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<th>$q$-deformed Araki-Woods factors (Theory of Operator Algebras and its Applications)</th>
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<tr>
<td>Author(s)</td>
<td>Hiai, Fumio</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2002, 1250: 82-88</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41788">http://hdl.handle.net/2433/41788</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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1 Construction of the $q$-deformed functor

Let $\mathcal{H}_R$ be a separable real Hilbert space and $U_t$ a strongly continuous one-parameter group of orthogonal transformations on $\mathcal{H}_R$. By linearity $U_t$ extends to a one-parameter unitary group on the complexified Hilbert space $\mathcal{H}_C := \mathcal{H}_R + i\mathcal{H}_R$. Write $U_t = A^t$ with the generator $A$ (a positive non-singular operator on $\mathcal{H}_C$) and define an inner product $\langle \cdot, \cdot \rangle_U$ on $\mathcal{H}_C$ by

$$\langle x, y \rangle_U = \langle 2A(1+A)^{-1}x, y \rangle,$$

$x, y \in \mathcal{H}_C$.

Let $\mathcal{H}$ be the complex Hilbert space obtained by completing $\mathcal{H}_C$ with respect to $\langle \cdot, \cdot \rangle_U$.

For $-1 < q < 1$, $\langle \cdot, \cdot \rangle_q$ is strictly positive and the $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{\text{finite}}(\mathcal{H})$ with respect to $\langle \cdot, \cdot \rangle_q$. Given $h \in \mathcal{H}$ the $q$-creation operator $a_q^*(h)$ and the $q$-annihilation operator $a_q(h)$ on $\mathcal{F}_q(\mathcal{H})$ are defined by

$$a_q^*(h)\Omega = h,$$

$$a_q^*(h)(f_1 \otimes \cdots \otimes f_n) = h \otimes f_1 \otimes \cdots \otimes f_n,$$

and

$$a_q(h)\Omega = 0,$$

$$a_q(h)(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^n q^{i-1} \langle h, f_i \rangle_U f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n.$$

The operators $a_q^*(h)$ and $a_q(h)$ are bounded operators on $\mathcal{F}_q(\mathcal{H})$ and they are adjoints of each other (see [BKS, Remark 1.2]).

Following [Sh1] we consider the von Neumann algebra $\Gamma_q(\mathcal{H}_R, U_t)'$, called a $q$-deformed Araki-Woods algebra, generated on $\mathcal{F}_q(\mathcal{H})$ by

$$s_q(h) := a_q^*(h) + a_q(h), \quad h \in \mathcal{H}_R.$$

The vacuum state $\varphi (= \varphi_{q,U}) := \langle \Omega, \cdot \Omega \rangle_q$ on $\Gamma_q(\mathcal{H}_R, U_t)'$ is called the $q$-quasi-free state.
Proposition 1.1 $\Omega$ is cyclic and separating for $\Gamma_q(\mathcal{H}_R, U_t)^\prime\prime$.

One can canonically extend $U_t$ on $\mathcal{H}$ to a one-parameter unitary group (the so-called second quantization) $\mathcal{F}_q(U_t)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\mathcal{F}_q(U_t)\Omega = \Omega,$$

$$\mathcal{F}_q(U_t)(f_1 \otimes \cdots \otimes f_n) = (U_t f_1) \otimes \cdots \otimes (U_t f_n).$$

Notice $\mathcal{F}_q(U_t) a_q^*(h) \mathcal{F}_q(U_t)^* = a_q^*(U_t h)$ for $h \in \mathcal{H}$ so that

$$\mathcal{F}_q(U_t) s_q(h) \mathcal{F}_q(U_t)^* = s_q(U_t h), \quad h \in \mathcal{H}_R.$$

Thus, $\alpha_t := \text{Ad}\mathcal{F}_q(U_t)$ defines a strongly continuous one-parameter automorphism group on $\Gamma_q(\mathcal{H}_R, U_t)^\prime\prime$.

Proposition 1.2 The $q$-quasi-free state $\varphi$ on $\Gamma_q(\mathcal{H}_R, U_t)^\prime\prime$ satisfies the KMS condition with respect to $\alpha_t$ at $\beta = 1$.

Let $(\mathcal{K}_R, V_t)$ be another pair of a separable real Hilbert space and a one-parameter group $V_t$ of orthogonal transformations on $\mathcal{K}_R$. Let $T : \mathcal{H}_R \to \mathcal{K}_R$ be a contraction such that $TU_t = V_t T$ for all $t \in \mathbb{R}$. By linearity $T$ extends to a contraction $T : \mathcal{H}_C \to \mathcal{K}_C$ and it satisfies $TU_t = V_t T$ on $\mathcal{H}_C$. Let $B$ be the generator of $V_t$ so that $V_t = B^t$. Since

$$TA(1+A)^{-1} = B(1+B)^{-1}T,$$

$T$ can further extend to a contraction from $(\mathcal{H}, \langle \cdot, \cdot \rangle_U)$ to $(\mathcal{K}, \langle \cdot, \cdot \rangle_V)$. Then:

Proposition 1.3 There is a unique completely positive normal contraction $\Gamma_q(T) : \Gamma_q(\mathcal{H}_R, U_t)^\prime\prime \to \Gamma_q(\mathcal{K}_R, V_t)^\prime\prime$ such that

$$(\Gamma_q(T)x)\Omega = \mathcal{F}_q(T)(x\Omega), \quad x \in \Gamma_q(\mathcal{H}_R, U_t)^\prime\prime,$$

where $\mathcal{F}_q(T) : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{K})$ is given by

$$\mathcal{F}_q(T)(f_1 \otimes \cdots \otimes f_n) = (T f_1) \otimes \cdots \otimes (T f_n).$$

In this way, we have presented a $q$-analogue of Shlyakhtenko’s free CAR functor; namely, a von Neumann algebra with a specified state, $(\Gamma_q(\mathcal{H}_R, U_t)^\prime\prime, \varphi)$, is associated to each real Hilbert space with a one-parameter group of orthogonal transformations, $(\mathcal{H}_R, U_t)$, and a unital completely positive state-preserving map $\Gamma_q(T) : \Gamma_q(\mathcal{H}_R, U_t)^\prime\prime \to \Gamma_q(\mathcal{K}_R, V_t)^\prime\prime$ to every contraction $T : (\mathcal{H}_R, U_t) \to (\mathcal{K}_R, V_t)$.

When $q = 0$, $\Gamma(\mathcal{H}_R, U_t)^\prime\prime \equiv \Gamma_0(\mathcal{H}_R, U_t)^\prime\prime$ is a free Araki-Woods factor (of type III) in $[\text{Sh1}]$. On the other hand, when $U_t = \text{id}$ a trivial action, $\Gamma_q(\mathcal{H}_R)^\prime\prime \equiv \Gamma_q(\mathcal{H}_R, \text{id})$ is a $q$-deformation of the free group factor in $[\text{BKS}]$; in particular, $\Gamma_0(\mathcal{H}_R)^\prime\prime \cong L(\mathbb{F}_{\text{dim} \mathcal{H}_R})$ a free group factor.
2 Factoriality and non-injectivity of $\Gamma_q(\mathcal{H}_R, U_t)''$

The following were proven in [BS2, BKS], but it is still open whether $\Gamma_q(\mathcal{H}_R)''$ is a non-injective type II$_1$ factor whenever $\dim \mathcal{H}_R \geq 2$.

(i) If $-1 < q < 1$ and $\dim \mathcal{H}_R > 16/(1 - |q|)^2$, then $\Gamma_q(\mathcal{H}_R)''$ is not injective.

(ii) If $\dim \mathcal{H}_R = \infty$, then $\Gamma_q(\mathcal{H}_R)$ is a factor (of type II$_1$) for all $-1 < q < 1$.

These results can be extended to $\Gamma_q(\mathcal{H}_R, U_t)''$ as follows.

**Theorem 2.1** If there is $T \in [1, \infty)$ such that

$$\frac{\dim E_A([1,T]) \mathcal{H}_C}{T} > \frac{16}{(1 - |q|)^2}$$

where $E_A$ is the spectral measure of $A$, then $\Gamma_q(\mathcal{H}_R, U_t)''$ is not injective. In particular, $\Gamma_q(\mathcal{H}_R, U_t)''$ is not injective if $A$ has a continuous spectrum.

**Theorem 2.2** Assume that the almost periodic part of $(\mathcal{H}_R, U_t)$ is infinite dimensional, that is, $A$ has infinitely many mutually orthogonal eigenvectors. Then

$$(\Gamma_q(\mathcal{H}_R, U_t)''\big|\mathcal{H}_C)_{\varphi}' \cap \Gamma_q(\mathcal{H}_R, U_t)'' = \mathcal{C}1,$$

where $(\Gamma_q(\mathcal{H}_R, U_t)''\big|\mathcal{H}_C)_{\varphi}'$ is the centralizer of $\Gamma_q(\mathcal{H}_R, U_t)''$ with respect to the vacuum state $\varphi$. In particular, $\Gamma_q(\mathcal{H}_R, U_t)''$ is a factor.

3 Type classification of $\Gamma_q(\mathcal{H}_R, U_t)''$

As usual let $S_{\varphi}$ be the closure of the operator given by

$$S_{\varphi}(x\Omega) = x^*\Omega, \quad x \in \Gamma_q(\mathcal{H}_R, U_t)''$$

and let $\Delta_{\varphi}, J_{\varphi}$ be the associated modular operator and the modular conjugation. Then the following are seen as in [Sh1]: For $h_1, \ldots, h_n \in \mathcal{H}_R$,

$$S_{\varphi}(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = h_n \otimes h_{n-1} \otimes \cdots \otimes h_1,$$

and for $h_1, \ldots, h_n \in \mathcal{H}_R \cap \text{dom} A^{-1}$,

$$\Delta_{\varphi}(h_1 \otimes \cdots \otimes h_n) = (A^{-1}h_1) \otimes \cdots \otimes (A^{-1}h_n).$$

Noting that $D := \{h + ig : h, g \in \mathcal{H}_R \cap \text{dom} A^{-1}\}$ is a core of $A^{-1}$ (on $\mathcal{H}$) such that $U_tD = D$ for all $t \in \mathbb{R}$, we see that

$$\Delta^u_{\varphi} = \mathcal{F}_q(A^{-it}) = \mathcal{F}_q(U_{-t}), \quad t \in \mathbb{R}.$$ 

By this and Theorem 2.2 we obtain the following type classification result:
Theorem 3.1 Assume that $A$ has infinitely many mutually orthogonal eigenvectors. Let $G$ be the closed multiplicative subgroup of $\mathbb{R}_+$ generated by the spectrum of $A$ ($U_t = A^{it}$). Then $\Gamma_q(\mathcal{H}_R, U_t)''$ is a non-injective factor of type $II_1$ or type $III_\lambda$ $(0 < \lambda \leq 1)$, and

$$\Gamma_q(\mathcal{H}_R, U_t)'' = \begin{cases} 
  \text{type } II_1 & \text{if } G = \{1\}, \\
  \text{type } III_\lambda & \text{if } G = \{\lambda^n : n \in \mathbb{Z}\} \ (0 < \lambda < 1), \\
  \text{type } III_1 & \text{if } G = \mathbb{R}_+. 
\end{cases}$$

This result for free Araki-Woods factors (in case of $q = 0$) was shown in [Sh1, Sh2] generally when $\dim \mathcal{H}_R \geq 2$. Moreover, it was shown as a consequence of Barnett’s theorem that free Araki-Woods factors are full whenever $U_t$ is almost periodic (i.e. the eigenvectors of $A$ span $\mathcal{H}$). The assumption of Theorems 2.2 and 3.1 is a bit too restrictive while the following opposite extreme case is easy to see:

Proposition 3.2 If $U_t$ has no eigenvectors, then $\Gamma_q(\mathcal{H}_R, U_t)''$ is a type $III_1$ factor.

It is worthwhile to note that the type $III_0$ case does not appear in the above type classifications.

For example, let $(\mathcal{H}_R, U_t) = \bigoplus_{k=1}^{\infty} (\mathbb{R}^2, V_t)$ where $V_t := \begin{bmatrix} \cos(t \log \lambda) & - \sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{bmatrix}$, $0 < \lambda \leq 1$, and write $(T_{q,\lambda}, \varphi_{q,\lambda}) := \Gamma_q(\mathcal{H}_R, U_t)'', \varphi)$ with two parameters $q \in (-1, 1)$ and $\lambda \in (0, 1]$. For $0 < \lambda < 1$, $T_{q,\lambda}$ is a type $III_\lambda$ $q$-deformed Araki-Woods factor. In particular when $q = 0$, $(T_{0,\lambda}, \varphi_{0,\lambda})$ coincides with the type $III_\lambda$ free Araki-Woods factor $(T_{\lambda}, \varphi_{\lambda})$ discussed in [Ra, Sh1]. For $\lambda = 1$, $T_{q,1}$ is the $q$-deformed type $II_1$ factor treated in [BKS].

The $C^*$-algebra $\Gamma_q(\mathcal{H}_R, U_t)$, $-1 < q < 1$, generated by $\{s_q(h) : h \in \mathcal{H}_R\}$ on $\mathcal{F}_q(\mathcal{H})$ is considered as the $q$-analogue of the CAR algebra. From this point of view, the above $T_{q,\lambda}$ $(0 < \lambda < 1)$ may be considered as the $q$-analogue of Powers’ $III_\lambda$ factor. In fact, we remark that, for $q = -1$, our construction of $T_{q,\lambda}$ provides Powers’ $III_\lambda$ factor. To be more precise, for given $(\mathcal{H}_R, U_t)$, let $\Gamma_-(\mathcal{H}_R, U_t)''$ denote the von Neumann algebra generated by $s_-(h) := a^*_+(h) + a_-(h)$ $(h \in \mathcal{H}_R)$ on the Fermion Fock space $\mathcal{F}_-(\mathcal{H})$, where $a^*_+(h)$ and $a_-(h)$ are the Fermion (CAR) creation and annihilation operators. If $(\mathcal{H}_R, U_t) = \bigoplus_{k=1}^{\infty} (\mathcal{H}_R^{(k)}, U_t^{(k)})$ where $\mathcal{H}_R^{(k)} = \mathbb{R}^2$, $U_t^{(k)} = \begin{bmatrix} \cos(t \log \lambda_k) & - \sin(t \log \lambda_k) \\ \sin(t \log \lambda_k) & \cos(t \log \lambda_k) \end{bmatrix}$ with $\lambda_k \leq 1$, then $(\Gamma_-(\mathcal{H}_R, U_t)'', \varphi := (\Omega, \cdot \Omega)_-) \text{ becomes an Araki-Woods factor}$

$$\bigotimes_{k=1}^{\infty} \left( M_2(\mathbb{C}), \text{Tr} \left( \begin{bmatrix} \lambda_k & 0 \\ 0 & 1/\lambda_k+1 \end{bmatrix} \right) \right).$$

Upon these considerations we called $\Gamma_q(\mathcal{H}_R, U_t)''$ a $q$-deformed Araki-Woods algebra.
4 Hypercontractivity of $\Gamma_q(T)$

When $T = e^{-t}1_{\mathcal{H}_R}$ ($t > 0$), we obtain a semigroup $\Gamma_q(e^{-t})$ ($t > 0$) of completely positive normal contractions on $\Gamma_q(\mathcal{H}_R, U_t)'$. This is a non-tracial extension of $q$-Ornstein-Uhlenbeck semigroup discussed in [Bi, Bo]. In the tracial case (i.e. the case of $U_t$ being trivial), the ultracontractivity for $\Gamma_q(e^{-t})$ was proven in [Bo] as follows:

$$||\Gamma_q(e^{-t})x|| \leq C_{|q|}^{3/2} \sqrt{\frac{1 + e^{-2t}}{(1-e^{-2t})^3}} ||x\Omega||,$$

with $C_{|q|}$ given below. In the non-tracial type III case, we have the following hypercontractivity property. This reduces to the above ultracontractivity when $A = 1$ or $\gamma = 0$.

**Theorem 4.1** Assume that $A$ is bounded (in particular, this is the case if $\dim \mathcal{H}_R < +\infty$), and let $\gamma := \frac{1}{2} \log ||A||$. If $-1 < q < 1$ and $t > \gamma$, then

$$||\Gamma_q(e^{-t})x|| \leq C_{|q|}^{3/2} \sqrt{\frac{1 + e^{-(2t-\gamma)}}{(1-e^{-2t})(1-e^{-(2t-\gamma)})(1-e^{-2(t-\gamma)})}} ||\Delta_{\varphi}^{\theta/2}x\Omega||,$$

for all $x \in \Gamma_q(\mathcal{H}_R, U_t)'$ and $0 \leq \theta \leq 1$, where

$$C_{|q|} := \frac{1}{\prod_{m=1}^{\infty} (1-|q|^m)}.$$

It might be expected that the hypercontractivity given in the above theorem is valid for the whole $t > 0$. However, the next proposition says that it is impossible to remove the assumption $t > \gamma$, so Theorem 4.1 seems more or less best possible. Also, it says that the hypercontractivity in the sense that $||\Gamma_q(e^{-t})x|| \leq C||x\Omega||_q$ holds for some $t > 0$ and for all $x \in \Gamma_q(\mathcal{H}_R, U_t)'$ is impossible when $A$ is unbounded; for example, this is the case when $U_t f = f(\cdot + t)$ on $\mathcal{H}_R = L^2(\mathbb{R}; \mathbb{R})$.

**Proposition 4.2** Let $-1 < q < 1$, $0 \leq \theta \leq 1$ and $t > 0$. If there exists a constant $c > 0$ such that

$$||\Gamma_q(e^{-t})x|| \leq c||\Delta_{\varphi}^{\theta/2}x\Omega||,$$

then $A$ is bounded and

$$||A|| \leq \exp\left(\frac{2t}{\max\{\theta, 1-\theta\}}\right).$$

It seems that it is convenient to consider the hypercontractivity of $\Gamma_q(T)$ in the setting of Kosaki's interpolated $L^p$-spaces. For a general von Neumann algebra $\mathcal{M}$ and $1 \leq p \leq \infty$ let $L^p(\mathcal{M})$ be Haagerup's $L^p$-space. Given a faithful normal state $\varphi$ on $\mathcal{M}$ let $h_\varphi$ denote the element of $L^1(\mathcal{M})$ ($\cong \mathcal{M}_\varphi$) corresponding to $\varphi$. For each $1 < p < \infty$
and $0 \leq \theta \leq 1$, Kosaki's $L^p$-space $L^p(\mathcal{M}; \varphi)_\theta$ with respect to $\varphi$ is introduced as the complex interpolation space

$$C_{1/p}(h^\theta_\varphi \mathcal{M} h^{1-\theta}_\varphi, L^1(\mathcal{M}))$$

equipped with the complex interpolation norm $\| \cdot \|_{p, \theta} = (\| \cdot \|_{C_{1/p}})$.

Let $T : \mathcal{H}_R \to \mathcal{K}_R$ be a contraction with $TU_t = V_t T$, $t \in \mathbb{R}$. The adjoint operator $T^* : \mathcal{K}_R \to \mathcal{H}_R$ is also a contraction satisfying $T^* V_t = U_t T^*$, $t \in \mathbb{R}$. For each $-1 < q < 1$ let

$$\mathcal{M} := \Gamma_q(\mathcal{H}_R, U_t)^{\prime\prime} \text{ with } \varphi = (\Omega, \cdot \Omega)_q, \quad \mathcal{N} := \Gamma_q(\mathcal{K}_R, V_t)^{\prime\prime} \text{ with } \psi = (\Omega, \cdot \Omega)_q,$$

where the vacuums in $\mathcal{F}_q(\mathcal{H})$ and in $\mathcal{F}_q(\mathcal{K})$ are denoted by the same $\Omega$. Then, by Proposition 1.3 the completely positive normal contractions

$$\Gamma_q(T) : \mathcal{M} \to \mathcal{N} \quad \text{and} \quad \Gamma_q(T^*) : \mathcal{N} \to \mathcal{M}$$

are determined by

$$(\Gamma_q(T)x)\Omega = \mathcal{F}_q(T)(x\Omega), \quad x \in \mathcal{M},$$

$$(\Gamma_q(T^*)y)\Omega = \mathcal{F}_q(T^*)(y\Omega), \quad y \in \mathcal{N}.$$ 

One can define the contraction $\omega \to \omega \circ \Gamma_q(T^*)$ of $\mathcal{M}_*$ into $\mathcal{N}_*$. Via $\mathcal{M}_* \cong L^1(\mathcal{M})$ and $\mathcal{N}_* \cong L^1(\mathcal{N})$ this induces the contraction $\tilde{\Gamma}_q(T)$ of $L^1(\mathcal{M})$ into $L^1(\mathcal{N})$ as follows:

$$\tilde{\Gamma}_q(T)h_\omega = h_\omega \mathcal{F}_q(T^*)\Omega, \quad \omega \in \mathcal{M}_*.$$ 

We see that for every $0 \leq \theta \leq 1$ and $x \in \mathcal{M}$,

$$\tilde{\Gamma}_q(T)(h^\theta_\varphi x h^{1-\theta}_\varphi) = h^\theta_\psi (\Gamma_q(T)x) h^{1-\theta}_\psi,$$

so that $\tilde{\Gamma}_q(T) : L^1(\mathcal{M}) \to L^1(\mathcal{N})$ is the (unique) continuous extension of the linear mapping from $h^\theta_\varphi \mathcal{M} h^{1-\theta}_\varphi (\subset L^1(\mathcal{M}))$ into $h^\theta_\psi \mathcal{N} h^{1-\theta}_\psi (\subset L^1(\mathcal{N}))$ given by

$$h^\theta_\varphi x h^{1-\theta}_\varphi \mapsto h^\theta_\psi (\Gamma_q(T)x) h^{1-\theta}_\psi, \quad x \in \mathcal{M}.$$ 

Moreover, the Riesz-Thorin theorem implies that for each $0 \leq \theta \leq 1$ and $1 \leq p \leq \infty$, $\tilde{\Gamma}_q(T)$ maps $L^p(\mathcal{M}; \varphi)_\theta$ into $L^p(\mathcal{N}; \psi)_\theta$ and

$$\|\tilde{\Gamma}_q(T)a\|_{p, \theta} \leq \|a\|_{p, \theta}, \quad a \in L^p(\mathcal{M}; \varphi)_\theta.$$ 

The next theorem is shown by using Theorem 4.1.

**Theorem 4.3** Assume that either $A (U_t = A^{it})$ or $B (V_t = B^{it})$ is bounded, and let $\rho := \min\{\|A\|, \|B\|\}$. Let $T : \mathcal{H}_R \to \mathcal{K}_R$ be a bounded operator such that $TU_t = V_t T$ for all $t \in \mathbb{R}$ and $\|T\| < \rho^{-1}$. Then $\tilde{\Gamma}_q(T)$ maps $L^1(\mathcal{M})$ into $\bigcap_{0 \leq \theta \leq 1} h^\theta_\varphi \mathcal{M} h^{1-\theta}_\varphi$ and

$$\|\tilde{\Gamma}_q(T)a\|_{\infty, \theta} \leq C_{1/[\|T\|]}^{2} \frac{1 + \rho^{1/2}\|T\|}{(1 - \|T\|)(1 - \rho^{1/2}\|T\|)(1 - \rho\|T\|)} \|a\|_1$$

for all $a \in L^1(\mathcal{M})$, $0 \leq \theta \leq 1$. 

References


