On Michael's and Vaughan's examples on product spaces

筑波大学数学系 中村 恵美 (Megumi Nakamura)
Institute of Mathematics, University of Tsukuba

The Michael's example given in [2] and the Vaughan's example given in [3] are famous ones showing fundamentally that the product of a paracompact space and a metric space need not be normal.

Let $\mathbb{R}$ be the real line and $\mathbb{P}$ the set of irrationals. Let $\mathbb{M}$ be the Michael line in [2], that is, the space $(\mathbb{R}, \tau_p)$ with a topology $\tau_p$ on $\mathbb{R}$ by defining all subsets of the form $G \cup K$ to be open, where $G$ is open in $\mathbb{R}$ and $K \subset \mathbb{P}$.

To recall the Vaughan's example we use the same notations as in [3]. Let $D(\omega_1)$ denote the set $\omega_1$ with the discrete topology. Let $\hat{D}(\omega_1)$ be the set $\omega_1 + 1$ with the topology so that every $\alpha$ with $\alpha < \omega_1$ is isolated and the sets $(\gamma, \omega_1] = \{ \beta \mid \gamma < \beta \leq \omega_1 \}, \gamma \in \omega_1$ are basic neighborhoods of $\omega_1$. Let $B_1 = \square \omega \hat{D}(\omega_1)$ denote the box product of countably many copies of $\hat{D}(\omega_1)$. For a space $X$, $X^\omega$ denotes the usual product space of countably many copies of $X$.

Example 1 (Michael [2]). $\mathbb{M} \times \mathbb{P}$ is not normal.

Example 2 (Vaughan [3]). $B_1 \times D(\omega_1)^\omega$ is not normal.

In reviewing the original proofs of the above two examples, although the situations of the spaces are quite similar, but the basic ideas of the proofs are completely different. The purpose of this paper is "to exchange these ideas", that is, to give our proofs to Example 1 under Vaughan's idea and to Example 2 under Michael's idea.

Unless otherwise indicated, spaces are assumed to be Hausdorff. For undefined notions and terminologies, referred to Engelking's book [1].

Our proof of Example 2.

For brevity, set $X = B_1$ and $Y = D(\omega_1)^\omega$. Sometimes we consider $Y$ as a subset of $X$.

Let $\pi_i$ be the $i$th projection map on $X$. For each $x = \langle x_1, x_2, \cdots \rangle \in X$ and for each $\alpha < \omega_1$, let

$$\alpha(x) = \cap \{ \pi_i^{-1}(x_i) \mid x_i < \omega_1 \} \cap \{ \pi_i^{-1}(\{\alpha, \omega_1\}) \mid x_i = \omega_1 \}.$$  

For each $y = \langle y_1, y_2, \cdots \rangle \in Y$ and for each positive integer $m$, let

$$m(y) = \cap \{ \pi_i^{-1}(y_i) \mid i \leq m \}.$$  

Note that $\alpha(x), \alpha < \omega_1$ and $m(y), m \in \mathbb{N}$ are basic open neighborhoods of $x$ and $y$ in $X$ and $Y$, respectively.
Let
\[ K_0 = (X \setminus Y) \times Y, \quad K_1 = \{ (y, y) \in X \times Y \mid y \in Y \}. \]

Then \( K_0 \) and \( K_1 \) are disjoint closed subsets of \( X \times Y \). Suppose that there exist disjoint open subsets \( U \) and \( V \) of \( X \times Y \) for which \( K_0 \subset U \) and \( K_1 \subset V \). For each natural number \( n \), let us put
\[ P_n = \{ y \in Y \mid \{ y \} \times n(y) \subset V \} \]
and
\[ M_n = D(\omega_1) \times D(\omega_1) \times \cdots D(\omega_1) \times D(\omega_1) \times D(\omega_1) \times \cdots \]
\( n \) times

Then we have \( Y = \cup\{ P_n \mid n \in \mathbb{N} \} \). We claim that there exists a natural number \( n \) such that
\[ (P_n \cap M_n) \cap (X \setminus Y) \neq \emptyset . \]
To see this, assume the contrary. Pick a countable ordinal \( x_0 \) and define \( z_1 = \langle x_0, \omega_1, \omega_1, \cdots \rangle \), then \( z_1 \notin P_1 \) because \( z_1 \in M_1 \cap (X \setminus Y) \). Therefore there exists an ordinal \( \alpha_1 < \omega_1 \) such that \( \alpha_1(z_1) \cap P_1 = \emptyset \). Pick a countable ordinal \( x_1 \) such that \( x_1 > \max\{ x_0, \alpha_1 \} \) and put \( z_2 = \langle x_0, x_1, \omega_1, \omega_1, \cdots \rangle \). Then similarly we have \( z_2 \notin P_2 \). Therefore there exists an ordinal \( \alpha_2 < \omega_1 \) such that \( \alpha_2(z_2) \cap P_1 = \emptyset \). Pick a countable ordinal \( x_2 > \max\{ x_1, \alpha_2 \} \). Assume we have constructed \( x_0, x_1, x_2, \cdots, x_{k-1} \). Put \( z_k = \langle x_0, x_1, \cdots, x_{k-1}, \omega_1, \omega_1, \cdots \rangle \), then \( z_k \notin \overline{P_k} \). Therefore there exists an ordinal \( \alpha_k < \omega_1 \) such that \( \alpha_k(z_k) \cap P_k = \emptyset \). Pick a countable ordinal \( x_k > \max\{ x_{k-1}, \alpha_k \} \). By induction, we can construct a point \( x = \langle x_0, x_1, \cdots \rangle \) such that \( x \in Y = \cup\{ P_n \mid n \in \mathbb{N} \} \). On the other hand, we must have
\[ x \in \cap\{ \alpha_i(z_i) \mid i \in \mathbb{N} \} \subset X \setminus \cup\{ P_n \mid n \in \mathbb{N} \}, \]
a contradiction.

Therefore there exists an \( n \) so that we can take a point
\[ z = \langle z_1, z_2, \cdots \rangle \in \overline{P_n} \cap M_n \setminus Y. \]
Take an arbitrary \( \alpha < \omega_1 \). Since \( Y \) is dense in \( X \), we can take a point \( y = \langle y_1, y_2, \cdots \rangle \in \alpha(z) \cap Y \). Then \( (z, y) \in K_0 \subset U \). Hence there exist an ordinal \( \beta < \omega_1 \) and a natural number \( k \) such that
\[ \beta(z) \times k(y) \subset U. \]
Since \( z \in \overline{P_n} \), we can take a point
\[ y' = \langle y'_1, y'_2, \cdots \rangle \in \beta(z) \cap P_n. \]
Then we have
\[ \langle y', y \rangle \in \beta(z) \times k(y) \subset U. \]
On the other hand, \( z \in M_n \) implies that
\[ \pi_i(\alpha(z)) = \{ z_i \} = \pi_i(\beta(z)) \]
for \( i = 1, 2, \cdots n \)
and since \( y \in \alpha(z) \) and \( y' \in \beta(z) \),
\[ y_i = z_i = y'_i \]
for \( i = 1, 2, \cdots, n \).
Therefore \( y \in n(y') \). Since \( \langle y', y \rangle \in \{ y' \} \times n(y') \) and by the definition of \( P_n \), we can conclude that
\[ \langle y', y \rangle \in U \cap V. \]
It is a contradiction. \( \Box \)
Our proof of Example 1.

For each natural number $k$, let $\varphi_k$ be a function from the product space $\mathbb{N}^k$ to $(0, 1) \cap \mathbb{Q}$, defined by

$$\varphi_k((n_1, n_2, \cdots, n_k)) = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \cdots + \frac{1}{n_k}}}}$$

for each $(n_1, n_2, \cdots, n_k) \in \mathbb{N}^k$. Then we notice that

$$(*) \quad \varphi_{2k+1}((n_1, n_2, \cdots, n_{2k+1})) - \varphi_{2k}((n_1, n_2, \cdots, n_{2k})) < \frac{1}{4^{k-1}} \frac{1}{n_{2k+1}}$$

for each natural number $k$.

Let $B(\mathbb{N})$ denote the Baire’s zero-dimensional space with respect to $\mathbb{N}$. Let $\varphi$ be a function from $B(\mathbb{N})$ for which

$$\varphi((n_1, n_2, \cdots)) = \lim_{k \to \infty} \varphi_k((n_1, n_2, \cdots, n_k))$$

for each $(n_1, n_2, \cdots) \in B(\mathbb{N})$. Then it follows that $\varphi$ is a homeomorphism between $B(\mathbb{N})$ and the space $\mathbb{P} \cap (0, 1)$.

Let $M_Q$ denote the rational points of $M$ and $M_P$ the irrational ones. Put $K_0 = M_Q \times \mathbb{P}$ and $K_1 = \{(p, p) \mid p \in M_P\}$. They are disjoint closed sets of $M \times \mathbb{P}$. Let $U$ be any open set of $M \times \mathbb{P}$ containing $K_0$. We need only to show that $\overline{U} \cap K_1 \neq \emptyset$.

Put

$$q_0 = 0, \quad p_0 = \varphi((1, 1, \cdots)),$$

where $\varphi$ is the above homeomorphism between $B(\mathbb{N})$ and the space $\mathbb{P} \cap (0, 1)$. Since $(q_0, p_0) \in K_0 \subset U$, there exist $m_0$ and $n_0 \in \mathbb{N}$ such that $S_{\frac{1}{m_0}}(q_0) \times S'_{\frac{1}{n_0}}(p_0) \subset U$ where $S'_{\frac{1}{n_0}}(p_0) = S_{\frac{1}{n_0}}(p_0) \cap \mathbb{P}$. Pick a natural number $x_0 > m_0$. Put

$$q_1 = \varphi_1(x_0), \quad p_1 = \varphi((x_0, 1, 1, \cdots)).$$

Since $(q_1, p_1) \in K_0 \subset U$, there exist $m_1$ and $n_1 \in \mathbb{N}$ such that $S_{\frac{1}{m_1}}(q_1) \times S'_{\frac{1}{n_1}}(p_1) \subset U$. Pick a natural number $x_1 > m_1$.

Assume we have reached the $k$th step in this construction, and have constructed $x_0, x_1, \cdots, x_{k-1}$. Put

$$q_k = \varphi_k((x_0, x_1, \cdots, x_{k-1})), \quad p_k = \varphi((x_0, x_1, \cdots, x_{k-1}, 1, 1, \cdots)).$$
Since $\langle q_k, p_k \rangle \in K_0 \subset U$, there exist $m_k$ and $n_k \in \mathbb{N}$ such that $S_{\frac{1}{m_k}}(q_k) \times S'_{\frac{1}{n_k}}(p_k) \subset U$.

Pick a natural number $x_k > m_k$.

By induction, we can construct a point $x = \varphi(x_0, x_1, \cdots)$ such that $\langle x, x \rangle \in K_1$. For any positive number $\varepsilon$, $\{x\} \times S'_\varepsilon(x)$ is a neighborhood of $\langle x, x \rangle$. Since $\varphi$ is continuous, there exists a positive and even integer $k(=2j)$ such that $\varphi(B_k(\langle x_0, x_1, \cdots \rangle)) \subset S'_\varepsilon(x)$. Therefore

$$p_k = \varphi(\langle x_0, x_1, \cdots, x_{k-1}, 1, 1, \cdots \rangle) \in S'_\varepsilon(x).$$

On the other hand,

$$q_k = \varphi_k(\langle x_0, x_1, \cdots, x_{k-1} \rangle) < x < \varphi_{k+1}(\langle x_0, x_1, \cdots, x_k \rangle) = q_{k+1}$$

because $k$ is even. And

$$q_{k+1} - q_k < \frac{1}{4^{j-1}} \cdot \frac{1}{x_k} < \frac{1}{x_k} < \frac{1}{m_k}$$

by (*). Then $x \in S_{\frac{1}{m_k}}(q_k)$. Therefore

$$\langle x, p_k \rangle \in (\{x\} \times S'_\varepsilon(x)) \cap \left(S_{\frac{1}{m_k}}(q_k) \times S'_{\frac{1}{n_k}}(p_k)\right).$$

Since $S_{\frac{1}{m_k}}(q_k) \times S'_{\frac{1}{n_k}}(p_k) \subset U$, every neighborhood of $\langle x, x \rangle$ hits $U$. \qed

References

