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Some Dynamical Aspects of Vortices in the Ginzburg-Landau Equation

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1 Introduction

We are concerned with the time-dependent Ginzburg-Landau (G-L) equation in a planar bounded domain Ω with the Neumann boundary condition:

$$u_t = \Delta u + \frac{1}{\epsilon^2}(1-|u|^2)u = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.2}$$

where $u = (u_1(x,t), u_2(x,t))^T$, ϵ is a small positive parameter and $\partial/\partial\nu$ denotes the outer normal derivative on the smooth boundary $\partial\Omega$. We easily verify that this equation is a gradient equation of the functional

$$E_\epsilon(u) := \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\epsilon^2}(1-|u|^2)^2 \right\} dx \tag{1.3}$$

in $H^1(\Omega;\mathbb{R}^2)$. Thus the asymptotic state of any solution to (1.1) are determined by the elliptic equation,

$$\Delta u + \frac{1}{\epsilon^2}(1-|u|^2)u = 0, \quad x \in \Omega, \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad x \in \partial\Omega, \tag{1.5}$$

which is called a Ginzburg-Landau equation. We suppose that the domain Ω is simply connected. For convenience of notation we allow the complex expression $u = u_1(x,t) + iu_2(x,t)$ for a solution $u = (u_1(x,t), u_2(x,t))^T$ to (1.1).

In order to know the asymptotic state of the solution, it suffices to investigate (1.4). We, however, observe some interesting transient dynamics of a solution to (1.1) with appropriate initial data if ϵ is sufficiently small. Before stating it, we first note that since a solution $u(x,t)$ of (1.1) has two components, a vector field on Ω can be defined by the solution for each time $t > 0$ and that the zero set of the solution at time t is consist of discrete points in generic. Then the degree of each zero y can be defined by $\deg(y, \partial B_\rho(y))$, where $B_\rho(y)$ is a disk centered at y with a small radius ρ. Zeros of the equation (1.1) (or (1.4)) are called vortices.

Now observe some characteristic dynamics of a solution to (1.1). When ϵ (> 0) is very small, the coefficient of the nonlinear term is so large that $|u(t,x)|$ goes close to 1 quickly as t grows in a finite time, except for a small neighborhood of zeros (vortices) if exist. Hence a sharp layer arises around each vortex and we see from the expression of E_ϵ that a large contribution to the energy comes from the neighborhood of the vortices. By virtue of the continuity of the solution and invariance of the degree around each zero, these vortices persist unless they collide one another or they reach the boundary. This vortex dynamics was studied by using formal perturbation methods (see [4], [18]). In addition the collision and annihilation of vortices can be verified by numerical computations (for instance, see [17]).

We are interested in a mathematically rigorous description of the vortex dynamics or the motion law of vortices. Lin [12, 13] and Jerrard-Soner [5] derived a singular limit equation describing the motion law as a sequence ϵ_n tends to zero, when the boundary condition is given by

$$u = g(x), \quad x \in \partial\Omega, \quad \deg(g, \partial\Omega) = d \neq 0, \tag{1.6}$$

later in [14] the limit equation for the Neumann case was done. Those results shed light on the study of vortex dynamics. We, however, see that since the limit equations are given in an implicit form there, it is not so easy to handle them except for a special case (for instance, the single vortex case in a disk). On the other hand, in the Neumann case, Jimbo-Morita [9] succeeded to write the limit equation using the Green function of the Laplacian (with Dirichlet zero condition) and the Robin function of it. By virtue of this nice form the vortex dynamics can be investigated in a general domain. In particular the collision of two vortices with opposite signs of degree and the annihilation on the boundary of a vortex are verified in the limit equation. We introduce the limit equation obtained in [9] together with some dynamical aspects of it in the next section.

We remark that the results of [9] for the dynamics in the limit equation do not guarantee that those hold for the original equation. In order to prove that such results are certainly true in the equation (1.1) -(1.2), we have to develop the study further. Nevertheless we believe that their study would be helpful in the future.

As for the annihilation, we give a remark on the study in [1], where it is shown that for an appropriately chosed initial data a solution in the whole space $\mathbb{R}^2$ eventually goes to a constant solution with modulus one in the uniform topology, thus all the zeros of the solutions disappear in a finite time. There, however, is no analytical description of the annihilation in their study.

We finally remark on the dynamical law of vortices for small but positive ϵ. In this direction of the study we refer to [10] and [16], where the motion laws locally in time are derived with the aid of approximate solutions.

2 Singular limit equation

We assume that the domain $\Omega \subset \mathbb{R}^2$ is simply connected. Let $u^\epsilon(x,t)$ be a solution of (1.1), (1.2). We rescale the time variable as

$$v^\epsilon(x,t) := u^\epsilon(x, |\log\epsilon| t). \tag{2.1}$$

Then v^ϵ satisfies

$$\frac{1}{|\log\epsilon|} v^\epsilon_t = \Delta v^\epsilon + \frac{1}{\epsilon^2}(1-|v^\epsilon|^2)v^\epsilon \tag{2.2}$$

with the Neumann boundary condition. Note that this scaling was used in [20], which is the first mathematically rigorous study of the vortex motion.

Let $G(x,p)$ be the Green function of Δ with the Dirichlet condition, that is,

$$\begin{cases} \Delta_x G = 0, & x \in \Omega \setminus \{p\}, \\ G = 0, & x \in \partial\Omega, \\ G(x,p) \sim \log|x-p| + O(1), & x \approx p, x \neq p \end{cases} \tag{2.3}$$

We decompose the Green function as

$$G(x,p) = \log|x-p| + S(x,p)$$

where $S(x,p)$ is a harmonic function over Ω. We put

$$S(x) := S(x,x)$$

which is called the Robin function of $G(x,p)$.

We denote a disk with radius ρ centered at $x = y$ by

$$B_\rho(y) := \{|x-y| < \rho\},$$

and denote the degree of a function $u = (u_1(x), u_2(x))^T$ around $x = y$ by $\deg(u(\cdot); \partial B_\rho(y))$. We write the configuration of m vortices by

$$\boldsymbol{y}(t) := (y^{(1)}(t), y^{(2)}(t), \cdots, y^{(m)}(t)) \in \hat{\Omega} := \Omega \times \Omega \times \ldots \times \Omega.$$

Then the limit equation describing the motion law of vortices is given as follows:

Theorem 2.1 *Let $v^\epsilon(x,t)$ be a solution of (2.2) with Neumann boundary condition. Then for appropriately chosen initial data v^ϵ_0, a finite number $T > 0$ and a subsequence $\epsilon_n, \epsilon_n \to 0$ such that each solution $v^\epsilon(x,t)$ has distinct m zeros $y^{(j)}_\epsilon(t), j = 1, \ldots, m$, for $t \in [0,T)$ and by taking $\epsilon_n \to 0$, those zeros converge to $y^{(j)}(t), j = 1, \ldots, m$, which are solutions of*

$$\dot{y}^{(j)}(t) = \nabla S(y^{(j)}(t)) + 2\sum_{k\neq j} d_k d_j \nabla_x G(y^{(j)}(t), y^{(k)}(t)) \quad (1 \leq j \leq m). \tag{2.4}$$

where $\dot{} = d/dt$,

$$d_j = \deg(u^\epsilon(\cdot,t); B_\rho(y_\epsilon^{(j)})), \qquad j = 1, \ldots, m.$$

Moreover we can write the above equation as

$$\begin{aligned} \dot{\boldsymbol{y}} &= -\frac{1}{\pi}\mathrm{grad}V(\boldsymbol{y}), \\ V(\boldsymbol{y}) &:= -\pi\sum_{j=1}^{m} S(y^{(j)}) - \pi\sum_{j=1}^{m}\sum_{k\neq j} d_k d_j G(y^{(j)}, y^{(k)}). \end{aligned} \tag{2.5}$$

For the proof see [9]. Here we state some dynamical aspects of the vortices showed up in the study of (2.4).

First consider the single vortex case, that is, $m = 1$ in (2.4). Then the equation is written as

$$\dot{y} = \nabla S(y).$$

It is known that the Robin function satisfies

$$\Delta S = 4e^{2S} \quad (x \in \Omega), \qquad S \to -\infty \quad (x \to \partial\Omega)$$

(see [19]) for a simply connected domain Ω. Therefore we easily see that any equilibrium solution of this equation is unstable. Moreover if the domain is convex, the result in [3] tells level sets of the Robin function are strictly convex. Thus there is a unique equilibrium state and any solution away from the equilibrium reaches the boundary in a finite time.

Next we consider two vortices with opposite sings of degree. Then we can prove that there is an invariant region $U_0 \subset \Omega$ for the solutions $y^{(1)}(t), y^{(2)}(t)$ around a critical point of $S(x)$, that is, if $y^{(1)}(0), y^{(2)}(0) \in U_0$, then $y^{(1)}(t), y^{(2)}(t) \in U_0$ for $t > 0$ as long as the solutions are defined. In addition it is proved that these solutions $y^{(1)}(t)$ and $y^{(2)}(t)$ in U_0 must collide in a finite time. For the detail of the argument, see [9].

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