q-deformed Araki-Woods factors

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1 Construction of the q-deformed functor

Let \( \mathcal{H}_R \) be a separable real Hilbert space and \( U_t \) a strongly continuous one-parameter group of orthogonal transformations on \( \mathcal{H}_R \). By linearity \( U_t \) extends to a one-parameter unitary group on the complexified Hilbert space \( \mathcal{H}_C := \mathcal{H}_R + i\mathcal{H}_R \). Write \( U_t = A^t \) with the generator \( A \) (a positive non-singular operator on \( \mathcal{H}_C \)) and define an inner product \( \langle \cdot, \cdot \rangle_U \) on \( \mathcal{H}_C \) by

\[
\langle x, y \rangle_U = \langle 2A(1+A)^{-1}x, y \rangle,
\]

where \( x, y \in \mathcal{H}_C \).

Let \( \mathcal{H} \) be the complex Hilbert space obtained by completing \( \mathcal{H}_C \) with respect to \( \langle \cdot, \cdot \rangle_U \).

For \(-1 < q < 1\), the \( q \)-Fock space \( \mathcal{F}_q(\mathcal{H}) \) was introduced in [BS1, BKS] as follows. Let \( \mathcal{F}^{\text{finite}}(\mathcal{H}) \) be the linear span of \( f_1 \otimes \cdots \otimes f_n \) for \( n = 0, 1, \ldots \) where \( \mathcal{H}^{\text{even}} = \mathbb{C}\Omega \) with vacuum \( \Omega \). The sesquilinear form \( \langle \cdot, \cdot \rangle_q \) on \( \mathcal{F}^{\text{finite}}(\mathcal{H}) \) is given by

\[
\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q = \delta_{nm} \sum_{\pi \in S_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle_U \cdots \langle f_n, g_{\pi(n)} \rangle_U,
\]

where \( i(\pi) \) denotes the number of inversions of the permutation \( \pi \in S_n \). For \(-1 < q < 1\), \( \langle \cdot, \cdot \rangle_q \) is strictly positive and the \( q \)-Fock space \( \mathcal{F}_q(\mathcal{H}) \) is the completion of \( \mathcal{F}^{\text{finite}}(\mathcal{H}) \) with respect to \( \langle \cdot, \cdot \rangle_q \). Given \( h \in \mathcal{H} \) the \( q \)-creation operator \( a_q^*(h) \) and the \( q \)-annihilation operator \( a_q(h) \) on \( \mathcal{F}_q(\mathcal{H}) \) are defined by

\[
a_q^*(h)\Omega = h,
\]

\[
a_q^*(h)(f_1 \otimes \cdots \otimes f_n) = h \otimes f_1 \otimes \cdots \otimes f_n,
\]

and

\[
a_q(h)\Omega = 0,
\]

\[
a_q(h)(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^n q^{i-1} \langle h, f_i \rangle_U f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n.
\]

The operators \( a_q^*(h) \) and \( a_q(h) \) are bounded operators on \( \mathcal{F}_q(\mathcal{H}) \) and they are adjoints of each other (see [BKS, Remark 1.2]).

Following [Sh1] we consider the von Neumann algebra \( \Gamma_q(\mathcal{H}_R, U_t)^\prime \), called a q-deformed Araki-Woods algebra, generated on \( \mathcal{F}_q(\mathcal{H}) \) by

\[
s_q(h) := a_q^*(h) + a_q(h), \quad h \in \mathcal{H}_R.
\]

The vacuum state \( \varphi (= \varphi_{q,U}) := \langle \Omega, \cdot \rangle_q \) on \( \Gamma_q(\mathcal{H}_R, U_t)^\prime \) is called the q-quasi-free state.
**Proposition 1.1** Ω is cyclic and separating for \( \Gamma_q(\mathcal{H}_R, U_t)'' \).

One can canonically extend \( U_t \) on \( \mathcal{H} \) to a one-parameter unitary group (the so-called second quantization) \( \mathcal{F}_q(U_t) \) on \( \mathcal{F}_q(\mathcal{H}) \) by

\[
\mathcal{F}_q(U_t)\Omega = \Omega,
\]

\[
\mathcal{F}_q(U_t)(f_1 \otimes \cdots \otimes f_n) = (U_tf_1) \otimes \cdots \otimes (U_tf_n).
\]

Notice \( \mathcal{F}_q(U_t)a_q^*(h)\mathcal{F}_q(U_t)^* = a_q^*(U_th) \) for \( h \in \mathcal{H} \) so that

\[
\mathcal{F}_q(U_t)s_q(h)\mathcal{F}_q(U_t)^* = s_q(U_th), \quad h \in \mathcal{H}_R.
\]

Thus, \( \alpha_t := \text{Ad} \mathcal{F}_q(U_t) \) defines a strongly continuous one-parameter automorphism group on \( \Gamma_q(\mathcal{H}_R, U_t)'' \).

**Proposition 1.2** The \( q \)-quasi-free state \( \varphi \) on \( \Gamma_q(\mathcal{H}_R, U_t)'' \) satisfies the KMS condition with respect to \( \alpha_t \) at \( \beta = 1 \).

Let \( (\mathcal{K}_R, V_t) \) be another pair of a separable real Hilbert space and a one-parameter group \( V_t \) of orthogonal transformations on \( \mathcal{K}_R \). Let \( T : \mathcal{H}_R \rightarrow \mathcal{K}_R \) be a contraction such that \( TU_t = V_tT \) for all \( t \in \mathbb{R} \). By linearity \( T \) extends to a contraction \( T : \mathcal{H}_C \rightarrow \mathcal{K}_C \) and it satisfies \( TU_t = V_tT \) on \( \mathcal{H}_C \). Let \( B \) be the generator of \( V_t \) so that \( V_t = B^t \). Since

\[
TA(1 + A)^{-1} = B(1 + B)^{-1}T,
\]

\( T \) can further extend to a contraction from \( (\mathcal{H}, \langle \cdot, \cdot \rangle_U) \) to \( (\mathcal{K}, \langle \cdot, \cdot \rangle_V) \). Then:

**Proposition 1.3** There is a unique completely positive normal contraction \( \Gamma_q(T) : \Gamma_q(\mathcal{H}_R, U_t)'' \rightarrow \Gamma_q(\mathcal{K}_R, V_t)'' \) such that

\[
(\Gamma_q(T)x)\Omega = \mathcal{F}_q(T)(x\Omega), \quad x \in \Gamma_q(\mathcal{H}_R, U_t)'',
\]

where \( \mathcal{F}_q(T) : \mathcal{F}_q(\mathcal{H}) \rightarrow \mathcal{F}_q(\mathcal{K}) \) is given by

\[
\mathcal{F}_q(T)(f_1 \otimes \cdots \otimes f_n) = (Tf_1) \otimes \cdots \otimes (Tf_n).
\]

In this way, we have presented a \( q \)-analogue of Shlyakhtenko's free CAR functor; namely, a von Neumann algebra with a specified state, \( (\Gamma_q(\mathcal{H}_R, U_t)'', \varphi) \), is associated to each real Hilbert space with a one-parameter group of orthogonal transformations, \( (\mathcal{H}_R, U_t) \), and a unital completely positive state-preserving map \( \Gamma_q(T) : \Gamma_q(\mathcal{H}_R, U_t)'' \rightarrow \Gamma_q(\mathcal{K}_R, V_t)'' \) to every contraction \( T : (\mathcal{H}_R, U_t) \rightarrow (\mathcal{K}_R, V_t) \).

When \( q = 0 \), \( \Gamma(\mathcal{H}_R, U_t)'' \equiv \Gamma_0(\mathcal{H}_R, U_t)'' \) is a free Araki-Woods factor (of type III) in [Sh1]. On the other hand, when \( U_t = \text{id} \) a trivial action, \( \Gamma_q(\mathcal{H}_R)'' \equiv \Gamma_q(\mathcal{H}_R, \text{id}) \) is a \( q \)-deformation of the free group factor in [BKS]; in particular, \( \Gamma_0(\mathcal{H}_R)'' \cong L(\mathcal{F}_{\dim \mathcal{H}_R}) \) a free group factor.
2 Factoriality and non-injectivity of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)$

The following were proven in [BS2, BKS], but it is still open whether $\Gamma_q(\mathcal{H}_\mathbb{R})''$ is a non-injective type $\Pi_1$ factor whenever $\dim \mathcal{H}_\mathbb{R} \geq 2$.

(i) If $-1 < q < 1$ and $\dim \mathcal{H}_\mathbb{R} > 16/(1-|q|)^2$, then $\Gamma_q(\mathcal{H}_\mathbb{R})''$ is not injective.

(ii) If $\dim \mathcal{H}_\mathbb{R} = \infty$, then $\Gamma_q(\mathcal{H}_\mathbb{R})$ is a factor (of type $\Pi_1$) for all $-1 < q < 1$.

These results can be extended to $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)$ as follows.

Theorem 2.1 If there is $T \in [1, \infty)$ such that

$$\frac{\dim E_A([1, T]) \mathcal{H}_\mathbb{C}}{T} > \frac{16}{(1-|q|)^2}$$

where $E_A$ is the spectral measure of $A$, then $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)$'' is not injective. In particular, $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)$'' is not injective if $A$ has a continuous spectrum.

Theorem 2.2 Assume that the almost periodic part of $(\mathcal{H}_\mathbb{R}, U_t)$ is infinite dimensional, that is, $A$ has infinitely many mutually orthogonal eigenvectors. Then

$$\left(\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''\right)_{\varphi}' \cap \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' = \mathbb{C}1,$$

where $\left(\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''\right)_{\varphi}'$ is the centralizer of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'''$ with respect to the vacuum state $\varphi$. In particular, $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is a factor.

3 Type classification of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)$

As usual let $S_\varphi$ be the closure of the operator given by

$$S_\varphi(x\Omega) = x^*\Omega, \quad x \in \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'',$$

and let $\Delta_\varphi, J_\varphi$ be the associated modular operator and the modular conjugation. Then the following are seen as in [Sh1]: For $h_1, \ldots, h_n \in \mathcal{H}_\mathbb{R}$,

$$S_\varphi(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = h_n \otimes h_{n-1} \otimes \cdots \otimes h_1,$$

and for $h_1, \ldots, h_n \in \mathcal{H}_\mathbb{R} \cap \text{dom} A^{-1}$,

$$\Delta_\varphi(h_1 \otimes \cdots \otimes h_n) = (A^{-1}h_1) \otimes \cdots \otimes (A^{-1}h_n).$$

Noting that $D := \{h + ig : h, g \in \mathcal{H}_\mathbb{R} \cap \text{dom} A^{-1}\}$ is a core of $A^{-1}$ (on $\mathcal{H}$) such that $U_t D = D$ for all $t \in \mathbb{R}$, we see that

$$\Delta_\varphi^t = \mathcal{F}_q(A^{-it}) = \mathcal{F}_q(U_{-t}), \quad t \in \mathbb{R}.$$

By this and Theorem 2.2 we obtain the following type classification result:
Theorem 3.1 Assume that $A$ has infinitely many mutually orthogonal eigenvectors. Let $G$ be the closed multiplicative subgroup of $\mathbb{R}_+$ generated by the spectrum of $A$ ($U_t = A^{it}$). Then $\Gamma_q(\mathcal{H}_R, U_t)$ is a non-injective factor of type $II_1$ or type $III_\lambda$ ($0 < \lambda \leq 1$), and

$$\Gamma_q(\mathcal{H}_R, U_t)$$

is

\[
\begin{cases}
type\ II_1 & \text{if } G = \{1\}, \\
type\ III_\lambda & \text{if } G = \{\lambda^n : n \in \mathbb{Z}\} (0 < \lambda < 1), \\
type\ III_1 & \text{if } G = \mathbb{R}_+.
\end{cases}
\]

This result for free Araki-Woods factors (in case of $q = 0$) was shown in [Sh1, Sh2] generally when $\dim \mathcal{H}_R \geq 2$. Moreover, it was shown as a consequence of Barnett’s theorem that free Araki-Woods factors are full whenever $U_t$ is almost periodic (i.e. the eigenvectors of $A$ span $\mathcal{H}$). The assumption of Theorems 2.2 and 3.1 is a bit too restrictive while the following opposite extreme case is easy to see:

Proposition 3.2 If $U_t$ has no eigenvectors, then $\Gamma_q(\mathcal{H}_R, U_t)$ is a type $III_1$ factor.

It is worthwhile to note that the type $III_0$ case does not appear in the above type classifications.

For example, let $(\mathcal{H}_R, U_t) = \bigoplus_{k=1}^\infty (\mathbb{R}^2, V_t)$ where $V_t := \begin{bmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{bmatrix}$, $0 < \lambda \leq 1$, and write $(T_{q,\lambda}, \varphi_{q,\lambda}) := (\Gamma_q(\mathcal{H}_R, U_t)^\prime, \varphi)$ with two parameters $q \in (-1, 1)$ and $\lambda \in (0, 1]$. For $0 < \lambda < 1$, $T_{q,\lambda}$ is a type $III_\lambda$ $q$-deformed Araki-Woods factor. In particular when $q = 0$, $(T_{0,\lambda}, \varphi_{0,\lambda})$ coincides with the type $III_\lambda$ free Araki-Woods factor $(T_{\lambda}, \varphi_{\lambda})$ discussed in [Ra, Sh1]. For $\lambda = 1$, $T_{q,1}$ is the $q$-deformed type $II_1$ factor treated in [BKS].

The $C^*$-algebra $\Gamma_q(\mathcal{H}_R, U_t)$, $-1 < q < 1$, generated by $\{s_q(h) : h \in \mathcal{H}_R\}$ on $\mathcal{F}_q(\mathcal{H})$ is considered as the $q$-analogue of the CAR algebra. From this point of view, the above $T_{q,\lambda} (0 < \lambda < 1)$ may be considered as the $q$-analogue of Powers’ $III_\lambda$ factor. In fact, we remark that, for $q = -1$, our construction of $T_{q,\lambda}$ provides Powers’ $III_\lambda$ factor. To be more precise, for given $(\mathcal{H}_R, U_t)$, let $\Gamma_-(\mathcal{H}_R, U_t)$ denote the von Neumann algebra generated by $s_-(h) := a^*_-(h) + a_-(h)$ ($h \in \mathcal{H}_R$) on the Fermion Fock space $\mathcal{F}_-(\mathcal{H})$, where $a^*_-(h)$ and $a_-(h)$ are the Fermion (CAR) creation and annihilation operators. If

$$(\mathcal{H}_R, U_t) = \bigoplus_{k=1}^\infty (\mathcal{H}_R^{(k)}, U_t^{(k)})$$

where $\mathcal{H}_R^{(k)} = \mathbb{R}^2$, $U_t^{(k)} = \begin{bmatrix} \cos(t \log \lambda_k) & -\sin(t \log \lambda_k) \\ \sin(t \log \lambda_k) & \cos(t \log \lambda_k) \end{bmatrix}$ with $\lambda_k \leq 1$, then $(\Gamma_-(\mathcal{H}_R, U_t), \varphi := (\Omega, \cdot \Omega)_-)$ becomes an Araki-Woods factor

$$\bigoplus_{k=1}^\infty \left( M_2(\mathbb{C}), \text{Tr} \left( \begin{bmatrix} \frac{\lambda_k}{\lambda_k+1} & 0 \\ 0 & \frac{1}{\lambda_k+1} \end{bmatrix} \right) \right).$$

Upon these considerations we called $\Gamma_q(\mathcal{H}_R, U_t)$ a $q$-deformed Araki-Woods algebra.
4 Hypercontractivity of $\Gamma_q(T)$

When $T = e^{-t}1_{\mathcal{H}_R}$ $(t > 0)$, we obtain a semigroup $\Gamma_q(e^{-t})$ $(t > 0)$ of completely positive normal contractions on $\Gamma_q(\mathcal{H}_R, U_t)'$. This is a non-tracial extension of $q$-Ornstein-Uhlenbeck semigroup discussed in [Bi, Bo]. In the tracial case (i.e. the case of $U_t$ being trivial), the ultracontractivity for $\Gamma_q(e^{-t})$ was proven in [Bo] as follows:

$$||\Gamma_q(e^{-t})x|| \leq C_{|q|}^{3/2} \sqrt{\frac{1+e^{-2t}}{(1-e^{-2t})^{3}}} ||x\Omega||$$

with $C_{|q|}$ given below. In the non-tracial type III case, we have the following hypercontractivity property. This reduces to the above ultracontractivity when $A = 1$ or $\gamma = 0$.

**Theorem 4.1** Assume that $A$ is bounded (in particular, this is the case if $\dim \mathcal{H}_R < +\infty$), and let $\gamma := \frac{1}{2} \log ||A||$. If $-1 < q < 1$ and $t > \gamma$, then

$$||\Gamma_q(e^{-t})x|| \leq C_{|q|}^{3/2} \sqrt{\frac{1+e^{-2(t-\gamma)}}{(1-e^{-2t})(1-e^{-2(t-\gamma)})}} ||\Delta^\theta/2 x\Omega||$$

for all $x \in \Gamma_q(\mathcal{H}_R, U_t)'$ and $0 \leq \theta \leq 1$, where

$$C_{|q|} := \frac{1}{\prod_{m=1}^{\infty}(1-|q|^m)}.$$

It might be expected that the hypercontractivity given in the above theorem is valid for the whole $t > 0$. However, the next proposition says that it is impossible to remove the assumption $t > \gamma$, so Theorem 4.1 seems more or less best possible. Also, it says that the hypercontractivity in the sense that $||\Gamma_q(e^{-t})x|| \leq C||x\Omega||_q$ holds for some $t > 0$ and for all $x \in \Gamma_q(\mathcal{H}_R, U_t)'$ is impossible when $A$ is unbounded; for example, this is the case when $U_tf = f(\cdot + t)$ on $\mathcal{H}_R = L^2(\mathbb{R}; \mathbb{R})$.

**Proposition 4.2** Let $-1 < q < 1$, $0 \leq \theta \leq 1$ and $t > 0$. If there exists a constant $c > 0$ such that

$$||\Gamma_q(e^{-t})x|| \leq c||\Delta^\theta/2 x\Omega||,$$

then $A$ is bounded and

$$||A|| \leq \exp\left(\frac{2t}{\max\{\theta, 1-\theta\}}\right).$$

It seems that it is convenient to consider the hypercontractivity of $\Gamma_q(T)$ in the setting of Kosaki's interpolated $L^p$-spaces. For a general von Neumann algebra $\mathcal{M}$ and $1 \leq p \leq \infty$ let $L^p(\mathcal{M})$ be Haagerup's $L^p$-space. Given a faithful normal state $\varphi$ on $\mathcal{M}$ let $h_{\varphi}$ denote the element of $L^1(\mathcal{M})$ ($\cong \mathcal{M}_\ast$) corresponding to $\varphi$. For each $1 < p < \infty$
and $0 \leq \theta \leq 1$, Kosaki's $L^p$-space $L^p(M; \varphi)_\theta$ with respect to $\varphi$ is introduced as the complex interpolation space

$$C_{1/p}(h_\varphi^\theta M h_\varphi^{1-\theta}, L^1(M))$$

equipped with the complex interpolation norm $\| \cdot \|_{p, \theta} (= \| \cdot \|_{C_{1/p}})$.

Let $T : \mathcal{H}_R \rightarrow \mathcal{K}_R$ be a contraction with $TU_t = V_tT$, $t \in \mathbb{R}$. The adjoint operator $T^* : \mathcal{K}_R \rightarrow \mathcal{H}_R$ is also a contraction satisfying $T^*V_t = U_tT^*$, $t \in \mathbb{R}$. For each $-1 < q < 1$ let

\[
\begin{align*}
\mathcal{M} &:= \Gamma_q(\mathcal{H}_R, U_t)^{''} \quad \text{with} \quad \varphi = \langle \Omega, \cdot \Omega \rangle_q, \\
\mathcal{N} &:= \Gamma_q(\mathcal{K}_R, V_t)^{''} \quad \text{with} \quad \psi = \langle \Omega, \cdot \Omega \rangle_q,
\end{align*}
\]

where the vacuums in $\mathcal{F}_q(\mathcal{H})$ and in $\mathcal{F}_q(\mathcal{K})$ are denoted by the same $\Omega$. Then, by Proposition 1.3 the completely positive normal contractions

$$\Gamma_q(T) : \mathcal{M} \rightarrow \mathcal{N} \quad \text{and} \quad \Gamma_q(T^*) : \mathcal{N} \rightarrow \mathcal{M}$$

are determined by

\[
\begin{align*}
(\Gamma_q(T)x)\Omega &= \mathcal{F}_q(T)(x\Omega), \quad x \in \mathcal{M}, \\
(\Gamma_q(T^*)y)\Omega &= \mathcal{F}_q(T^*)(y\Omega), \quad y \in \mathcal{N}.
\end{align*}
\]

One can define the contraction $\omega \mapsto \omega \circ \Gamma_q(T^*)$ of $\mathcal{M}_*$ into $\mathcal{N}_*$. Via $\mathcal{M}_* \cong L^1(\mathcal{M})$ and $\mathcal{N}_* \cong L^1(\mathcal{N})$ this induces the contraction $\tilde{\Gamma}_q(T)$ of $L^1(\mathcal{M})$ into $L^1(\mathcal{N})$ as follows:

$$\tilde{\Gamma}_q(T)h_\omega = h_{\omega 0\Gamma_q(T^{*})}, \quad \omega \in \mathcal{M}_*.$$  

We see that for every $0 \leq \theta \leq 1$ and $x \in \mathcal{M}$,

$$\tilde{\Gamma}_q(T)(h_\varphi^\theta x h_\varphi^{1-\theta}) = h_\psi^\theta(\Gamma_q(T)x)h_\psi^{1-\theta},$$

so that $\tilde{\Gamma}_q(T) : L^1(\mathcal{M}) \rightarrow L^1(\mathcal{N})$ is the (unique) continuous extension of the linear mapping from $h_\varphi^\theta M h_\varphi^{1-\theta} (\subseteq L^1(\mathcal{M}))$ into $h_\psi^\theta N h_\psi^{1-\theta} (\subseteq L^1(\mathcal{N}))$ given by

$$h_\varphi^\theta x h_\varphi^{1-\theta} \mapsto h_\psi^\theta(\Gamma_q(T)x)h_\psi^{1-\theta}, \quad x \in \mathcal{M}.$$ 

Moreover, the Riesz-Thorin theorem implies that for each $0 \leq \theta \leq 1$ and $1 \leq p \leq \infty$, $\tilde{\Gamma}_q(T)$ maps $L^p(\mathcal{M}; \varphi)_\theta$ into $L^p(\mathcal{N}; \psi)_\theta$ and

$$\|\tilde{\Gamma}_q(T)a\|_{p, \theta} \leq \|a\|_{p, \theta}, \quad a \in L^p(\mathcal{M}; \varphi)_\theta.$$ 

The next theorem is shown by using Theorem 4.1.

**Theorem 4.3** Assume that either $A$ ($U_t = A^{it}$) or $B$ ($V_t = B^{it}$) is bounded, and let $\rho := \min\{|\lambda(A)|, |\lambda(B)|\}$. Let $T : \mathcal{H}_R \rightarrow \mathcal{K}_R$ be a bounded operator such that $TU_t = V_tT$ for all $t \in \mathbb{R}$ and $\|T\| < \rho^{-1}$. Then $\tilde{\Gamma}_q(T)$ maps $L^1(\mathcal{M})$ into $\bigcap_{0 \leq \theta \leq 1} h_\varphi^\theta \mathcal{F}_q(\mathcal{K})$ and

$$\|\tilde{\Gamma}_q(T)a\|_{\infty, \theta} \leq C_{|\lambda|, 1}(1 + \rho^{1/2} \|T\|) \frac{1}{(1 - \|T\|)(1 - \rho^{1/2} \|T\|)(1 - \rho \|T\|)} \|a\|_1$$ 

for all $a \in L^1(\mathcal{M})$, $0 \leq \theta \leq 1$. 
References


