数理解析研究所講究録 1249 巻 2002 年 117-125

Interfaces of Solutions to an Inhomogeneous Filtration Equation with Absorption *

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Abstract

We review some recent results, obtained jointly with R. Kersner and G. Reyes, concerning qualitative properties of solutions to the Cauchy problem for the equation $\rho(x)u_t=(u^m)_{xx}-c_0u^p$, where $m>1$ and $c_0,p>0$. The initial data are nonnegative with compact support and the density $\rho(x)>0$ satisfies suitable decay conditions as $|x|\to\infty$.

1 Introduction

We discuss some recent results (see [KRT]) concerning support properties of solutions to the Cauchy problem:

$$(1.1)\qquad \begin{cases} \rho(x)u_t=(u^m)_{xx}-c_0u^p & \text{in } I\!R\times(0,\infty)\\[2ex] u=u_0 & \text{in } I\!R\times\{0\}, \end{cases}$$

where

$$(i)\ m>1,\quad c_0>0,\quad p>0;$$

$$(ii)\ \rho\in C^1(I\!R)\cap L^\infty(I\!R),\quad \rho>0 \text{ in } I\!R;$$

$$(iii)\ u_0\in C_c(I\!R),\quad u_0\geq 0 \ \text{ in } \ I\!R.$$

(by $C_c(I\!R)$ we denote the set of continuous, compactly supported functions on $I\!R$). A typical choice for the density $\rho=\rho(x)$ is

$$(1.2)\qquad \rho(x)=\frac{\bar\rho}{(1+|x|)^k}\qquad(\bar\rho,k>0).$$

Physical motivations of the model can be found in [KR1], [KR2], [GuHP] and references therein; for instance, it arises in connection with a parabolic system suggested by plasma

*Work partially supported through TMR Programme NPE # FMRX-CT98-0201.

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physics (see [BK]). Let us also mention that the related problem of *uniqueness* of solutions to the positive Cauchy problem for diffusion equations with variable density has raised much attention in recent years (*e.g.*, see [E], [GuHP], [KKT], [T] and references therein).

Solutions to problem (1.1) are always meant in the following weak sense.

Definition 1.1 *By a* solution *to problem (1.1) we mean any bounded, nonnegative and continuous function u on $I\!R \times [0, \infty)$ such that*

$$\iint_{D^\tau}\{\rho\, u\, \phi_t \; + u^m\, \phi_{xx} - c_0 u^p\, \phi\}\, dx\, dt =$$

$$= \int_{-r}^{r} \rho\, u(x,\, \tau)\phi(x,\, \tau)\, dx - \int_{-r}^{r} \rho\, u_0(x)\phi(x,\, 0)\, dx +$$

$$+ \int_0^\tau [u^m(r,\, t)\, \phi_x(r,\, t) - u^m(-r,\, t)\, \phi_x(-r,\, t)]\, dt$$

for any $r, T > 0$, $\tau \in [0, T]$ and $\phi \in C^{2,1}_{x,t}(\overline{D}^T)$, $\phi \geq 0$ such that $\phi(-r,\, t) = \phi(r,\, t) = 0$ in $[0, T]$ (here $D^\tau := (-r,\, r) \times (0,\, \tau]$, $\tau \in [0, T]$) .

Subsolutions of problem (1.1) are similarly defined, replacing "$=$" by "$\geq$" in the above equality. On the other hand, supersolutions are meant in the following more restricted sense (introduced in [B] to deal with the case $0 < p < 1$).

Definition 1.2 *By a* supersolution *to problem (1.1) we mean any solution $\bar{u}$ to the problem*

$$\begin{cases} \rho(x)u_t = (u^m)_{xx} - c_0 u^p + h & \text{in } I\!R \times (0, T] \\ u(x,\, 0) = \hat{u}_0(x), & x \in I\!R, \end{cases}$$

for some $h \in L^\infty(S^T)$, $h \geq 0$, $\hat{u}_0 \geq u_0$ and $T > 0$.

Finally, let $u \geq 0$ be any solution to problem (1.1); its *interfaces* are defined as follows:

$$\zeta^+(t) := \sup\{x : u(x, t) > 0\}, \quad \zeta^-(t) := \inf\{x : u(x,t) > 0\} \qquad (t \geq 0).$$

We also set:

$$\zeta(t) := \sup\{|x| : u(x,\, t) > 0\} = \max\{|\zeta^-(t)|, |\zeta^+(t)|\}\ .$$

2 Previous Results

Let us recall some well-known results for problem (1.1) in two particular cases: $\rho = 1$, $c_0 > 0$, respectively $\rho = \rho(x)$, $c_0 = 0$.

2.1 Porous Medium Equation with Absorption

When $\rho = 1$, $c_0 > 0$ it is well known that (see [BKP], [CMM], [Ka]):

- the following estimates hold:

$$
\begin{array}{lll}
|\zeta^{\pm}(t)| \leq \mathit{constant} & \text{for} & 1 \leq p < m, \\
|\zeta^{\pm}(t)| \sim \log t & \text{for} & p = m, \\
|\zeta^{\pm}(t)| \sim t^{\alpha}, \quad \alpha > 0 & \text{for} & p > m
\end{array}
$$

  (*localization* of the solution if $p < m$, respectively *positivity* if $p \geq m$);
- if $0 < p < 1$, there is *extinction* of the solution in finite time - namely, there exists $T^* \in (0, \infty)$ such that $u \equiv 0$ in $I\!R \times (T^*, \infty)$.

Let us mention that the case $\rho = 1$, $c_0 = c_0(x)$ was also investigated (see [PT1], [PT2]).

2.2 Inhomogeneous Porous Medium Equation

New interesting phenomena arise when ρ depends on the space variable. Consider the case $\rho = \rho(x)$, $c_0 = 0$. The main qualitative novelty is that, if $\rho(x) \to 0$ "fast enough" as $|x| \to \infty$, interfaces can *run off in finite time* - namely there possibly exists $\bar{T} \in (0, \infty)$ such that

$$\zeta(t) \to \infty \ \text{ as } t \to \bar{T}^- .$$

In fact, the following holds ([KK], [GuHP], [GKK]; see also [GK], [P]).

Theorem 2.1 *Let $c_0 = 0$, $|x|\,\rho(x) \in L^1(I\!R)$. Then for any solution to problem (1.1) there exists $\bar{T} \in (0, \infty)$ such that*

$$\zeta(t) \to \infty \ \text{ as } \ t \to \bar{T}^- .$$

The above theorem is related to the following convergence result ([KR2]).

Theorem 2.2 *If $\rho \in L^1(I\!R)$, there holds*

$$u(\cdot, t) \to \overline{u} := \frac{\|\rho u_0\|_1}{\|\rho\|_1} \quad \text{as} \quad t \to \infty ,$$

the convergence being uniform on compact subsets of $I\!R$.

Sketch of the Proof of Theorem 2.1: If *supp* $u(\cdot,t)$ is compact for any $t>0$, the following equalities can be proven:

$$\int_{\mathbb{R}} \rho(x)u(x,t)dx = \int_{\mathbb{R}} \rho(x)u_0(x)dx,$$

$$\frac{1}{t}\int_0^\infty x\rho(x)[u(x,t)-u_0(x)]dx = \frac{1}{t}\int_0^t u^m(0,\tau)d\tau.$$

$(t>0)$. Since $u(\cdot,t) \to \bar{u} > 0$ as $t\to\infty$ by Theorem 2.2, a contradiction arises if $\zeta(t)<\infty$ for any $t>0$. This proves the result. □

Remark 2.3 *The above proof cannot be adapted to the case $c_0>0$.*

The criterion of Theorem 2.1 is extended to higher space dimension as follows. Let $\rho\sigma \in L^1(\mathbb{R}^n)$, where

$$\sigma(x) := \begin{cases} |x| & \text{if } n=1, \\ log(|x|) & \text{if } n=2, \\ |x|^{\frac{2-n}{m}} & \text{if } n\geq 3\,; \end{cases}$$

then *supp u* becomes unbounded in finite time ([GuHP], [GiT]).

For the choice

$$\rho(x) = \frac{\bar{\rho}}{(1+|x|)^k}$$

it can be proved by comparison methods that blow-up occurs if and only if $k>2$ (see [KK]). In fact, for any $k\leq 2$ there exist a_0, $b_0>0$ such that:

$$|\zeta^{\pm}(t)| \sim a_0 t^{\frac{1}{2-k}} \text{ as } t\to\infty \text{ if } k<2,$$

$$|\zeta^{\pm}(t)| \sim e^{b_0 t} \text{ as } t\to\infty \text{ if } k=2\,.$$

Hence *critical value* for the case $c_0=0$ is $k=2$.

3 Results

Now the question arises: How do absorption and variable density cooperate to influence the situation depicted in Section 2? In particular, which phenomena arising when $\rho=\rho(x)$ and $c_0=0$ are *structurally stable* with respect to the parameter $c_0\geq 0$?

As we shall see below, the following answer can be given:

(i) As in the case of constant ρ, there is localization of the solution if $p<m$, positivity if $p>m$. In the latter event, the behaviour of interfaces is the same as in the case $c_0=0$, yet with a different critical value of k, namely

$$k^* := 2\frac{p-1}{p-m}$$

(observe that $k^* \to 2$ as $p \to \infty$).

(ii) The convergence result in Theorem 2.2 is not structurally stable; in fact, solutions to problem (1.1) go to zero uniformly as $t \to \infty$ for any $c_0 > 0$.

The results summarized above are proved by comparing solutions to problem (1.1) with suitable sub- and supersolutions (possibly suggested by a proper splitting of domains). We only sketch belowthe proof of Theorem 3.9, referring the reader to [KRT] for complete proofs. As already remarked (see Remark 2.3) the techniques used for the case $c_0 = 0$ (which rely on mass conservation; see [KK]), cannot be adapted to the present situation.

Concerning the critical values $k = 2$ (if $c_0 = 0$) and $k = k^*$ (if $c_0 > 0$, $p > m$), observe that:

(i') the function $u(x,t) = f(xt^{-\frac{1}{2-k}})$ is a similarity solution to the equation

$$|x|^{-k}u_t = (u^m)_{xx}$$

if $k = 2$;

(ii') the function $u(x,t) = t^\alpha f(xt^{-\beta})$ with

$$\alpha := -\frac{2}{(p-m)(k^*-k)}, \qquad \beta := \frac{1}{k^*-k}$$

is a similarity solution to the equation

$$|x|^{-k}u_t = (u^m)_{xx} - c_0 u^p$$

in the case $c_0 > 0$, $p > m$ if $k = k^*$.

Analogous results hold at $k = 2$ and $k = k^*$.

3.1 Well-Posedness and Comparison

The basic theory for problem (1.1) - as well as for initial-boundary value problems related to the differential equation in (1.1) - was studied in [RT]; in particular, this includes comparison results for solutions to the first boundary value and to the Cauchy-Dirichlet problems, in regions whose lateral boundaries may be *curvilinear*, which are needed to prove several statements listed below. Let us mention the following results.

Theorem 3.1 *Let $m > 1, p > 0$, $\rho \in C^3(I\!R) \cap L^\infty(I\!R)$, $\rho > 0$ and $u_0 \in C(I\!R) \cap L^\infty(I\!R)$. Then there exists a unique solution to the Cauchy problem (1.1).*

Theorem 3.2 *Let $\underline{u}$ be a subsolution, $\overline{u}$ a supersolution to the Cauchy problem (1.1). Then $\underline{u} \le \overline{u}$.*

3.2 Asymptotic Behaviour

As already mentioned, the convergence result in Theorem 2.2 is not structurally stable; moreover, there is extinction of the solution in finite time if $0 < p < 1$. In fact, the following result can be proven.

Theorem 3.3 *Let u be any solution to problem (1.1). Then*

$$\|u(\cdot,t)\|_\infty \to 0 \quad as \quad t \to \infty.$$

Moreover, if $0 < p < 1$ there exists $T^* \in (0,\infty)$ such that $u(\cdot,\, t) = 0$ for $t > T^*$.

3.3 Localization and Positivity

The results of this subsection show that there is localization in the range $p < m$, positivity in the range $p > m$; namely, in this respect the qualitative situation is the same as in the case $\rho = 1$ (see Subsection 2.1).

Theorem 3.4 *Let $p < m$. Then for any solution u to problem (1.1) there exists $L > 0$ (depending on m, p, c_0, u_0) such that*

$$|\zeta^{\pm}(t)| \leq L \quad \textit{for any} \quad t \geq 0.$$

Theorem 3.5 *Let $p > m$. Then for any solution u to problem (1.1) there exist a, $b > 0$ (depending on m, p, $\|\rho\|_\infty$, c_0, u_0) such that*

$$|\zeta^{\pm}(t)| \geq b\,[\log(at+3)]^{1/2} \quad \textit{for any} \quad t \geq 0.$$

3.4 Global Existence of Interfaces

Now suppose $p > m$, so that positivity prevails (see Theorem 3.5 above). Does the support of the solution remain bounded for any positive time? On the strength of the case $c_0 = 0$ (see Subsection 2.2), it is expected that the dependence of the support on time is influenced by the decay rate of the density ρ as $|x| \to \infty$. In fact, if the exponent k in (1.2) is "small" - namely, if $k \leq k^*$ - both interfaces exist for any $t > 0$; this is the content of the following two theorems. On the other hand, the interfaces can blow up in finite time if the exponent k is "large" (namely, if $k > k^*$), as we shall see in the following subsection.

Theorem 3.6 *Let $p > m$. Moreover, let ρ satisfy the condition:*

$$\frac{\rho_1}{(1+|x|)^k} \leq \rho(x) \leq \rho_0, \quad (\rho_0,\, \rho_1 > 0) \tag{3.3}$$

where $0 < k < k^* := 2(p-1)/(p-m)$. Then there exists $C_1 > 0$ such that

$$|\zeta^{\pm}(t)| \leq C_1 t^{\frac{1}{k^*-k}} \quad \textit{for any} \quad t \geq 0.$$

Theorem 3.7 *Let $p > m$. Moreover, let ρ satisfy condition (3.3) with $k = k^*$. Then there exist C_2, $\beta > 0$ such that*

$$|\zeta^{\pm}(t)| \leq C_2 e^{\beta t} \quad \textit{for any} \quad t \geq 0.$$

Theorem 3.8 *Let $p = m$. Moreover, let ρ satisfy condition (3.3) with $k > 0$. Then for any $\beta > 0$ there exists $C_3 > 0$ such that*

$$|\zeta^{\pm}(t)| \leq C_3 t^{\beta} \quad \textit{for any} \quad t \geq 0.$$

3.5 Blow-Up of Interfaces

In contrast with the previous situation, we prove below that the interfaces can blow up in finite time if $k > k^*$, at least for a suitable class of initial data.

Theorem 3.9 *Let $p > m$. Moreover, let ρ satisfy the inequalities*

$$\frac{\rho_1}{(1+|x|)^k} \leq \rho(x) \leq \frac{\rho_2}{(1+|x|)^k}, \qquad (\rho_1, \rho_2 > 0),$$

with $k > k^*$. Then for any $h > 0$ there exists $b_0 = b_0(h) > 0$ such that, if

$$\frac{m}{m-1} u_0^{m-1}(x) \geq h \left[1 - \frac{|x|}{b}\right]_+$$

with $b > b_0$, then

$$u(x, t) > 0 \quad \textit{for any} \quad x \in I\!R, t > 1.$$

Sketch of the Proof: Consider the auxiliary function

$$w(x, t) := (1 + at)^{-\alpha} \left[b^2 - \frac{x^2}{(1+at)^{2\beta}} \right]_+ ,$$

where a, b, α and β are positive parameters.

The following claim can be proved: There exist a, b, α, $\beta > 0$ such that

$$v \leq w \quad \text{in} \quad G := \left\{ |x| > \sqrt{\frac{\alpha}{\alpha+\beta}} b(1+at)^{\beta}, t > 0 \right\},$$

where $v := \frac{m}{m-1} u^{m-1}$. In fact, we can achieve

$$-M \leq \mathcal{L}w \equiv -\rho(x) w_t + (m-1) w w_{xx} + w_x^2 - c w^q \leq 0.$$

This implies

$$\textit{supp}\ v(\cdot, t) \subseteq (-b(1+at)^{\beta}, b(1+at)^{\beta})$$

for any $t \geq 0$. It is possible to choose $\beta = \frac{1}{k^*-k}$; then the conclusion follows. $\square$

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