数理解析研究所講究録 1248 巻 2002 年 96-99

On Michael's and Vaughan's examples on product spaces

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The Michael's example given in [2] and the Vaughan's example given in [3] are famous ones showing fundamentally that the product of a paracompact space and a metric space need not be normal.

Let $\mathbf{R}$ be the real line and $\mathbf{P}$ the set of irrationals. Let $\mathbf{M}$ be the Michael line in [2], that is, the space $(\mathbf{R}, \tau_p)$ with a topology τ_p on $\mathbf{R}$ by defining all subsets of the form $G \cup K$ to be open, where G is open in $\mathbf{R}$ and $K \subset \mathbf{P}$.

To recall the Vaughan's example we use the same notations as in [3]. Let $D(\omega_1)$ denote the set ω_1 with the discrete topology. Let $\hat{D}(\omega_1)$ be the set $\omega_1 + 1$ with the topology so that every α with $\alpha < \omega_1$ is isolated and the sets $(\gamma, \omega_1] = \{ \beta \mid \gamma < \beta \leq \omega_1\}, \gamma \in \omega_1$ are basic neighborhoods of ω_1. Let $\mathbf{B_1} = \underset{\omega}{\square}\, \hat{D}(\omega_1)$ denote the box product of countably many copies of $\hat{D}(\omega_1)$. For a space X, X^ω denotes the usual product space of countably many copies of X.

Example 1 (Michael [2]). $\mathbf{M} \times \mathbf{P}$ *is not normal.*

Example 2 (Vaughan [3]). $\mathbf{B_1} \times D(\omega_1)^\omega$ *is not normal.*

In reviewing the original proofs of the above two examples, although the situations of the spaces are quite similar, but the basic ideas of the proofs are completely different. The purpose of this paper is "to exchange these ideas", that is, to give our proofs to Example 1 under Vaughan's idea and to Example 2 under Michael's idea.

Unless otherwise indicated, spaces are assumed to be Hausdorff. For undefined notions and terminologies, referred to Engelking's book [1].

Our proof of Example 2.

For brevity, set $X = \mathbf{B_1}$ and $Y = D(\omega_1)^\omega$. Sometimes we consider Y as a subset of X.

Let π_i be the ith projection map on X. For each $x = \langle x_1, x_2, \cdots \rangle \in X$ and for each $\alpha < \omega_1$, let

$$\alpha(x) = [\cap\{\pi_i^{-1}(x_i) \mid x_i < \omega_1\}] \cap [\cap\{\pi_i^{-1}((\alpha, \omega_1]) \mid x_i = \omega_1\}].$$

For each $y = \langle y_1, y_2, \cdots \rangle \in Y$ and for each positive integer m, let

$$m(y) = \cap\{\pi_i^{-1}(y_i) \mid i \leq m\}.$$

Note that $\alpha(x), \alpha < \omega_1$ and $m(y), m \in N$ are basic open neighborhoods of x and y in X and Y, respectively.

Let

$$K_0 = (X \setminus Y) \times Y, \qquad K_1 = \{\langle y, y\rangle \in X \times Y \mid y \in Y\}.$$

Then K_0 and K_1 are disjoint closed subsets of $X \times Y$. Suppose that there exist disjoint open subsets U and V of $X \times Y$ for which $K_0 \subset U$ and $K_1 \subset V$. For each natural number n, let us put

$$P_n = \{y \in Y \mid \{y\} \times n(y) \subset V\}$$

and

$$M_n = \underbrace{D(\omega_1) \times D(\omega_1) \times \cdots D(\omega_1)}_{n \text{ times}} \times \hat{D}(\omega_1) \times \hat{D}(\omega_1) \times \cdots .$$

Then we have $Y = \cup\{P_n \mid n \in \mathbf{N}\}$. We claim that there exists a natural number n such that

$$(\overline{P_n} \cap M_n) \cap (X \setminus Y) \neq \emptyset.$$

To see this, assume the contrary. Pick a countable ordinal x_0 and define $z_1 = \langle x_0, \omega_1, \omega_1, \cdots\rangle$, then $z_1 \notin \overline{P_1}$ because $z_1 \in M_1 \cap (X \setminus Y)$. Therefore there exists an ordinal $\alpha_1 < \omega_1$ such that $\alpha_1(z_1) \cap P_1 = \emptyset$. Pick a countable ordinal $x_1 > \max\{x_0, \alpha_1\}$ and put $z_2 = \langle x_0, x_1, \omega_1, \omega_1, \cdots\rangle$, then similarly we have $z_2 \notin \overline{P_2}$. Therefore there exists an ordinal $\alpha_2 < \omega_1$ such that $\alpha_2(z_2) \cap P_1 = \emptyset$. Pick a countable ordinal $x_2 > \max\{x_1, \alpha_2\}$. Assume we have constructed $x_0, x_1, x_2, \cdots, x_{k-1}$. Put $z_k = \langle x_0, x_1, \cdots, x_{k-1}, \omega_1, \omega_1, \cdots\rangle$, then $z_k \notin \overline{P_k}$. Therefore there exists an ordinal $\alpha_k < \omega_1$ such that $\alpha_k(z_k) \cap P_k = \emptyset$. Pick a countable ordinal $x_k > \max\{x_{k-1}, \alpha_k\}$. By induction, we can construct a point $x = \langle x_0, x_1, \cdots\rangle$ such that $x \in Y = \cup\{P_n \mid n \in \mathbf{N}\}$. On the other hand, we must have

$$x \in \cap\{\alpha_i(z_i) \mid i \in \mathbf{N}\} \subset X \setminus \cup\{P_n \mid n \in \mathbf{N}\},$$

a contradiction.

Therefore there exists an n so that we can take a point

$$z = \langle z_1, z_2, \cdots\rangle \in \overline{P_n} \cap M_n \setminus Y.$$

Take an arbitrary $\alpha < \omega_1$. Since Y is dense in X, we can take a point $y = \langle y_1, y_2, \cdots\rangle \in \alpha(z) \cap Y$. Then $\langle z, y\rangle \in K_0 \subset U$. Hence there exist an ordinal $\beta < \omega_1$ and a natural number k such that

$$\beta(z) \times k(y) \subset U.$$

Since $z \in \overline{P_n}$, we can take a point

$$y' = \langle y'_1, y'_2, \cdots\rangle \in \beta(z) \cap P_n.$$

Then we have

$$\langle y', y\rangle \in \beta(z) \times k(y) \subset U.$$

On the other hand, $z \in M_n$ implies that

$$\pi_i(\alpha(z)) = \{z_i\} = \pi_i(\beta(z)) \qquad \text{for } i = 1, 2, \cdots n$$

and since $y \in \alpha(z)$ and $y' \in \beta(z)$,

$$y_i = z_i = y'_i \qquad \text{for } i = 1, 2, \cdots, n.$$

Therefore $y \in n(y')$. Since $\langle y', y\rangle \in \{y'\} \times n(y')$ and by the definition of P_n, we can conclude that

$$\langle y', y\rangle \in U \cap V.$$

It is a contradiction. $\square$

Our proof of Example 1.

For each natural number k, let φ_k be a function from the product space $\mathbf{N}^k$ to $(0,1)\cap\mathbf{Q}$, defined by

$$\varphi_k(\langle n_1, n_2, \cdots, n_k\rangle) = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}}}$$

for each $\langle n_1, n_2, \cdots, n_k\rangle \in \mathbf{N}^k$. Then we notice that

$$(*)\ \varphi_{2k+1}(\langle n_1, n_2, \cdots, n_{2k+1}\rangle) - \varphi_{2k}(\langle n_1, n_2, \cdots, n_{2k}\rangle) < \frac{1}{4^{k-1}}\frac{1}{n_{2k+1}}.$$

for each natural number k

Let $B(\mathbf{N})$ denote the Baire's zero-dimensional space with respect to $\mathbf{N}$. Let φ be a function from $B(\mathbf{N})$ for which

$$\varphi(\langle n_1, n_2, \cdots\rangle) = \lim_{k\to\infty} \varphi_k(\langle n_1, n_2, \cdots, n_k\rangle)$$

for each $\langle n_1, n_2, \cdots\rangle \in B(\mathbf{N})$. Then it follows that φ is a homeomorphism between $B(\mathbf{N})$ and the space $\mathbf{P}\cap(0,1)$.

Let M_Q denotes the rational points of M and M_P the irrational ones. Put $K_0 = M_Q\times\mathbf{P}$ and $K_1 = \{\langle p,p\rangle \mid p\in M_P\}$. They are disjoint closed sets of $M\times\mathbf{P}$. Let U be any open set of $M\times\mathbf{P}$ containing K_0. We need only to show that $\overline{U}\cap K_1 \neq \emptyset$.

Put

$$q_0 = 0, \quad p_0 = \varphi(\langle 1, 1, \cdots\rangle),$$

where φ is the above homeomorphism between $B(\mathbf{N})$ and the space $\mathbf{P}\cap(0,1)$. Since $\langle q_0, p_0\rangle \in K_0 \subset U$, there exist m_0 and $n_0\in\mathbf{N}$ such that $S_{\frac{1}{m_0}}(q_0)\times S'_{\frac{1}{n_0}}(p_0)\subset U$ where $S'_{\frac{1}{n_0}}(p_0) = S_{\frac{1}{n_0}}(p_0)\cap\mathbf{P}$. Pick a natural number $x_0 > m_0$. Put

$$q_1 = \varphi_1(x_0), \quad p_1 = \varphi(\langle x_0, 1, 1, \cdots\rangle).$$

Since $\langle q_1, p_1\rangle \in K_0\subset U$, there exist m_1 and $n_1\in\mathbf{N}$ such that $S_{\frac{1}{m_1}}(q_1)\times S'_{\frac{1}{n_1}}(p_1)\subset U$. Pick a natural number $x_1 > m_1$.

Assume we have reached the kth step in this construction, and have constructed $x_0, x_1, \cdots, x_{k-1}$. Put

$$q_k = \varphi_k(\langle x_0, x_1, \cdots, x_{k-1}\rangle), \quad p_k = \varphi(\langle x_0, x_1, \cdots, x_{k-1}, 1, 1, \cdots\rangle).$$

Since $\langle q_k, p_k \rangle \in K_0 \subset U$, there exist m_k and $n_k \in \mathbf{N}$ such that $S_{\frac{1}{m_k}}(q_k) \times S'_{\frac{1}{n_k}}(p_k) \subset U$. Pick a natural number $x_k > m_k$.

By induction, we can construct a point $x = \varphi(x_0, x_1, \cdots)$ such that $\langle x, x \rangle \in K_1$. For any positive number ε, $\{x\} \times S'_\varepsilon(x)$ is a neighborhood of $\langle x, x \rangle$. Since φ is continuous, there exists a positive and even integer $k(= 2j)$ such that $\varphi(B_k(\langle x_0, x_1, \cdots \rangle)) \subset S'_\varepsilon(x)$. Therefore

$$p_k = \varphi(\langle x_0, x_1, \cdots, x_{k-1}, 1, 1, \cdots \rangle) \in S'_\varepsilon(x).$$

On the other hand,

$$q_k = \varphi_k(\langle x_0, x_1, \cdots, x_{k-1} \rangle) < x < \varphi_{k+1}(\langle x_0, x_1, \cdots, x_k \rangle) = q_{k+1}$$

because k is even. And

$$q_{k+1} - q_k < \frac{1}{4^{j-1}} \cdot \frac{1}{x_k} < \frac{1}{x_k} < \frac{1}{m_k}$$

by $(*)$. Then $x \in S_{\frac{1}{m_k}}(q_k)$. Therefore

$$\langle x, p_k \rangle \in (\{x\} \times S'_\varepsilon(x)) \cap \left(S_{\frac{1}{m_k}}(q_k) \times S'_{\frac{1}{n_k}}(p_k) \right).$$

Since $S_{\frac{1}{m_k}}(q_k) \times S'_{\frac{1}{n_k}}(p_k) \subset U$, every neighborhood of $\langle x, x \rangle$ hits U. $\square$

References

[1] R. Engelking, *General Topology*, Polish Scientific Publishers, Warszawa (1977).

[2] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc., **69** (1963), 375-376.

[3] J. E. Vaughan, *Non-normal products of ω_μ-metrizable spaces*, Proc. Amer. Math. Soc., **51** (1975), 203-208.