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Kyoto University
Interfaces of Solutions
to an Inhomogeneous Filtration Equation
with Absorption *

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Abstract

We review some recent results, obtained jointly with R. Kersner and G. Reyes, concerning qualitative properties of solutions to the Cauchy problem for the equation \( \rho(x)u_t = (u^m)_{xx} - c_0 u^p \), where \( m > 1 \) and \( c_0, p > 0 \). The initial data are nonnegative with compact support and the density \( \rho(x) > 0 \) satisfies suitable decay conditions as \( |x| \to \infty \).

1 Introduction

We discuss some recent results (see [KRT]) concerning support properties of solutions to the Cauchy problem:

\[
\begin{cases}
\rho(x)u_t = (u^m)_{xx} - c_0 u^p & \text{in } \mathbb{R} \times (0, \infty) \\
 u = u_0 & \text{in } \mathbb{R} \times \{0\},
\end{cases}
\]

where

(i) \( m > 1, \ c_0 > 0, \ p > 0 \);

(ii) \( \rho \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \ \rho > 0 \) in \( \mathbb{R} \);

(iii) \( u_0 \in C_c(\mathbb{R}), \ u_0 \geq 0 \) in \( \mathbb{R} \).

(by \( C_c(\mathbb{R}) \) we denote the set of continuous, compactly supported functions on \( \mathbb{R} \)). A typical choice for the density \( \rho = \rho(x) \) is

\[
(1.2) \quad \rho(x) = \frac{\bar{\rho}}{(1 + |x|)^k} \quad (\bar{\rho}, k > 0).
\]

Physical motivations of the model can be found in [KR1], [KR2], [GuHP] and references therein; for instance, it arises in connection with a parabolic system suggested by plasma

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physics (see [BK]). Let us also mention that the related problem of uniqueness of solutions
to the positive Cauchy problem for diffusion equations with variable density has raised
much attention in recent years (e.g., see [E], [GuHP], [KKT], [T] and references therein).

Solutions to problem (1.1) are always meant in the following weak sense.

**Definition 1.1** By a solution to problem (1.1) we mean any bounded, nonnegative and
continuous function $u$ on $\mathbb{R} \times [0, \infty)$ such that

$$\int_{D^{\tau}} \{ \rho u \phi_t + u^m \phi_{xx} - c_0 u^p \phi \} \, dx \, dt =$$

$$= \int_{-r}^{r} \rho u(x, \tau) \phi(x, \tau) \, dx - \int_{-r}^{r} \rho u_0(x) \phi(x, 0) \, dx +$$

$$+ \int_{0}^{\tau} [u^m(r, t) \phi_x(r, t) - u^m(-r, t) \phi_x(-r, t)] \, dt$$

for any $r, T > 0$, $\tau \in [0, T]$ and $\phi \in C^{2,1}_{x,t}(D^T)$, $\phi \geq 0$ such that $\phi(-r, t) = \phi(r, t) = 0$
in $[0, T]$ (here $D^r := (-r, r) \times (0, \tau]$, $\tau \in [0, T]$).

Subsolutions of problem (1.1) are similarly defined, replacing "$=$" by "$\geq" in the
above equality. On the other hand, supersolutions are meant in the following more re-
stricted sense (introduced in [B] to deal with the case $0 < p < 1$).

**Definition 1.2** By a supersolution to problem (1.1) we mean any solution $\bar{u}$ to the prob-
lem

$$\left\{ \begin{array}{l}
\rho(x) u_t = (u^m)_{xx} - c_0 u^p + h \\
u(x, 0) = \bar{u}_0(x),
\end{array} \right. \quad in \ \mathbb{R} \times (0, T]$$

for some $h \in L^\infty(S^T)$, $h \geq 0$, $\bar{u}_0 \geq u_0$ and $T > 0$.

Finally, let $u \geq 0$ be any solution to problem (1.1); its interfaces are defined as follows:

$$\zeta^+(t) := \sup \{ x : u(x, t) > 0 \}, \quad \zeta^-(t) := \inf \{ x : u(x, t) > 0 \} \quad (t \geq 0).$$

We also set:

$$\zeta(t) := \sup \{ |x| : u(x, t) > 0 \} = \max \{|\zeta^-(t)|, |\zeta^+(t)|\}.$$
2 Previous Results

Let us recall some well-known results for problem (1.1) in two particular cases: \( \rho = 1 \), \( c_0 > 0 \), respectively \( \rho = \rho(x) \), \( c_0 = 0 \).

2.1 Porous Medium Equation with Absorption

When \( \rho = 1 \), \( c_0 > 0 \) it is well known that (see [BKP], [CMM], [Ka]):

- the following estimates hold:
  \[ |\zeta^\pm(t)| \leq \text{constant} \quad \text{for} \quad 1 \leq p < m, \]
  \[ |\zeta^\pm(t)| \sim \log t \quad \text{for} \quad p = m, \]
  \[ |\zeta^\pm(t)| \sim t^\alpha, \quad \alpha > 0 \quad \text{for} \quad p > m \]

  \( (\text{localization of the solution if} \ p < m, \ \text{respectively positivity if} \ p \geq m) \);

- if \( 0 < p < 1 \), there is extinction of the solution in finite time - namely, there exists \( T^* \in (0, \infty) \) such that \( u \equiv 0 \) in \( \mathbb{R} \times (T^*, \infty) \).

Let us mention that the case \( \rho = 1 \), \( c_0 = c_0(x) \) was also investigated (see [PT1], [PT2]).

2.2 Inhomogeneous Porous Medium Equation

New interesting phenomena arise when \( \rho \) depends on the space variable. Consider the case \( \rho = \rho(x) \), \( c_0 = 0 \). The main qualitative novelty is that, if \( \rho(x) \to 0 \) "fast enough" as \( |x| \to \infty \), interfaces can run off in finite time - namely there possibly exists \( \bar{T} \in (0, \infty) \) such that

\[ \zeta(t) \to \infty \quad \text{as} \quad t \to \bar{T}^- . \]

In fact, the following holds ([KK], [GuHP], [GKK]; see also [GK], [P]).

**Theorem 2.1** Let \( c_0 = 0 \), \( |x| \rho(x) \in L^1(\mathbb{R}) \). Then for any solution to problem (1.1) there exists \( \bar{T} \in (0, \infty) \) such that

\[ \zeta(t) \to \infty \quad \text{as} \quad t \to \bar{T}^- . \]

The above theorem is related to the following convergence result ([KR2]).

**Theorem 2.2** If \( \rho \in L^1(\mathbb{R}) \), there holds

\[ u(\cdot, t) \to \bar{u} := \frac{\|\rho u_0\|_1}{\|\rho\|_1} \quad \text{as} \quad t \to \infty , \]

the convergence being uniform on compact subsets of \( \mathbb{R} \).
Sketch of the Proof of Theorem 2.1: If $\text{supp } u(\cdot, t)$ is compact for any $t > 0$, the following equalities can be proven:

\[ \int_{\mathbb{R}} \rho(x)u(x,t)dx = \int_{\mathbb{R}} \rho(x)u_0(x)dx, \]
\[ \frac{1}{t} \int_{0}^{\infty} x\rho(x)[u(x,t) - u_0(x)]dx = \frac{1}{t} \int_{0}^{t} u^{m}(0,\tau)d\tau. \]

$(t > 0)$. Since $u(\cdot, t) \rightarrow \bar{u} > 0$ as $t \rightarrow \infty$ by Theorem 2.2, a contradiction arises if $\zeta(t) < \infty$ for any $t > 0$. This proves the result. \[\square\]

Remark 2.3 The above proof cannot be adapted to the case $c_0 > 0$.

The criterion of Theorem 2.1 is extended to higher space dimension as follows. Let $\rho \sigma \in L^1(\mathbb{R}^n)$, where

\[ \sigma(x) := \begin{cases} |x| & \text{if } n = 1, \\ \log(|x|) & \text{if } n = 2, \\ |x|^{2-n} & \text{if } n \geq 3; \end{cases} \]

then $\text{supp } u$ becomes unbounded in finite time ([GuHP], [GiT]).

For the choice

\[ \rho(x) = \frac{\bar{\rho}}{(1 + |x|)^k} \]

it can be proved by comparison methods that blow-up occurs if and only if $k > 2$ (see [KK]). In fact, for any $k \leq 2$ there exist $a_0, b_0 > 0$ such that:

\[ |\zeta^{\pm}(t)| \sim a_0 t^{\frac{1}{2-k}} \text{ as } t \rightarrow \infty \text{ if } k < 2, \]
\[ |\zeta^{\pm}(t)| \sim e^{b_0 t} \text{ as } t \rightarrow \infty \text{ if } k = 2. \]

Hence critical value for the case $c_0 = 0$ is $k = 2$.

3 Results

Now the question arises: How do absorption and variable density cooperate to influence the situation depicted in Section 2? In particular, which phenomena arising when $\rho = \rho(x)$ and $c_0 = 0$ are structurally stable with respect to the parameter $c_0 \geq 0$?

As we shall see below, the following answer can be given:

(i) As in the case of constant $\rho$, there is localization of the solution if $p < m$, positivity if $p > m$. In the latter event, the behaviour of interfaces is the same as in the case $c_0 = 0$, yet with a different critical value of $k$, namely

\[ k^* := 2 \frac{p-1}{p - m} \]
(observe that $k^* \to 2$ as $p \to \infty$).

(ii) The convergence result in Theorem 2.2 is not structurally stable; in fact, solutions to problem (1.1) go to zero uniformly as $t \to \infty$ for any $c_0 > 0$.

The results summarized above are proved by comparing solutions to problem (1.1) with suitable subs- and supersolutions (possibly suggested by a proper splitting of domains). We only sketch below the proof of Theorem 3.9, referring the reader to [KRT] for complete proofs. As already remarked (see Remark 2.3) the techniques used for the case $c_0 = 0$ (which rely on mass conservation; see [KK]), cannot be adapted to the present situation.

Concerning the critical values $k = 2$ (if $c_0 = 0$) and $k = k^*$ (if $c_0 > 0$, $p > m$), observe that:

(i') the function $u(x, t) = f(xt^{-\frac{1}{2-k}})$ is a similarity solution to the equation

$$|x|^{-k}u_t = (u^m)_{xx}$$

if $k = 2$;

(ii') the function $u(x, t) = t^\alpha f(xt^{-\beta})$ with

$$\alpha := -\frac{2}{(p - m)(k^* - k)}, \quad \beta := \frac{1}{k^* - k}$$

is a similarity solution to the equation

$$|x|^{-k}u_t = (u^m)_{xx} - c_0 u^p$$

in the case $c_0 > 0$, $p > m$ if $k = k^*$.

Analogous results hold at $k = 2$ and $k = k^*$.

### 3.1 Well-Posedness and Comparison

The basic theory for problem (1.1) - as well as for initial-boundary value problems related to the differential equation in (1.1) - was studied in [RT]; in particular, this includes comparison results for solutions to the first boundary value and to the Cauchy-Dirichlet problems, in regions whose lateral boundaries may be curvilinear, which are needed to prove several statements listed below. Let us mention the following results.

**Theorem 3.1** Let $m > 1$, $p > 0$, $\rho \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\rho > 0$ and $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists a unique solution to the Cauchy problem (1.1).

**Theorem 3.2** Let $u$ be a subsolution, $\bar{u}$ a supersolution to the Cauchy problem (1.1). Then $u \leq \bar{u}$. 
3.2 Asymptotic Behaviour

As already mentioned, the convergence result in Theorem 2.2 is not structurally stable; moreover, there is extinction of the solution in finite time if $0 < p < 1$. In fact, the following result can be proven.

**Theorem 3.3** Let $u$ be any solution to problem (1.1). Then

$$
\|u(\cdot,t)\|_\infty \to 0 \quad \text{as} \quad t \to \infty.
$$

Moreover, if $0 < p < 1$ there exists $T^* \in (0, \infty)$ such that $u(\cdot, t) = 0$ for $t > T^*$.

3.3 Localization and Positivity

The results of this subsection show that there is localization in the range $p < m$, positivity in the range $p > m$; namely, in this respect the qualitative situation is the same as in the case $\rho = 1$ (see Subsection 2.1).

**Theorem 3.4** Let $p < m$. Then for any solution $u$ to problem (1.1) there exists $L > 0$ (depending on $m$, $p$, $c_0$, $u_0$) such that

$$
|\zeta^\pm(t)| \leq L \quad \text{for any} \quad t \geq 0.
$$

**Theorem 3.5** Let $p > m$. Then for any solution $u$ to problem (1.1) there exist $a$, $b > 0$ (depending on $m$, $p$, $\|\rho\|_\infty$, $c_0$, $u_0$) such that

$$
|\zeta^\pm(t)| \geq b [\log(at + 3)]^{1/2} \quad \text{for any} \quad t \geq 0.
$$

3.4 Global Existence of Interfaces

Now suppose $p > m$, so that positivity prevails (see Theorem 3.5 above). Does the support of the solution remain bounded for any positive time? On the strength of the case $c_0 = 0$ (see Subsection 2.2), it is expected that the dependence of the support on time is influenced by the decay rate of the density $\rho$ as $|x| \to \infty$. In fact, if the exponent $k$ in (1.2) is ”small” - namely, if $k \leq k^*$ - both interfaces exist for any $t > 0$; this is the content of the following two theorems. On the other hand, the interfaces can blow up in finite time if the exponent $k$ is ”large” (namely, if $k > k^*$), as we shall see in the following subsection.

**Theorem 3.6** Let $p > m$. Moreover, let $\rho$ satisfy the condition:

$$
(3.3) \quad \frac{\rho_1}{(1 + |x|)^k} \leq \rho(x) \leq \rho_0, \quad (\rho_0, \rho_1 > 0)
$$

where $0 < k < k^* := 2(p - 1)/(p - m)$. Then there exists $C_1 > 0$ such that

$$
|\zeta^\pm(t)| \leq C_1 t^{\frac{1}{k - k^*}} \quad \text{for any} \quad t \geq 0.
$$
Theorem 3.7 Let $p > m$. Moreover, let $\rho$ satisfy condition (3.3) with $k = k^*$. Then there exist $C_2, \beta > 0$ such that

$$|\zeta^\pm(t)| \leq C_2 e^{\beta t} \text{ for any } t \geq 0.$$ 

Theorem 3.8 Let $p = m$. Moreover, let $\rho$ satisfy condition (3.3) with $k > 0$. Then for any $\beta > 0$ there exists $C_3 > 0$ such that

$$|\zeta^\pm(t)| \leq C_3 t^\beta \text{ for any } t \geq 0.$$ 

3.5 Blow-Up of Interfaces

In contrast with the previous situation, we prove below that the interfaces can blow up in finite time if $k > k^*$, at least for a suitable class of initial data.

Theorem 3.9 Let $p > m$. Moreover, let $\rho$ satisfy the inequalities

$$\frac{\rho_1}{(1 + |x|)^k} \leq \rho(x) \leq \frac{\rho_2}{(1 + |x|)^k}, \quad (\rho_1, \rho_2 > 0),$$

with $k > k^*$. Then for any $h > 0$ there exists $b_0 = b_0(h) > 0$ such that, if

$$\frac{m}{m-1} u_0^{m-1}(x) \geq h [1 - \frac{|x|}{b}]_+$$

with $b > b_0$, then

$$u(x, t) > 0 \text{ for any } x \in \mathbb{R}, t > 1.$$ 

Sketch of the Proof: Consider the auxiliary function

$$w(x, t) := (1 + at)^{-\alpha} \left[ b^2 - \frac{x^2}{(1 + at)^{2\beta}} \right]_+,$$

where $a, b, \alpha$ and $\beta$ are positive parameters.

The following claim can be proved: There exist $a, b, \alpha, \beta > 0$ such that

$$v \leq w \text{ in } G := \left\{ |x| > \sqrt{\frac{\alpha}{\alpha + \beta} b(1 + at)^\beta}, t > 0 \right\},$$

where $v := \frac{m}{m-1} u^{m-1}$. In fact, we can achieve

$$-M \leq \mathcal{L}w \equiv -\rho(x) w_t + (m-1) w w_{xx} + w_x^2 - cw^q \leq 0.$$ 

This implies

$$\text{supp } v(\cdot, t) \subseteq (-b(1 + at)^\beta, b(1 + at)^\beta)$$

for any $t \geq 0$. It is possible to choose $\beta = \frac{1}{k^* - k}$; then the conclusion follows. \square
References


