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Certain covering-maps and \(k\)-networks

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The characterization for nice images of metric spaces is one of the most important problems in General Topology. Various kinds of characterizations have been obtained by means of certain \(k\)-networks. For a survey in this field, see [T5], for example.

In this paper, we shall introduce a general type of covering-maps, \(\sigma\)-(P)-maps associated with certain covering properties (P), in terms of \(\sigma\)-maps defined by [L1]. Then, we unify lots of characterizations and obtain new ones by means of these maps.

All spaces are regular and \(T_1\), and all maps are continuous and onto.

Let \(\mathcal{P}\) be a cover of a space \(X\). Let (P) be a certain covering-property of \(\mathcal{P}\). Let us say that \(\mathcal{P}\) has property \(\sigma\)-(P) if \(\mathcal{P}\) can be expressed as \(\bigcup\{\mathcal{P}_i : i \in N\}\), where each \(\mathcal{P}_i\) is a cover of \(X\) having the property (P) such that \(\mathcal{P}_i \subset \mathcal{P}_{i+1}\), and \(\mathcal{P}_i\) is closed under finite intersections. (Sometimes, we may assume that \(X \in \mathcal{P}_i\). When \(\mathcal{P} = \mathcal{P}_i = \mathcal{P}_{i+1}\) for all \(i \in N\), we shall say that \(\mathcal{P}\) has property (P) (instead of \(\sigma\)-(P)).

In this paper, we shall restrict (P) to the covering-property which is (*): Locally finite; Countable; Locally countable; Star-countable; or Point-countable.

Let us say that a map \(f : X \to Y\) is a \(\sigma\)-(P)-map (resp. (P)-map) if, for some base \(B = \{B_\alpha : \alpha\}\) in \(X\), the family \(f(B) = \{f(B_\alpha) : \alpha\}\) has property \(\sigma\)-(P) (resp. (P)).

**Remark 1.** In the above definition, we assume that the family \(f(B) = \{f(B_\alpha) : \alpha\}\) is to be interpreted in the strict " indexed " sense, hence, the sets \(f(B_\alpha)\) are not required to be different. Thus, by the restriction (*), the base \(B = \{B_\alpha : \alpha\}\) must be at least point-countable, and \(f\) be an \(s\)-map (i.e., every \(f^{-1}(y)\) is separable). When \(f(B)\) is \(\sigma\)-locally finite, then \(X\) is a metrizable space with the \(\sigma\)-locally finite base \(B\); \(Y\) is a \(\sigma\)-space with the \(\sigma\)-locally finite network \(f(B)\); and \(f^{-1}(L)\) is Lindelöf for every Lindelöf subset \(L\) of \(Y\). When \(f(B)\) is locally countable or star-countable, then \(X\) is a locally separable, metrizable space with the locally countable base \(B\).

For map \(f : X \to Y\), the following hold in view of the above.

(a) If \(f\) is a \(\sigma\)-(locally finite)-map, then \(X\) is metrizable.

(b) If \(f\) a (locally countable)-map or a (star-countable)-map, then \(X\) is locally separable, metrizable.

(c) (i) \(f\) is a (countable)-map if \(X\) is separable metric.

(ii) \(f\) is a (locally-finite)-map if \(X\) and \(Y\) are discrete.

We do not consider a trivial case of (locally finite)-maps.

S. Lin [L1] introduced the concept of \(\sigma\)-maps; that is, a map is a \(\sigma\)-map if it is a \(\sigma\)-(locally finite)-map. Related to \(\sigma\)-maps, let us review certain maps which are useful in the theory of networks. K. Nagami [N] introduced a \(\sigma\)-map \(f : X \to Y\) in the following sense: For every \(\sigma\)-locally finite open cover \(\mathcal{G}\) of \(X\), \(f(\mathcal{G})\) has a refinement \(\mathcal{F}\) such that \(\mathcal{F}\) is a \(\sigma\)-locally finite closed cover of \(Y\). Let us call such a map \(f\) a weak \(\sigma\)-map here, but we need not the closedness of the cover \(\mathcal{F}\). Related to \(\sigma\)-maps of [N], E. Michael [E1] (or [E2]) defined a \(\sigma\)-locally finite map \(f : X \to Y\) as follows: Every \(\sigma\)-locally finite (not
necessarily open) cover of $X$ has a refinement $\mathcal{P}$ such that $f(\mathcal{P})$ is a $\sigma$-locally finite cover of $Y$.

The following implication holds: $\sigma$-maps $\rightarrow$ $\sigma$-locally finite maps $\rightarrow$ weak $\sigma$-maps, but each converse need not hold; see Remark 2 below.

For a cover $\mathcal{P}$ of a space $X$, we recall the following definitions. These are generalizations of bases. For a survey around $k$-networks, see [T5], for example.

$\mathcal{P}$ is a $k$-network if, for any compact set $K$ and for any open set $U$ such that $K \subset U$, $K \subset \cup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$. (When $K$ is a single point, such a cover $\mathcal{P}$ is called a network (or net)). As is well-known, a space $X$ is called an $\aleph_0$-space (resp. $\aleph_0$-space) if $X$ has a $\sigma$-locally finite $k$-network (resp. countable $k$-network).

$\mathcal{P}$ is a $cs$-network (resp. $cs*$-network) if, for each $x \in X$, each nbd $V$ of $x$, and each convergent sequence $L$ with the limit point $x$, there exists $P \in \mathcal{P}$ such that $x \in P \subset V$, and $P$ contains $L$ eventually (resp. frequently).

$\mathcal{P} = \cup \{P_x : x \in X\}$ with each $P_x$ closed under finite intersections is a weak base if (a) each $P \in P_x$ contains $x$; (b) for each $x \in X$, and each nbd $G$ of $x$, there exists $P(x) \in P_x$ such that $P(x) \subset G$; and (c) $G \subset X$ is open in $X$ if, for each $x \in G$, there exists $P(x) \in P_x$ such that $P(x) \subset G$. A space $X$ is called $g$-metrizable [S2] if $X$ has a $\sigma$-locally finite weak base.

$\mathcal{P} = \cup \{P_x : x \in X\}$ satisfying the above (a) and (b) is an $sn$-network [L2] if, for each $x \in X$, any $P \in P_x$ is a sequential neighborhood of $x$ (i.e., any sequence converging to $x$ is eventually contained in $P$).

**Remark 2.** (i) A map $f : X \rightarrow Y$ is a weak $\sigma$-map if the following (a) or (b) holds.

(a) $f$ is a closed map such that $X$ is a $\sigma$-space.

(b) $f$ is an open map such that $Y$ is subparacompact.

(In fact, for case (a), every open cover $\mathcal{G}$ of $X$ has a refinement $\mathcal{P}$ which is a $\sigma$-locally finite closed network for $X$. But, $f(\mathcal{P})$ is a $\sigma$-closure preserving closed network for $Y$. Thus, $f(\mathcal{P})$ has a refinement which is a $\sigma$-discrete closed network $\mathcal{F}$ in view of the proof of [S1; Theorem]. Then, $\mathcal{F}$ is a $\sigma$-locally finite refinement of $f(\mathcal{G})$).

(ii) Let $f : X \rightarrow Y$ be a map. If (a) or (b) below holds, then $f$ is $\sigma$-locally finite ([M1] or [M2]). Conversely, if $f : X \rightarrow Y$ is $\sigma$-locally finite, then for any closed, and $\omega_1$-compact subset $L$ of $Y$ (i.e., every uncountable subset of $L$ has an accumulation point), $f^{-1}(L)$ is $\omega_1$-compact.

(a) $f$ is a closed map with every $f^{-1}(y)$ Lindelöf, and $X$ or $Y$ is subparacompact.

(b) $f(\mathcal{P})$ is $\sigma$-locally finite for some network $\mathcal{P}$ in $X$. (Thus, $X$ and $Y$ must be $\sigma$-spaces).

(iii) Let $f : X \rightarrow Y$ be a map such that $X$ is a $\sigma$-space. Then (a) $\leftrightarrow$ (b) $\leftrightarrow$ (c) holds. When $f$ is closed, (a), (b), and (c) are equivalent, and (a) and (c) are equivalent under $X$ being subparacompact. (In fact, these hold by means of (ii) and [E2; Proposition 2.2]).

(a) $f$ is a $\sigma$-locally finite map.

(b) $f(\mathcal{P})$ is $\sigma$-locally finite for some network $\mathcal{P}$ in $X$.

(c) Every $f^{-1}(y)$ is Lindelöf.

The above shows that every $\sigma$-locally finite image of a $\sigma$-space is a $\sigma$-space. But, every weak $\sigma$-image (actually, open $s$-image) of a metric space need not be a $\sigma$-space (by the Michael-Line).

For closed maps, we have the following. In (a) or (b), $f$ can not been weaken to be a
weak $\sigma$-map in view of (i).

(iv) For a closed map $f : X \to Y$ with $X$ metric, the following are equivalent.
   (a) $f$ is a $\sigma$-map.
   (b) $f$ is a $\sigma$-locally finite map.
   (c) $f$ is an $s$-map.
   (d) $X$ has a point-countable $k$-network consisting of closed subsets.
   (e) $X$ is an $\aleph$-space.

(Indeed, (a) $\to$ (b) $\to$ (c) is already shown. For (c) $\leftrightarrow$ (d), see [T2], For (c) $\to$ (a), since $f$ is a closed $s$-map with $Y$ paracompact, every $\sigma$-locally finite base for $X$ has a refinement $B$ such that $B$ is a base for $X$ and $f(B)$ is $\sigma$-locally finite in $Y$. (c) $\to$ (e) holds by [Ga; Theorem 1]).

Concerning characterizations for $\sigma$-spaces by means of maps, the following holds. (a) $\leftrightarrow$ (b); (a) $\leftrightarrow$ (d) $\leftrightarrow$ (e); and (a) $\leftrightarrow$ (c) is respectively due to [L1]; [N]; and [E1] or [E2].

(v) For a space $X$, the following are equivalent. In (b), (c), and (e), the map can be chosen to be one-to-one. In (d) and (e), the condition of the weak $\sigma$-map is essential; see (iii).

(a) $X$ is a $\sigma$-space.
(b) $X$ is the image of a metric space under a $\sigma$-map.
(c) $X$ is the image of a metric space under a $\sigma$-locally finite map.
(d) $X$ is the image of a metric space under a one-to-one, weak $\sigma$-map.
(e) $X$ is the image of a metric space under a weak $\sigma$-map $f$ such that $f^{-1}(x)$ is compact for every $x \in X$.

**Proposition:** For a map $f : X \to Y$, (1), (2), and (3) below hold.

1. The following are equivalent.
   (a) $f$ is a (point-countable)-map.
   (b) $X$ has a point-countable base, and $f$ is an $s$-map.
   (c) $X$ has a point-countable base, and $f(B)$ is point-countable for any point-countable base $B$ in $X$.

2. Let $X$ be locally separable, metric. Then the following are equivalent.
   (a) $f$ is a (locally countable)-map (resp. (star-countable)-map).
   (b) Each point $y \in Y$ has a nbd $V_y$ with $f^{-1}(V_y)$ (resp. each point $x \in X$ has a nbd $W_x$ with $f^{-1}(f(W_x)))$ separable in $X$.
   (c) $f(B)$ is locally countable (resp. star-countable) for any locally countable (resp. star-countable) base $B$ in $X$.
   (d) $f(B)$ is locally countable (resp. star-countable) for any star-countable base $B$ in $X$.

3. Let $X$ be locally separable, metric. Then the implications (a) $\to$ (b) $\to$ (c); and (d) $\to$ (e) $\to$ (b) and (c) hold. When $f$ is quotient, (a) $\sim$ (f) are equivalent.
   (a) $f$ is a (locally countable)-map.
   (b) $f^{-1}(L)$ is Lindelöf for every Lindelöf subset $L$ of $Y$.
   (c) $f$ is a (star-countable)-map.
   (d) $f$ is a $\sigma$-map.
   (e) $f$ is a $\sigma$-locally finite map.
   (f) $f^{-1}(L)$ is separable for every separable subset $L$ of $Y$.

(Indeed, (1) holds in view of Remark 1(i). (2) would be routinely shown (cf. [TX;
Proposition 1.1], but note that any star-countable base for $X$ is locally countable. We show (3) holds, but the implication (a) $\rightarrow$ (b) $\rightarrow$ (c) is routine, and (d) $\rightarrow$ (e) is already shown. (e) $\rightarrow$ (b) holds by Remark 2(ii). For (e) $\rightarrow$ (c), let $f$ be a $\sigma$-locally finite map, and let $B$ be a $\sigma$-locally finite base for $X$ consisting of hereditarily Lindelöf subsets. Then, $B$ has a refinement $F$ such that $f(F)$ is $\sigma$-locally finite. For each $B \in B$, $f(B)$ meets only countably many $f(F_n) \in f(F)$ with $F_n \in F$, for $f(B)$ is Lindelöf. While, each Lindelöf subset $F_n$ meets only countably many elements of $B$. Hence, each $f(B)$ meets only countably many elements of $f(B)$. Then, $f(B)$ is a star-countable cover of $Y$. Thus, $f$ is a (star-countable)-map. For the latter part, let (c) hold. Since $f$ is quotient, $Y$ is determined by a star-countable cover $C = f(B)$ for some base $B$ in $X$. Thus, as in the proof of [T3; Theorem 1], $Y$ is the topological sum of subspaces, where each subspace is a countable union of elements of $C$. Thus, the cover $C$ is locally countable and $\sigma$-locally finite in $Y$. Thus (c) implies (a), (d), and (f). (f) $\rightarrow$ (c) would be routine.

Remark 3. In view of (a) $\leftrightarrow$ (d) in (2), (locally countable)-maps (resp. (star-countable)-maps) coincide with locally countable maps (resp. star-countable maps) discussed in [TX].

We note that it is impossible to replace “any star-countable base” by “any locally countable base” in (d) for the parenthetic part.

Corollary 1. For a quotient map $f : X \to Y$ such that $X$ is a locally separable, metric space, the following are equivalent.

(a) $f$ is a (locally countable)-map.
(b) $f$ is a (star-countable)-map.
(c) $f$ is a $\sigma$-map.
(d) $f$ is a $\sigma$-locally finite map.
(e) $f^{-1}(L)$ is Lindelöf for every Lindelöf subset $L$ of $Y$.
(f) $f^{-1}(S)$ is separable for every separable subset $S$ of $Y$.

For a map $f : X \to Y$, let us recall the following definitions around compact-covering maps.

$f$ is sequence-covering [S1], if each convergent sequence in $Y$ is the image of some convergent sequence in $X$.

$f$ is sequence-covering of [GMT], if each convergent sequence $L$ in $Y$ is the image of some compact subset of $X$. In this paper, let us call such a sequence-covering map of [GMT] pseudo-sequence-covering as in [ILuT]. (When “convergent sequence $L$” is replaced by “compact set $L$”, as is well-known, such a map $f$ is called compact-covering).

$f$ is subsequence-covering [LLuD], if for each $y \in Y$, and each sequence $L$ in $Y$ converging to $y$, there exists a convergent sequence $K$ in $X$ such that $f(K)$ is a subsequence of $L$.

$f$ is 1-sequence-covering [L3], if for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each sequence $K$ converging to $y$, there exists a sequence $L$ converging to $x$ such that $f(L) = K$. For 1-sequence-covering maps, see [LY], for example.

Let $f : X \to Y$ be a map such that $X$ is sequential. If $f$ is pseudo-sequence-covering, then $f$ is subsequence-covering. Also, $f$ is quotient iff $f$ is subsequence-covering such that $Y$ is sequential ([T4]).
Lemma: Let $f: X \to Y$ be a $\sigma$-(P)-map. Then the following hold.

(i) If $f$ is quotient, then $Y$ has a $k$-network having property $\sigma$-(P).

(ii) If $f$ is subsequence-covering (resp. sequence-covering; 1-sequence covering), then $Y$ has a $cs*$-network (resp. $cs$-network; $sn$-network) having property $\sigma$-(P).

(Indeed, for (i), let $f(B)$ have property $\sigma$-(P) for some base $B$ in $X$. Let $K \subset U$ with $K$ compact and $U$ open in $Y$. Since $f|f^{-1}(U)$ is quotient, $U$ is determined by a point-countable cover $U = \{f(B) : B \in B, f(B) \subset U\}$. Thus, $K \subset \cup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset U$ by [GMT: Proposition 2.1]. This shows that $f(B)$ is a $k$-network. (ii) is routine).

Every $\sigma$-image of a metric space is a $\sigma$-space, but need not be an $\aleph$-space in view of Remark 2(v). But, we have the following by the previous lemma and Corollary 1.

Corollary 2. (1) Every quotient $\sigma$-image of a metric space is an $\aleph$-space.

(2) Every quotient $\sigma$-locally finite image of a locally separable, metric space is an $\aleph$-space.

Remark 4. (i) Every (1-sequence-covering) quotient $\sigma$-locally finite image of a metric space need not be an $\aleph$-space (by the open finite-to-one image of a metric space in Example 3.2 in [T1]). This shows that the local separability of the domain is essential in Corollary 2(2).

(ii) Every quotient, finite-to-one, weak $\sigma$-image of a locally compact, metric space need not be an $\aleph$-space, and need not satisfy each of (e) $\sim$ (f) in Corollary 1, even if the range is a paracompact $\sigma$-space (by the example in [LT; Remark 14(2)]). Hence, we can not replace " $\sigma$-locally finite " by " weak $\sigma$ " in Corollary 1 and Corollary 2(2).

The nice characterization for quotient $s$-images of metric spaces was obtained by [GMT], in 1984. Since then, lots of characterizations for certain images of metric spaces have been obtained by many topologists by using the analogous methods to the proof of [GMT; Theorem 6.1]. To unify these characterizations, we have General Theorem below. This theorem (resp. its latter part) could be shown by modifying the proof of [Li; Lemma 2.1] (resp. [L2; Theorem]). But, we shall omit the proof here.

General Theorem: For a space $X$, the following are equivalent. Also, it is possible to replace " subsequence-covering " by " pseudo-sequence-covering " in (b).

(a) $X$ has a $cs*$-network (resp. $cs$-network; $sn$-network) having property $\sigma$-(P).

(b) $X$ is the subsequence-covering (resp. sequence-covering; 1-sequence-covering) $\sigma$-(P)-image of a metric space.

The following is due to [Li]. Also, an analogous result for a $\sigma$-(locally countable)-property could be valid.

Corollary 3. A space $X$ is an $\aleph$-space iff $X$ is the sequence-covering $\sigma$-image of a metric space. Also, it is possible to replace " sequence-covering " by " subsequence-covering " or " pseudo-sequence-covering " (cf. [L1]).

In the following, (a) $\leftrightarrow$ (b) is due to [L2] (resp. [LLu]; [L3]).

Corollary 4. For a space $X$, the following are equivalent. Also, it is possible to
replace "subsequence-covering" by "pseudo-sequence-covering" in (b) and (c).

(a) $X$ has a point-countable $cs*$-network (resp. $cs$-network; $sn$-network).
(b) $X$ is the subsequence-covering (resp. sequence-covering; 1-sequence-covering), $s$-image of a metric space.
(c) $X$ is the subsequence-covering (resp. sequence-covering; 1-sequence-covering), (point-countable)-image of a metric space.

In the following, (1) is (well) known, and some parts of (2) are shown in [TX].

**Corollary 5.** For a space $X$, the following hold. Also, it is possible to replace "subsequence-covering" by "pseudo-sequence-covering" in (1) and (2), and to replace "locally countable" by "star-countable" in (2).

(1) $X$ has a countable $cs*$-network (resp. $cs$-network; $sn$-network) iff $X$ is the subsequence-covering (resp. sequence-covering; 1-sequence-covering) image of a separable metric space.

(2) $X$ has a locally countable $cs*$-network (resp. $cs$-network; $sn$-network) iff $X$ is the subsequence-covering (resp. sequence-covering; 1-sequence-covering), (locally-countable)-image of a locally separable metric space.

**Remark 5.** Related to (1), let us recall a result that, for a space $X$, $X$ has a countable $cs*$-network $\iff X$ has a countable $cs$-network $\iff X$ is an $\aleph_0$-space. Concerning (2), when $X$ is sequential, then $X$ has a locally countable $cs*$-network $\iff X$ has a locally countable $cs$-network $\iff X$ is the topological sum of $\aleph_0$-spaces. Also, we can replace "locally countable" by "star-countable" (cf. [T5]).

**Corollary 6.** (1) A space $X$ is a sequential space with a point-countable $cs*$-network iff $X$ is the quotient $s$-image of a metric space ([T4] or [L2]).

(2) A space $X$ is a sequential space with a point-countable $cs$-network iff $X$ is the sequence-covering, quotient $s$-image of a metric space ([LLu]).

(3) A space $X$ has a point-countable weak base iff $X$ is the 1-sequence-covering, quotient $s$-image of a metric space ([L2]).

**Corollary 7.** For a space $X$, the following are equivalent. It is possible to replace "locally countable" by "star-countable" in (a) or (b). Moreover, if we replace "$cs*$-network" by "$cs$-network (resp. $sn$-network)" in (a), then the same equivalence holds by adding the prefix "sequence-covering (resp. 1-sequence-covering)" before "quotient" in (b) $\sim$ (e).

(a) $X$ is a sequential space with a locally countable $cs*$-network.

(b) $X$ is the quotient (locally-countable)-image of a locally separable metric space.

(c) $X$ is the quotient $\sigma$-image of a locally separable metric space.

(d) $X$ is the quotient $\sigma$-locally finite image of a locally separable metric space.

(e) $X$ is the image of a locally separable metric space under a quotient map $f$ such that $f^{-1}(S)$ is separable for every separable (or Lindelöf) subset $S$ of $Y$.

**Corollary 8.** (1) A space $X$ is a $k$-and-$\aleph$-space iff $X$ is the (sequence-covering) quotient $\sigma$-image of a metric space.

(2) A space $X$ is $g$-metrizable iff $X$ is the quotient, 1-sequence-covering, $\sigma$-image of a metric space.
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