condition that \( \mathcal{G} = \mathcal{D}_N(M)_0 \). The symbol \( \# X \) denotes the cardinal of a set \( X \).

**Theorem 2.4.** Suppose \( M \) is a compact connected smooth 2-manifold, \( N \) is a compact smooth 2-submanifold of \( M \) with \( \partial M \subset N \), \( X \) is a subset of \( N \). Suppose \( (M, N, X) \) satisfies the following conditions:

(i) \( M \neq \mathrm{T}, \mathrm{P}, \mathrm{K} \) or \( X \neq \emptyset \).

(ii) (a) if \( H \) is a disk component of \( N \), then \( \# (H \cap X) \geq 2 \),
    (b) if \( H \) is an annulus or Möbius band component of \( N \), then \( H \cap X \neq \emptyset \),

(iii) (a) if \( L \) is a disk component of \( \text{cl}(M \setminus N) \), then \( \# (L \cap X) \geq 2 \),
    (b) if \( L \) is a Möbius band component of \( \text{cl}(M \setminus N) \), then \( L \cap X \neq \emptyset \).

Then we have:

(1) If \( h \in \mathcal{D}_N(M) \) is \( C^0 \)-isotopic to \( \text{id}_M \) rel \( X \), then \( h \) is \( C^\infty \)-isotopic to \( \text{id}_M \) rel \( N \).
(2) \( \mathcal{D}(M)_0 \cap \mathcal{D}_N(M) = \mathcal{D}_N(M)_0 \).

Theorem 2.4 follows from [17, Theorem 3.1] and Proposition 2.1.

2.4. Basic properties of ANR's and \( \ell^2 \)-manifolds.

The ANR-property of diffeomorphism groups and embedding spaces is also essential in our argument. Here we recall basic properties of ANR's [8, 10, 13] and a topological characterization theorem of \( \ell^2 \)-manifolds.

A metrizable space \( X \) is called an ANR (absolute neighborhood retract) for metric spaces if any map \( f : B \to X \) from a closed subset \( B \) of a metrizable space \( Y \) admits an extension to a neighborhood \( U \) of \( B \) in \( Y \). If we can always take \( U = Y \), then \( X \) is called an AR. It is known that \( X \) is an AR (an ANR) iff it is a retract of (an open subset of) a normed space. Any ANR has a homotopy type of CW-complex. An AR is exactly a contractible ANR.

We apply the following criterion of ANR's:

**Lemma 2.3.** (1) A space \( X \) is an ANR iff every point of \( X \) has an ANR neighborhood in \( X \).
(2) If \( X = \cup_{i=1}^\infty U_i \), \( U_i \) is open in \( X \) and \( U_i \subset U_{i+1} \) and if each \( U_i \) is an AR, then \( X \) is also an
(3) In a fiber bundle, the total space is an ANR iff both the base space and the fiber are ANR's.

(4) A metric space $X$ is an ANR iff for any $\varepsilon > 0$ there is an ANR $Y$ and maps $f : X \to Y$ and $g : Y \to X$ such that $gf$ is $\varepsilon$-homotopic to $\text{id}_X$.

Since any Fréchet space is an AR, every Fréchet manifold is an ANR.

Finally we recall a characterization of $\ell_2$-manifold topological groups [3, 19].

**Theorem 2.5.** A topological group is an $\ell_2$-manifold iff it is a separable, non locally compact, completely metrizable ANR.

The diffeomorphism group $D(M)_0$ satisfies all conditions except the ANR property (Lemma 2.1). Thus the proof of Theorem 1.1 (1) reduces to the verification of ANR property of $D(M)_0$. The latter follows from the ANR property of the diffeomorphism groups and embedding spaces of compact 2-manifolds (Lemma 2.2).

3. PROOF OF MAIN THEOREMS

In this section we give a sketch of proofs of Theorems 1.1 and 1.2 in the case where $M \neq$ a plane, an open Möbius band, an open annulus. Below we assume that $M$ is a noncompact connected smooth 2-manifold without boundary and that $M \neq$ a plane, an open Möbius band, an open annulus.

We can write as $M = \bigcup_{i=0}^{\infty} M_i$, where $M_0 = \emptyset$ and for each $i \geq 1$

(a) $M_i$ is a nonempty compact connected smooth 2-submanifold of $M$ and $M_{i-1} \subset \text{int} M_i$, 
(b) for each component $L$ of $\partial (M \setminus M_i)$, $L$ is noncompact and $L \cap M_{i+1}$ is connected.

Note that $M$ is a plane (an open Möbius band, an open annulus) iff infinitely many $M_i$'s are disks (Möbius bands, annuli respectively). Since $M \neq$ a plane, an open Möbius band, an open annulus, passing to a subsequence, we may assume that

(c) $M_i \neq$ a disk, an annulus, a Möbius band.

For each $i \geq 1$ let $U_i = \text{int} M_i$, and choose a small closed collar $E_i$ of $\partial M_i$ in $U_{i+1} \setminus U_i$, and set $M_i' = M_i \cup E_i \subset U_{i+1}$.

3.1. Proof of Theorem 1.1.

[1] For each $j > i > k \geq 0$, we have the following pullback diagram:
The restriction maps, 
\( \pi_{k,j} \equiv D_{M_{k} \cup (M \setminus U_{j})}(M)_{0} \cap D_{M} \).

Lemma 3.1. (1) \((\pi_{k,j})_{*}\) is a principal bundle with fiber \( \mathcal{G}_{k,j} \).

(2) \( \mathcal{G}_{k,j} = \mathcal{G}_{k} \) is an AR.

(3) \((\pi_{k,j})_{*}\) is a trivial bundle.

(4) \( E_{M_{k}'}(M_{j}', U_{j})_{0} \) is an AR.

In (2) we apply Theorem 2.4 to deduce \( \mathcal{G}_{k,j} \equiv D_{M_{k} \cup (M \setminus U_{j})}(M)_{0} \cap D_{M} \).

Lemma 3.2. (1) \((\pi_{k})_{*}\) is a principal bundle with fiber \( \mathcal{G}_{k} \).

(2) \( E_{M_{k}'}(M_{j}', M)_{0} \) is an AR.

(3) \((\pi_{k})_{*}\) is a trivial bundle.

(4) \( \mathcal{G}_{k} = D_{M_{k}}(M)_{0} \) and \( D_{M_{k}}(M)_{0} \) strongly deformation retracts onto \( D_{M_{k}}(M)_{0} \).

The assertion (2) follows from Lemma 2.3 (2), Lemma 3.1 (4) and the fact that \( E_{M_{k}'}(M_{j}', M)_{0} = \bigcup_{j>i} E_{M_{k}'}(M_{j}', U_{j})_{0} \).

Proof of Theorem 1.1.

(A) \( D(M)_{0} \simeq * \):

\( D_{M_{k}}(M)_{0} \) strongly deformation retracts onto \( D_{M_{k+1}}(M)_{0} \) for each \( i \geq 0 \) (Lemma 3.2 (4)).

Since \( \text{diam} D_{M_{k}}(M)_{0} \to 0 \) \((i \to \infty)\), it follows that \( D(M)_{0} \) strongly deformation retracts onto \( \{id_{M}\} \).

(B) \( D(M)_{0} \) is an \( \ell_{2} \)-manifold:

By Theorem 2.5 and Lemma 2.1 it remains to show that \( D(M)_{0} \) is an ANR. We apply Lemma 2.3 (4): For each \( i \geq 0 \), we have the following pullback diagram:

\[
(p_{i})^{*}(D(M)_{0}) \xrightarrow{(p_{i})_{*}} D(M)_{0} \xrightarrow{\pi_{i}} \mathcal{E}(M'_{i}, M)_{0} \xrightarrow{p_{i}} \mathcal{E}(M_{i}, M)_{0},
\]

\( \pi_{i}, p_{i} : \) the restriction maps, 
\( q_{i} : D(M)_{0} \to (p_{i})^{*}(D(M)_{0}) \)
\( q_{i}(h) = (h|_{M'_{i}}, h) \).
Since \((\pi_{i})_{*}\) is a trivial principal bundle with the contractible fiber \(\mathcal{D}_{M}(M)_{0}\) (Lemma 3.2 (3),(4), (A)), it follows that \((\pi_{i})_{*}\) admits a section \(s_{i}\) and \(s_{i}(\pi_{i})_{*}\) is \((\pi_{i})_{*}\)-fiber preserving homotopic to \(id\). Consider the two maps

\[
\varphi = (\pi_{i})_{*}q_{i} : \mathcal{D}(M)_{0} \to \mathcal{E}(M, M)_{0} \quad \text{and} \quad \psi = (p_{i})_{*}s_{i} : \mathcal{E}(M, M)_{0} \to \mathcal{D}(M)_{0}.
\]

Then \(\mathcal{E}(M, M)_{0}\) is an ANR (Lemma 2.2 (i)) and \(\psi \circ \varphi : \mathcal{D}(M)_{0} \to \mathcal{D}(M)_{0}\) is \(\pi_{i}\)-fiber preserving homotopic to \(id\). Since \(\text{diam} (\text{fibers of } \pi_{i}) \to 0 (i \to \infty)\), Lemma 2.3 (4) implies that \(\mathcal{D}(M)_{0}\) is an ANR. \(\square\)

3.2. Proof of Theorem 1.2.

We use the following notations:
\\
\[D_{j} = \mathcal{D}_{M \backslash U_{j}}(M)_{0}, \quad U_{i,j} = \mathcal{E}(M, \cdot, U_{j})_{0}, \quad U_{i,j'} = \mathcal{E}(M', U_{j})_{0} \quad (j > i \geq 1).\]
\\
We have the pullback diagram:

\[
\begin{array}{ccc}
(p_{i,j})^{*}D_{j} & \xrightarrow{(p_{i,j})_{*}} & D_{j} \\
\downarrow & & \downarrow \pi_{i,j} \\
U_{i,j'} & \xrightarrow{p_{i,j}} & U_{i,j},
\end{array}
\]

\[\pi'_{i,j} : \mathcal{D}(M)_{0} \to \mathcal{E}(M', M)_{0}, \quad \pi_{i,j}, \pi_{i,j'} : \text{the restriction maps}.\]

Lemma 3.3. (i) \((\pi_{i,j})_{*}\) is a trivial bundle with AR fiber.

(ii) \(\pi_{i,j}\) has the following lifting property:

\[(*) \text{ If } Y \text{ is a metric space, } B \text{ is a closed subset of } Y \text{ and } \varphi : Y \to U_{i,j'} \text{ and } \varphi_{0} : B \to D_{j} \text{ are maps with } p_{i,j}\varphi|_{B} = \pi_{i,j}\varphi_{0}, \text{ then there exists a map } \Phi : Y \to D_{j} \text{ such that } \pi_{i,j}\Phi = p_{i,j}\varphi \text{ and } \Phi|_{B} = \varphi_{0}.\]

For each \(j > i \geq 1\), we regard as \(U_{i,j'} \subset \mathcal{E}(M', M)_{0}\) and set \(V_{i,j'} = (\pi'_{i})^{-1}(U_{i,j'}) \subset \mathcal{D}(M)_{0}\). For each \(i \geq 1\) we have:

(i) \(\mathcal{E}(M, M)_{0} = \cup_{j>i} \text{cl}U_{i,j'} \quad (U_{i,j'} \text{ is open in } \mathcal{E}(M, M)_{0}, \text{ cl}U_{i,j'} \subset U_{i,j'+1'})\)

(ii) \(\mathcal{D}(M)_{0} = \cup_{j>i} \text{cl}V_{i,j'} \quad (V_{i,j'} \text{ is open in } \mathcal{D}(M)_{0}, \text{ cl}V_{i,j'} \subset V_{i,j'+1'}, \text{ cl}V_{i,j'+1'} \subset V_{i,j'} \quad (j > i + 1))\)

(iii) \(\mathcal{D}(M)_{0}^{c} = \cup_{j>i} D_{j} \quad (D_{j} \subset D_{j+1})\)

Proof of Theorem 1.2.

We construct a homotopy

\[F : \mathcal{D}(M)_{0} \times [1, \infty] \to \mathcal{D}(M)_{0} \quad \text{such that } F_{\infty} = id \text{ and } F_{t}(\mathcal{D}(M)_{0}) \subset \mathcal{D}(M)_{0}^{t} \quad (1 \leq t < \infty).\]

(1) \(F_{t} \quad (i \geq 1): \) Using Lemma 3.3(ii), inductively we can construct a map \(s_{j}^{i} : \text{cl}U_{i,j'} \to D_{j+1}\) such that \(s_{j}^{i}(f)|_{M_{i}} = f|_{M_{i}} \quad (f \in \text{cl}U_{i,j'})\) and \(s_{j+1}^{i}|_{\text{cl}U_{i,j'}} = s_{j}^{i} \quad (j > i)\). Define a map
$s^i : E(M'_i, M) \rightarrow D(M)^0_0$ by $s^i|_{\mathcal{U}_j} = s^i_j$, and set $F_i = s^i \pi_i$. We have $F_i(\partial \mathcal{V}_{i+1}^j) \subset D_{j+1}$ and $F_i(h)|_{M'_i} = h|_{M'_i}$.

(2) $F_i$ ($i \leq t \leq i + 1$): Inductively we can construct a sequence of homotopies $G^j : \partial \mathcal{V}_{i+1,j} \times [i, i + 1] \rightarrow D_{j+1}$ ($j > i + 1$) such that $G^i_1 = F_i$, $G^i_{i+1} = F_{i+1}$, $G^{j+1}_i|_{\partial \mathcal{V}_{i+1,j} \times [i, i+1]} = G^j$ and $G^j_i(h)|_{M'_i} = h|_{M'_i}$. If $G^j$ is given, then $G^{j+1}$ is obtained by applying Lemma 3.3(ii) to the diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi_0} & D_{j+2} \\
\cap & \downarrow & \varphi(h, t) = h|_{M'_i} \\
Y & \xrightarrow{\varphi} & \mathcal{U}_{i+2} \\
\end{array}
\]

$$(Y, B) = (\partial \mathcal{V}_{i+1,j+1} \times [i, i + 1], (\partial \mathcal{V}_{i+1,j} \times [i, i+1]) \cup (\partial \mathcal{V}_{i+1,j+1} \times \{i, i + 1\})).$$

Define $F : D(M)_0 \times [i, i + 1] \rightarrow D(M)^0_0$ by $F = G^j$ on $\partial \mathcal{V}_{i+1,j} \times [i, i + 1]$.

(3) $F_{\infty}$: Since $F_i(h)|_{M'_i} = h|_{M'_i}$ for $t \geq i$, we can continuously extend $F$ by $F_{\infty} = id$. This completes the proof. 

\[\square\]

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