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HOMOTOPY TYPES OF DIFFEOMORPHISM GROUPS
OF NONCOMPACT 2-MANIFOLDS

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1. Introduction

This is a report on the study of topological properties of the diffeomorphism groups of noncompact smooth 2-manifolds endowed with the compact-open $C^\infty$-topology [18].

When $M$ is a compact smooth 2-manifold, the diffeomorphism group $\mathcal{D}(M)$ with the compact-open $C^\infty$-topology is a smooth Fréchet manifold [6, Section I.4], and the homotopy type of the identity component $\mathcal{D}(M)_0$ has been classified by S. Smale [15], C. J. Earle and J. Eel [4], et al. In the $C^0$-category, for any compact 2-manifold $M$, the homeomorphism group $\mathcal{H}(M)$ with the compact-open topology is a topological Fréchet manifold [3, 11, 19], and the homotopy type of the identity component $\mathcal{H}(M)_0$ has been classified by M. E. Hamstrom [7].

Recently we have shown that $\mathcal{H}(M)_0$ is a topological Fréchet-manifold even if $M$ is a noncompact connected 2-manifold, and have classified its homotopy type [17]. The argument in [17] is based on the following ingredients: (i) the ANR-property and the contractibility of $\mathcal{H}(M)_0$ for compact $M$, (ii) the bundle theorem connecting the homeomorphism group $\mathcal{H}(M)_0$ and the embedding spaces of submanifolds into $M$ [16, Corollary 1.1], and (iii) a result on the relative isotopies of 2-manifolds [17, Theorem 3.1]. The same strategy based on the $C^\infty$-versions of these results implies a corresponding conclusion for the diffeomorphism groups of noncompact smooth 2-manifolds.

Suppose $M$ is a smooth 2-manifold and $X$ is a closed subset of $M$. We denote by $\mathcal{D}_X(M)$ the group of $C^\infty$-diffeomorphisms $h$ of $M$ onto itself with $h|_X = id_X$, endowed with the compact-open $C^\infty$-topology [9, Ch.2, Section 1], and by $\mathcal{D}_X(M)_0$ the identity connected component of $\mathcal{D}_X(M)$.

The following is our main result:

Theorem 1.1. Suppose $M$ is a noncompact connected smooth 2-manifold without boundary.

(1) $\mathcal{D}(M)_0$ is a topological $\ell_2$-manifold.

(2) (i) $\mathcal{D}(M)_0 \simeq S^1$ if $M$ is a plane, an open Möbius band or an open annulus.

(ii) $\mathcal{D}(M)_0 \simeq \ast$ in all other cases.

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Any separable infinite-dimensional Fréchet space is homeomorphic to the Hilbert space $\ell_2 = \{ (x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty \}$. A topological $\ell_2$-manifold is a separable metrizable space which is locally homeomorphic to $\ell_2$. Topological types of $\ell_2$-manifolds are classified by their homotopy types. Theorem 1.1 implies the following conclusion:

**Corollary 1.1.** (i) $D(M)_0 \cong S^1 \times \ell_2$ if $M = a$ plane, an open Möbius band or an open annulus. 
(ii) $D(M)_0 \cong \ell_2$ in all other cases.

For the subgroup of diffeomorphisms with compact supports, we have the following results: Let $D(M)_0^c$ denote the subgroup of $D(M)_0$ consisting of $h \in D(M)$ which admits a $C^\infty$-isotopy with a compact support, $h_t : M \to M$ such that $h_0 = id_M$ and $h_1 = h$.

We say that a subspace $A$ of a space $X$ has the homotopy negligible (h.n.) complement in $X$ if there exists a homotopy $\varphi_t : X \to X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ ($0 < t \leq 1$). In this case, the inclusion $A \subset X$ is a homotopy equivalence, and $X$ is an ANR iff $A$ is an ANR.

**Theorem 1.2.** Suppose $M$ is a noncompact connected smooth 2-manifold without boundary. Then $D(M)_0^c$ has the h.n. complement in $D(M)_0$.

**Corollary 1.2.** (1) $D(M)_0^c$ is an ANR.
(2) The inclusion $D(M)_0^c \subset D(M)_0$ is a homotopy equivalence.

Section 2 contains fundamental facts on diffeomorphism groups of 2-manifolds and $\ell_2$-manifolds. Section 3 contains a sketch of proofs of Theorems 1.1 and 1.2.

### 2. Fundamental properties of diffeomorphism groups

In this preliminary section we list fundamental facts on diffeomorphism groups of 2-manifolds (general properties, bundle theorem, homotopy type, relative isotopies, etc) and basic facts on ANR's and $\ell_2$-manifolds. Throughout the paper all spaces are separable and metrizable and maps are continuous.

#### 2.1. General property of diffeomorphism groups.

Suppose $M$ is a smooth $n$-manifold possibly with boundary and $X$ is a closed subset of $M$.

**Lemma 2.1.** (c.f. [9, Ch 2., Section 4], etc)

$D_X(M)$ is a topological group, which is separable, completely metrizable, infinite-dimensional and not locally compact.
When \( N \) is a smooth submanifold of \( M \), the symbol \( \mathcal{E}_X(N, M) \) denotes the space of \( C^\infty \)-embeddings \( f : N \hookrightarrow M \) with \( f|_X = id_X \) with the compact-open \( C^\infty \)-topology, and \( \mathcal{E}_X(N, M)_0 \) denotes the connected component of the inclusion \( i_N : N \subset M \) in \( \mathcal{E}_X(N, M) \).

**Lemma 2.2.** (i) Suppose \( M \) is a smooth manifold without boundary, \( N \) is a compact smooth submanifold of \( M \) and \( X \) is a closed subset of \( N \). Then \( \mathcal{E}_X(N, M) \) is a Fréchet manifold.

(ii) Suppose \( M \) is a compact smooth \( n \)-manifold and \( X \) is a closed subset of \( M \) with \( \partial M \subset X \) or \( \partial M \cap X = \emptyset \). Then \( \mathcal{D}_X(M) \) is a Fréchet manifold.

In Lemma 2.2 \( \mathcal{E}_X(N, M)_0 \) and \( \mathcal{D}_X(M)_0 \) are path-connected. Thus any \( h \in \mathcal{D}_X(M)_0 \) can be joined with \( id_M \) by a path \( h_t \) (\( t \in [0, 1] \)) in \( \mathcal{D}_X(M)_0 \).

### 2.2. Bundle theorems.

The bundle theorem asserts that the natural restriction maps from diffeomorphism groups to embedding spaces are principal bundles [2, 12]. This has been used to study the homotopy types of diffeomorphism groups. This theorem also plays an essential role in our argument.

Suppose \( M \) is a smooth \( m \)-manifold without boundary, \( N \) is a compact smooth \( n \)-submanifold of \( M \) and \( X \) is a closed subset of \( N \).

**Case 1:** \( n < m \) [2, 12]

Let \( U \) be any open neighborhood of \( N \) in \( M \).

**Theorem 2.1.** For any \( f \in \mathcal{E}_X(N, U) \) there exist a neighborhood \( U \) of \( f \) in \( \mathcal{E}_X(N, U) \) and a map \( \varphi : U \rightarrow \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \) such that \( \varphi(g)f = g \) (\( g \in U \)) and \( \varphi(f) = id_M \).

**Corollary 2.1.** The restriction map \( \pi : \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \rightarrow \mathcal{E}_X(N, U)_0 \), \( \pi(h) = h|_N \), is a principal bundle with fiber \( \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \cap \mathcal{D}_N(M) \).

**Case 2:** \( n = m \)

In this case we have a weaker conclusion: Suppose \( N' \) is a compact smooth \( n \)-submanifold of \( M \) obtained from \( N \) by attaching a closed collar \( \partial N \times [0, 1] \) to \( \partial N \). Let \( U \) be any open neighborhood of \( N' \) in \( M \). We can apply Theorem 2.1 to \( \partial N' \) to obtain the following result:

**Theorem 2.2.** For any \( f \in \mathcal{E}_X(N', U) \) there exist a neighborhood \( U' \) of \( f \) in \( \mathcal{E}_X(N', U) \) and a map \( \varphi : U' \rightarrow \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \) such that \( \varphi(g)f|_N = g|_N \) (\( g \in U' \)) and \( \varphi(f) = id_M \).

For the sake of simplicity, we set \( \mathcal{D}_0 = \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \), \( \mathcal{E}_0 = \mathcal{E}_X(N, U)_0 \), \( \mathcal{E}_0' = \mathcal{E}_X(N', U)_0 \).

Consider the restriction map \( p : \mathcal{E}_0' \rightarrow \mathcal{E}_0 \), \( p(f) = f|_N \) and \( \pi : \mathcal{D}_0 \rightarrow \mathcal{E}_0 \), \( \pi(h) = h|_N \). We have the pullback diagram:
\[ p^*(D_0) \xrightarrow{p_*} D_0 \]
\[ \pi_* \downarrow \downarrow \pi \]
\[ E'_0 \xrightarrow{p} E_0, \]

where \( p^*D_0 = \{(f, h) \in \mathcal{E}'_0 \times D_0 \mid f|_N = h|_N\} \), \( p_*(f, h) = h \) and \( \pi_*(f, h) = f \). The map \( p_* \)
ads a natural right inverse \( q : D_0 \rightarrow p^*(D_0) \), \( q(h) = (h|_{N'}, h) \). The group \( D_0 \cap D_N(M) \) acts on \( p^*D_0 \) by \( (f, h)g = (f, hg) \) (\( g \in D_0 \cap D_N(M) \)).

**Corollary 2.2.**

(1) \( \pi_* : p^*(D_0) \rightarrow \mathcal{E}'_0 \) is a principal bundle with fiber \( D_0 \cap D_N(M) \).

(2) \( p_* : p^*(D_0) \rightarrow D_0 \) is a homotopy equivalence with the homotopy inverse \( q : D_0 \rightarrow p^*(D_0) \).

(3) \( p : \mathcal{E}'_0 \rightarrow \mathcal{E}_0 \) is a homotopy equivalence if \( X \subset \text{int} N \).

The statements (2) and (3) exhibit a close relation between the restriction map \( \pi \) and the pullback \( \pi_* \).

**2.3. Diffeomorphism groups of 2-manifolds.**

Next we recall fundamental facts on diffeomorphism groups of compact 2-manifolds. The following theorem shows that \( D_X(M)_0 \simeq * \) except a few cases. The symbols \( S^1, S^2, T, P, K, D, A \) and \( M \) denote the 1-sphere, 2-sphere, torus, projective plane, Klein bottle, disk, annulus and Möbius band respectively.

**Theorem 2.3.** ([4, 15] etc.) Suppose \( M \) is a compact connected smooth 2-manifold. Then the homotopy type of \( D(M)_0 \) is classified as follows:

<table>
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<th>( M )</th>
<th>( D(M)_0 )</th>
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<tr>
<td>( S^2, P )</td>
<td>( SO(3) )</td>
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<td>( T )</td>
<td>( T )</td>
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<tr>
<td>( K, D, A, M )</td>
<td>( S^1 )</td>
</tr>
<tr>
<td>all other cases</td>
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\( \circ D_\theta(D) \simeq *, D_\theta(M) \simeq * \).

\( \circ \) If \( X \) is a disjoint union of a compact smooth 2-submanifold and finitely many smooth circles and points in \( M \) and \( \partial M \subset X \), then \( D_X(M)_0 \simeq * \).

For 2-manifolds there is no difference among the conditions: homotopic, \( C^0 \)-isotopic, \( C^\infty \)-isotopic and joinable by a path in the diffeomorphism group. By [4] and a \( C^\infty \)-analogue of [5] we have

**Proposition 2.1.** Suppose \( M \) is a compact smooth 2-manifold.

(1) Suppose \( N \) is a closed collar of \( \partial M \). If \( h \in D_N(M) \) is homotopic to \( \text{id}_M \text{ rel } N \), then \( h \) is \( C^\infty \)-isotopic to \( \text{id}_M \text{ rel } N \).

(2) Suppose \( N \) is a compact smooth 2-submanifold of \( M \) with \( \partial M \subset N \). For \( h \in D_N(M) \), the
The following conditions are equivalent:

(a) $h$ is $C^0$-isotopic to $\text{id}_M$ rel $N$.
(b) $h$ is $C^\infty$-isotopic to $\text{id}_M$ rel $N$.
(c) $h \in \mathcal{D}_N(M)_0$.

In Corollaries 2.1 and 2.2 we have a principal bundle with fiber $\mathcal{G} \equiv \mathcal{D}_X(M)_0 \cap \mathcal{D}_N(M)$. The next theorem gives us a sufficient condition that $\mathcal{G} = \mathcal{D}_N(M)_0$. The symbol $\# X$ denotes the cardinal of a set $X$.

**Theorem 2.4.** Suppose $M$ is a compact connected smooth 2-manifold, $N$ is a compact smooth 2-submanifold of $M$ with $\partial M \subset N$, $X$ is a subset of $N$. Suppose $(M,N,X)$ satisfies the following conditions:

(i) $M \neq T, P, K$ or $X \neq \emptyset$.
(ii) (a) if $H$ is a disk component of $N$, then $\# (H \cap X) \geq 2$,
    (b) if $H$ is an annulus or Möbius band component of $N$, then $H \cap X \neq \emptyset$,
(iii) (a) if $L$ is a disk component of $\text{cl}(M \setminus N)$, then $\# (L \cap X) \geq 2$,
    (b) if $L$ is a Möbius band component of $\text{cl}(M \setminus N)$, then $L \cap X \neq \emptyset$.

Then we have:

1. If $h \in \mathcal{D}_N(M)$ is $C^0$-isotopic to $\text{id}_M$ rel $X$, then $h$ is $C^\infty$-isotopic to $\text{id}_M$ rel $N$.
2. $\mathcal{D}(M)_0 \cap \mathcal{D}_N(M) = \mathcal{D}_N(M)_0$.

Theorem 2.4 follows from [17, Theorem 3.1] and Proposition 2.1.

2.4. Basic properties of ANR's and $\ell^2$-manifolds.

The ANR-property of diffeomorphism groups and embedding spaces is also essential in our argument. Here we recall basic properties of ANR's [8, 10, 13] and a topological characterization theorem of $\ell^2$-manifolds.

A metrizable space $X$ is called an ANR (absolute neighborhood retract) for metric spaces if any map $f : B \to X$ from a closed subset $B$ of a metrizable space $Y$ admits an extension to a neighborhood $U$ of $B$ in $Y$. If we can always take $U = Y$, then $X$ is called an AR. It is known that $X$ is an AR (an ANR) iff it is a retract of (an open subset of) a normed space. Any ANR has a homotopy type of CW-complex. An AR is exactly a contractible ANR.

We apply the following criterion of ANR's:

**Lemma 2.3.** (1) A space $X$ is an ANR iff every point of $X$ has an ANR neighborhood in $X$.
(2) If $X = \cup_{i=1}^{\infty} U_i$, $U_i$ is open in $X$ and $U_i \subset U_{i+1}$ and if each $U_i$ is an AR, then $X$ is also an
(3) In a fiber bundle, the total space is an ANR iff both the base space and the fiber are ANR's.

(4) A metric space $X$ is an ANR iff for any $\epsilon > 0$ there is an ANR $Y$ and maps $f : X \to Y$ and $g : Y \to X$ such that $gf$ is $\epsilon$-homotopic to $id_X$.

Since any Fréchet space is an AR, every Fréchet manifold is an ANR.

Finally we recall a characterization of $\ell_2$-manifold topological groups [3, 19].

**Theorem 2.5.** A topological group is an $\ell_2$-manifold iff it is a separable, non locally compact, completely metrizable ANR.

The diffeomorphism group $D(M)_0$ satisfies all conditions except the ANR property (Lemma 2.1). Thus the proof of Theorem 1.1 (1) reduces to the verification of ANR property of $D(M)_0$. The latter follows from the ANR property of the diffeomorphism groups and embedding spaces of compact 2-manifolds (Lemma 2.2).

### 3. Proof of Main Theorems

In this section we give a sketch of proofs of Theorems 1.1 and 1.2 in the case where $M \neq$ a plane, an open Möbius band, an open annulus. Below we assume that $M$ is a noncompact connected smooth 2-manifold without boundary and that $M \neq$ a plane, an open Möbius band, an open annulus.

We can write as $M = \bigcup_{i=0}^{\infty} M_i$, where $M_0 = \emptyset$ and for each $i \geq 1$

(a) $M_i$ is a nonempty compact connected smooth 2-submanifold of $M$ and $M_{i-1} \subset int M_i$,

(b) for each component $L$ of $cl (M \setminus M_i)$, $L$ is noncompact and $L \cap M_{i+1}$ is connected.

Note that $M$ is a plane (an open Möbius band, an open annulus) iff infinitely many $M_i$'s are disks (Möbius bands, annuli respectively). Since $M \neq$ a plane, an open Möbius band, an open annulus, passing to a subsequence, we may assume that

(c) $M_i \neq$ a disk, an annulus, a Möbius band.

For each $i \geq 1$ let $U_i = int M_i$, and choose a small closed collar $E_i$ of $\partial M_i$ in $U_{i+1} \setminus U_i$, and set $M'_i = M_i \cup E_i \subset U_{i+1}$.

#### 3.1. Proof of Theorem 1.1.

[1] For each $j > i > k \geq 0$, we have the following pullback diagram:
Lemma 3.1. (1) $(\pi_{k,j}^*_{\dot{j}})_{*}$ is a principal bundle with fiber $G_{k,j}^i$.
(2) $G_{k,j}^i$ is an AR.
(3) $(\pi_{k,j}^*_{\dot{j}})_{*}$ is a trivial bundle.
(4) $E_{M_{\dot{k}}}(M')_{0}$ is an AR.

In (2) we apply Theorem 2.4 to deduce $G_{k,j}^i \cong D_{M_{\dot{k}} \cup U_j}(M)'_0$. The latter is an AR (Lemma 2.2 (ii), Theorem 2.3).

[2] For each $i > k \geq 0$, we have the following pullback diagram:

\[
\begin{array}{ccc}
(p_{k,j}^i)^*(D_{M_{k}}(M)_{0}) & \xrightarrow{(p_{k,j}^i)^*} & D_{M_{k}}(M)_{0} \\
\downarrow & & \downarrow \pi_k^i \\
E_{M_{k}}(M_{\dot{k}}, M)_{0} & \xrightarrow{p_k^i} & E_{M_{k}}(M_{\dot{k}}, M)_{0},
\end{array}
\]

\[\pi_k^i, p_k^i : \text{the restriction maps}, \quad G_{k,j}^i \equiv D_{M_{k}}(M)_{0} \cap D_{M_{\dot{k}}}(M).\]

Lemma 3.2. (1) $(\pi_{k}^*_{\dot{k}})_{*}$ is a principal bundle with fiber $G_{k}^i$.
(2) $E_{M_{\dot{k}}}(M_{\dot{k}}', M)_{0}$ is an AR.
(3) $(\pi_{k}^*_{\dot{k}})_{*}$ is a trivial bundle.
(4) $G_{k}^i = D_{M_k}(M)'_0$ and $D_{M_k}(M)'_0$ strongly deformation retracts onto $D_{M_k}(M)'_0$.

The assertion (2) follows from Lemma 2.3 (2), Lemma 3.1 (4) and the fact that $E_{M_{\dot{k}}}(M_{\dot{k}}', M)_{0} = \cup_{j > i} E_{M_{\dot{k}}}(M_{\dot{k}}, U_{j})_{0}$.

Proof of Theorem 1.1.

(A) $D(M)'_0 \simeq \ast$:

$D_{M_k}(M)'_0$ strongly deformation retracts onto $D_{M_{k+1}}(M)'_0$ for each $i \geq 0$ (Lemma 3.2 (4)). Since $\text{diam} \ D_{M_k}(M)'_0 \to 0$ ($i \to \infty$), it follows that $D(M)'_0$ strongly deformation retracts onto $\{id_M\}$.

(B) $D(M)'_0$ is an $\ell_2$-manifold:

By Theorem 2.5 and Lemma 2.1 it remains to show that $D(M)'_0$ is an ANR. We apply Lemma 2.3 (4): For each $i \geq 0$, we have the following pullback diagram:

\[
\begin{array}{ccc}
(p_{i})^*(D(M)'_0) & \xrightarrow{(p_{i})^*} & D(M)'_0 \\
\downarrow & & \downarrow \pi_{i} \\
E(M_{\dot{i}}, M)'_0 & \xrightarrow{p_{i}} & E(M_{\dot{i}}, M)'_0,
\end{array}
\]

\[\pi_i, p_i : \text{the restriction maps}, \quad q_i : D(M)'_0 \to (p_i)^*(D(M)'_0) \quad q_i(h) = (h|_{M_{\dot{i}}}, h).\]
Since $(\pi_i)_*$ is a trivial principal bundle with the contractible fiber $\mathcal{D}_{M_i}(M)_0$ (Lemma 3.2 (3),(4). (A)), it follows that $(\pi_i)_*$ admits a section $s_i$ and $s_i(\pi_i)_*$ is $(\pi_i)_*$-fiber preserving homotopic to $id$. Consider the two maps

$$\varphi = (\pi_i)_*q_i : D(M)_0 \to \mathcal{E}(M_i, M)_0 \quad \text{and} \quad \psi = (p_i)_*s_i : \mathcal{E}(M_i, M)_0 \to D(M)_0.$$  

Then $\mathcal{E}(M_i, M)_0$ is an ANR (Lemma 2.2 (i)) and $\psi \varphi : D(M)_0 \to D(M)_0$ is $\pi_i$-fiber preserving homotopic to $id$. Since diam (fibers of $\pi_i$) $\to 0$ ($i \to \infty$), Lemma 2.3 (4) implies that $D(M)_0$ is an ANR. \hfill \Box

3.2. Proof of Theorem 1.2.

We use the following notations:

$$D_j = D_{M \backslash U_j}(M)_0, \quad U_{i,j} = \mathcal{E}(M_i, U_j)_0, \quad U_{\dot{i}j} = \mathcal{E}(M_{\dot{i}}, U_j)_0 \quad (j > i \geq 1).$$

We have the pullback diagram:

$$\begin{array}{ccc}
(p_{i,j})^*D_j & \xrightarrow{(p_{i,j})_*} & D_j \\
\downarrow & & \downarrow \pi_{i,j} \\
U_{\dot{i}j} & \xrightarrow{p_{\dot{i}j}} & U_{\dot{i}j},
\end{array}$$

$$\pi': D(M)_0 \to \mathcal{E}(M_{\dot{i}}, M)_0,$$

$$\pi_i, \pi_{i,j}, \pi'_i : \text{the restriction maps.}$$

Lemma 3.3. (i) $(\pi_{i,j})_*$ is a trivial bundle with AR fiber.

(ii) $\pi_{i,j}$ has the following lifting property:

$$(*) \quad \text{If} \ Y \text{ is a metric space,} \ B \text{ is a closed subset of} \ Y \text{ and} \ \varphi : Y \to U_{i,j}' \text{ and} \ \varphi_0 : B \to D_j \text{ are map with} \ p_{i,j}\varphi|_B = \pi_{i,j}\varphi_0, \text{then there exists a map} \ \Phi : Y \to D_j \text{ such that} \ \pi_{i,j}\Phi = p_{i,j}\varphi \text{ and} \ \Phi|_B = \varphi_0.$$  

For each $j > i \geq 1$, we regard as $U_{i,j} \subset \mathcal{E}(M'_i, M)_0$ and set $V_{i,j}' = (\pi'_i)^{-1}(U_{i,j}') \subset D(M)_0$.

For each $i \geq 1$ we have:

(i) $\mathcal{E}(M_i, M)_0 = \bigcup_{j > i} clU_{i,j}'$ ($U_{i,j}'$ is open in $\mathcal{E}(M_i, M)_0$, $clU_{i,j}' \subset U_{i,j+1}'$)

(ii) $\mathcal{D}(M)_0 = \bigcup_{j > i} clV_{i,j}'$ ($V_{i,j}'$ is open in $\mathcal{D}(M)_0$, $clV_{i,j}' \subset V_{i,j+1}'$, $V_{i+1,j}' \subset V_{i,j}'$ ($j > i + 1$))

(iii) $\mathcal{D}(M)_0 = \bigcup_{j > i} D_j$ ($D_j \subset D_{j+1}$)

Proof of Theorem 1.2.

We construct a homotopy

$$F : \mathcal{D}(M)_0 \times [1, \infty] \to \mathcal{D}(M)_0 \text{ such that} \ F_\infty = id \text{ and} \ F_t(\mathcal{D}(M)_0) \subset \mathcal{D}(M)_0 (1 \leq t < \infty).$$

(1) $F_i$ ($i \geq 1$): Using Lemma 3.3 (ii), inductively we can construct a map $s_{ij} : clU_{i,j}' \to D_{j+1}$ such that $s_{ij}(f)|_{M_i} = f|_{M_i}$ ($f \in clU_{i,j}'$) and $s_{i+1,j}|_{clU_{i,j}'} = s_{ij}$ ($j > i$). Define a map
$s^i : \mathcal{E}(M'_i, M)_0 \to \mathcal{D}(M)'_0$ by $s^i|_{\cup_i j'} = s^i_j$, and set $F_i = s^i\pi^i_i$. We have $F_i(\text{cl} V_{i+1}') \subset \mathcal{D}_{j+1}$ and $F_i(h)|_{M'_i} = h|_{M'_i}$.

(2) $F_i (i \leq t \leq i + 1)$: Inductively we can construct a sequence of homotopies $G^j : \text{cl} V_{i+1,j}' \times [i, i + 1] \to \mathcal{D}_{j+1} (j > i + 1)$ such that $G^j_i = F_i$, $G^j_{i+1} = F_{i+1}$, $G^{j+1}_i|_{\text{cl} V_{i+1,j}' \times [i, i + 1]} = G^j$ and $G^j_i(h)|_{M'_i} = h|_{M'_i}$. If $G^j_i$ is given, then $G^{j+1}_i$ is obtained by applying Lemma 3.3(ii) to the diagram:

$$
\begin{array}{ccc}
B & \xrightarrow{\varphi_0} & \mathcal{D}_{j+2} \\
\cap & \downarrow & \\
Y & \xrightarrow{\varphi} & U_{i+j+2}' \longrightarrow U_{i+j+2},
\end{array}
$$

$$(Y, B) = (\text{cl} V_{i+1,j+1}' \times [i, i + 1], (\text{cl} V_{i+1,j}' \times [i, i + 1]) \cup (\text{cl} V_{i+1,j+1}' \times \{i, i + 1\})).$$

Define $F : \mathcal{D}(M)'_0 \times [i, i + 1] \to \mathcal{D}(M)'_0$ by $F = G^j_i$ on $\text{cl} V_{i+1,j}' \times [i, i + 1]$.

(3) $F_{\infty}$: Since $F_i(h)|_{M'_i} = h|_{M'_i}$ for $t \geq i$, we can continuously extend $F$ by $F_{\infty} = \text{id}$. This completes the proof. \qed

References

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