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<td>COMPUTATIONS OF CHOW RINGS AND THE MOD $p$ MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES (Cohomology theory of finite groups and related topics)</td>
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ABSTRACT. In this note, we explain how to compute mod p motivic cohomology over \( \mathbb{C} \), the complex number field, by only using algebraic topology. Examples of algebraic spaces \( X \) are classifying spaces \( BG \) of algebraic groups.

1. CHOW RING, MILNOR K-THEORY, ÉTALE COHOMOLOGY

We use some category \( Spc \) of (algebraic) spaces, defined by Voevodsky, where schemes \( A \), quotients \( A_1/A_2 \) and \( cotim(A_\alpha) \) are all contained ([Vo2],[Mo-Vo]). Here schemes are defined over a field \( k \) with \( ch(k) = 0 \). The motivic cohomology is the double indexed cohomology defined by Suslin and Voevodsky directly related with the Chow ring, Milnor K-theory and étale cohomology.

\[
\text{(CH)} \quad H^{2n,n}(X) = CH^n(X) : \text{the classical Chow group.}
\]

\[
\text{(MK)} \quad H^{n,n}(Spc(k)) \cong K^n_M(k), \text{the Milnor } K \text{-group for the field } k.
\]

For a smooth variety \( X \) of \( \text{dim}(X) = n \). The Chow ring is the sum \( CH^*(X) = \oplus CH^i(X) \) where

\[
CH^i(X) = \{(n-i)\text{cycles in } X\}/(\text{rational equivalence}).
\]

Here the rational equivalence \( a \equiv b \) is defined if there is a codimension \( i \) subvariety \( W \) in \( X \times \mathbb{P}^1 \) such that \( a = p_*f^*(0) \) and \( b = p_*f^*(1) \) where \( \mathbb{P}^1 \) is the projective line, \( p\text{(resp. } f) \) is the projection for the first (resp. second) factor.

The multiplications in \( CH^*(X) \) is giving by intersections of cycles. Let \( k = \mathbb{C} \). Let \( \mathbb{P}^n \) be the \( n \)-dimensional projective space. Then \( CH^i(\mathbb{P}^n) \cong \mathbb{Z}\{L_{n-i}\} \) where \( L_{n-i} \cong \mathbb{P}^{n-1} \) is an \( n-i \)-dimensional subspace of \( \mathbb{P}^n \). Hence the product is \( L_{n-i} \cdot L_{n-j} = L_{n-i-j} \). This shows that

\[
CH^*(\mathbb{P}^n) \cong \mathbb{Z}[y]/(y^{n+1}) \cong H^*(\mathbb{C P}^n)
\]

identifying \( y^i = L_{n-i} \).

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Since $Spc$ contains colimit, we can consider the infinite projective space $P^\infty = B\mathbb{G}_m$ and the infinite Lens space $\varprojlim n(A_n - \{0\}/\mathbb{Z}/p) = L_p^\infty = B\mathbb{G}_m$. The Chow rings of $B\mathbb{G}_m$ are given in [To 1]

\[(1.1) \quad CH^*(P^\infty) \cong H^{2*}(P^\infty) \cong \mathbb{Z}[y], \quad CH^*(B\mathbb{G}_m) \cong H^{2*}(B\mathbb{G}_m) \cong \mathbb{Z}[y]/(py)\]

with $\text{deg}(y) = (2, 1)$. For product of these spaces

\[(1.2) \quad CH^*(P^\infty \times \ldots \times P^\infty) \cong \mathbb{Z}[y_1, \ldots, y_n]\]

\[(1.3) \quad CH^*(B\mathbb{G}_m \times \ldots \times B\mathbb{G}_m) \cong \mathbb{Z}[y_1, \ldots, y_n]/(py_1, \ldots py_n)\]

Here note that $CH^*(X) \not\cong H^{even}(X(C))$ for the last case. Even if $H^*(X(C))$ is generated by even dimensional elements, there are cases that $CH^*(X) \not\cong H^*(X(C))$, e.g., the K3-surfaces have the cohomology $H^2(X(C)) \cong \mathbb{Z}^2$ but there is a K3-surface such that $CH^1(X) \cong \mathbb{Z}^4$ for each $1 \leq i \leq 20$.

The Milnor K-theory is the graded ring $\oplus_n K_n^M(k)$ defined by $K_n^M(k) = (k^*)^\otimes n/J$ where the ideal $J$ is generated by elements $a \in (1 - a)$ for $a \in k^*$. Hence $K_0^M(k) = \mathbb{Z}$ and by definition $K_n^M(k)$ is just the multiplicative group $k^*$ but written additively in the ring $K_n^M(k)$. Hilbert's theorem 90, which is essentially said that the Galois cohomology $H^1(G(k_s/k); k^*_s) = 0$, implies the isomorphism $K_n^M(k)/p \cong k^*/(k^*)^p \cong H^1(G(k_s/k); \mathbb{Z}/p)$ for $1/p \in k$. Similarly we can define a map (the norm residue map) for any extension $F$ of $k$ of finite type

\[(BK) \quad K_n^M(F)/p \to H^n(G(F_s/F); \mu_p^\otimes n)\]

where $\mu_p^\otimes n$ is the discrete $G(F_s/F)$-module of $n$-th tensor power of the group of $p$-roots of 1.

The Bloch-Kato conjecture is that this map is an isomorphism for all field $k$ and the Milnor conjecture is its $p = 2$ case. This conjecture is solved when $n = 2$ by Merkurjev-Suslin[Me-Su], and for $p = 2$ by Voevodsky [Vo1] by using the motivic cohomology.

Notice that $H^n(G(k_s/k); \mu_p^\otimes n) \cong H^n_{et}(Spec(k), \mu_p^\otimes n)$ the étale cohomology of the point.

The étale cohomology $H^n_{et}(X; \mathbb{Z}/p)$ has the properties:

(E.1) If $k$ contains a primitive $p$-th root of 1, then there is the additive isomorphism

$$H^m_{et}(X, \mu_p^\otimes n) \cong H^m_{et}(X; \mathbb{Z}/p).$$

(E.2) For smooth $X$ over $k = \mathbb{C}$,

$$H^m_{et}(X; \mathbb{Z}/p^N) \cong H^m(X(C); \mathbb{Z}/p^N) \quad \text{for all} \ n \geq 1.$$
2. THE REALIZATION MAP

In this section we consider the relation to the usual ordinary cohomology. Let $R$ be $\mathbb{Z}$ or $\mathbb{Z}/p$. The motivic cohomology has the following properties [Vo2].

(C1) $H^{*,*}(X; R)$ is a bigraded ring natural in $X$.

(C2) There are maps (realization maps)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X; R) \to H^{m}(X(\mathbb{C}); R)$$

which sum up $t_{\mathbb{C}}^{m,n} = \oplus_{m,n} t_{\mathbb{C}}^{m,n}$ the natural ring homomorphism.

(C3) There are (the Bockstein, the reduced powers) operations

$$\beta : H^{*,*}(X; \mathbb{Z}/p) \to H^{*,+1,*}(X; \mathbb{Z}/p)$$

$$P^i : H^{*,*}(X; \mathbb{Z}/p) \to H^{*+2i(p-1),+i+1}(X; \mathbb{Z}/p)$$

which commutes with the realization map $t_{\mathbb{C}}$.

(C4) For the projective space $\mathbb{P}^n$, there is an isomorphism

$$H^{*,*}(\mathbb{P}^n; \mathbb{Z}/p) \cong H^{*,*}(\mathbb{P}^{n-1}; \mathbb{Z}/p) \cong H^{*,*}(X; R)(1, y')$$

with $\deg(y') = (2n, n)$ and $t_{\mathbb{C}}(y') \neq 0$.

Here we consider some examples. Recall $H^{*}(\mathbb{P}^\infty = \mathbb{P}^\infty(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p[y], \deg(y) = 2$ and $H^{*}(\mathbb{P}^2/\mathbb{P}(\mathbb{C}) = \mathbb{P}^2/\mathbb{P}; \mathbb{Z}/p) \cong \mathbb{Z}/p[y] \otimes \Lambda(x)$ with $\beta x = y$ (when $p = 2$, $y = x^2$). From the above properties (C1), (C2), we easily see that $t_{\mathbb{C}}$ is epic for $X = \mathbb{P}^\infty$. Moreover there is $x' \in H^{1,1}(\mathbb{P}^2/\mathbb{P}; \mathbb{Z}/p)$ such that $t_{\mathbb{C}}(x') = x$ and from (C2), we also see $t_{\mathbb{C}}$ is epic for $X = B\mathbb{Z}/p$.

To see these facts hold for other spaces, we recall the Lichtenbaum motivic cohomology [Vo2]. Lichtenbaun defined the similar cohomology $H^{*,*}_L(X; R)$ by using the étale topology, while $H^{*,*}(X; R)$ is defined by using Nisnevich topology. Since Nisnevich covers are some restricted étale covers, there is the natural map $H^{*,*}(X; R) \to H^{*,*}_L(X; R)$. We say that the condition $B(n, p)$ holds if

$$B(n, p) : H^{m,n}(X; \mathbb{Z}/(p)) \cong H^{m,n}_L(X; \mathbb{Z}/(p)) \quad for \ all \ m \leq n + 1$$

and all smooth $X$. The Beilinson-Lichtenbaum conjecture is that $B(n, p)$ holds for all $n$, $p$. It is proved that the $B(n, p)$ condition is equivalent the Bloch-Kato conjecture (BK) for degree $n$ and prime $p$. Hence $B(n, p)$ holds for $n \leq 2$ or $p = 2$. Moreover Suslin-Voevodsky proves

(L-E) If $1/p \in k$, then for all $X$,

$$H^{m,n}_L(X; \mathbb{Z}/(p)) \cong H^{m,n}_{et}(X; \mu_{\mathbb{Z}/(p)}^\oplus).$$

Now we compute $H^{*,*}(\mathbb{P}^2 = Spec(k); \mathbb{Z}/p)$. For a smooth $X$, it is known the following dimensional conditions:

(C5) For a smooth $X$, if $H^{m,n}(X; R) \neq 0$, then

$$m \leq n + \dim(X), \ m \leq 2n \ and \ m \geq 0.$$
Hereafter this paper, we assume that \( k \) contains a primitive \( p \)-th root of 1 and \( B(n,p) \) holds for all \( n \) but \( X = \text{Spec}(k) \). Then

\[
H^{m,n}(pt; \mathbb{Z}/p) \cong H^{m}_{\text{et}}(pt; \mu_{p}^{\otimes n}) \cong H^{m}_{\text{et}}(pt; \mathbb{Z}/p) \quad \text{if} \quad m \leq n
\]

and \( H^{m,n}(pt; \mathbb{Z}/p) \cong 0 \) otherwise. Let \( \tau \in H^{0,1}(pt; \mathbb{Z}/p) \) be the element corresponding a generator of \( H^{0}_{\text{et}}(\text{Spec}(k); \mu_{p}) \cong H^{0}_{\text{et}}(\text{Spec}(k); \mathbb{Z}/p) \). Then we get the isomorphism

\[
H^{**}(\text{Spec}(k); \mathbb{Z}/p) \cong H^{**}_{\text{et}}(\text{Spec}(k); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]
\]

since \( \tau : H^{0}_{\text{et}}(\text{Spec}(k); \mu_{p}) \cong H^{0}_{\text{et}}(\text{Spec}(k); \mathbb{Z}/p)^{(n+1)} \). In particular, for the real number field \( R \) and a local field \( F_{v} \) with the residue field \( k_{v} \) of \( ch(k_{v}) \neq 2 \)

\[
H^{**}(\text{Spec}(R); \mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(x) \beta
\]

where the degree is defined by \( \deg(x) = (m, m) \) if \( x \in H^{m}(\text{Spec}(C); \mathbb{Z}/p) \).

For \( k = C \), \( B(n,p) \) condition holds for \( X = \text{Spec}(C) \), indeed \( K^M_n(C) \cong 0 \) for \( n > 0 \).

Therefore

\[
H^{**}(\text{Spec}(C); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \quad \text{with} \quad \deg(\tau) = (0, 1).
\]

When \( k = C \), if \( B(n,p) \) condition holds for \( X \), then it is immediate that

\[
H^{**}(\text{Spec}(C); \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \quad \text{where} \quad \deg(\tau) = (0, 1).
\]

Next we compute cohomology of \( P^\infty \) and \( BZ/p \). For any (algebraic) map \( f : X \to Y \) in the category \( \text{Sp} \), we can construct the cofiber sequence

\[
X \to Y \to \text{cone}(f) = Y/X
\]

which induces the long exact sequence (Voevodsky [V2])

\[
H^{**}(X; R) \to H^{**}(Y; R) \to H^{**}(Y/X; R) \to H^{**-1}(X; R).
\]

In particular, we get the Mayer-Vietoris, Gysin and blow up long exact sequences.

By the cofiber sequence \( P^n \to P^n/P^{n-1} \) and (C4), we can inductively see that

\[
H^{*-1,*}(P^n; \mathbb{Z}/p) \cong H^{*(P^n; \mathbb{Z}/p)} \otimes \mathbb{Z}/p[\gamma]/(\gamma^{n+1}) \quad \text{with} \quad \deg(\gamma) = (2, 1)
\]

Since \( B(1,p) \) is always holds, \( H^{1,1}(L^n_p; \mathbb{Z}/p) \cong H^1(L^n_p; \mathbb{Z}/p) \). Hence there is the element \( x' \in H^{1,1}(L^n_p; \mathbb{Z}/p) \) with \( tc(x') = x \in H^{1}(L^n_p; \mathbb{Z}/p) \). The Lens space is identified with the sphere bundle associated with the line bundle

\[
(A^n - \{0\}) \times (A - \{0\}) \to (A^n - \{0\})/(A - \{0\}) = P^n.
\]

Where \( (A^n - \{0\}) \times (A - \{0\}) \to (A^n - \{0\})/(A - \{0\}) = P^n \). Hence we get the ring isomorphism for \( \rho = k^* \)

\[
H^{**}(L^n_p; \mathbb{Z}/p) \cong Z/p[\gamma]/(\gamma^{n+1}) \otimes \text{cone}(f) \quad \text{with} \quad \deg(\gamma) = (1, 1).
\]

However note that when \( p = 2 \), we see \( x^2 = y^2 + x^2 \rho \) [Vo3] where \( \rho \in H^{1,1}(pt; \mathbb{Z}/p) \cong k^*/k^2 \) represents \( -1 \). (hence \( \rho = 0 \) when \( \sqrt{-1} \in k^* \)). This is proved by the wellknown facts \( \{a, -1\} \in k^2(k) \).
Let us say that a space $X$ satisfies the Kunneth formula for a space $Y$ if $H^{**}(X \times Y; \mathbb{Z}/p) \cong H^{**}(X; \mathbb{Z}/p) \otimes H^{**}(Y; \mathbb{Z}/p)$.

By the above cofiber sequences, we can easily see that $P^\infty$ and $B\mathbb{Z}/p$ satisfy the Kunneth formula for all spaces. In particular, we have the ring isomorphisms

\begin{equation}
H^{**}(P^\infty \times \ldots \times P^\infty; \mathbb{Z}/p) \cong \mathbb{Z}[y_1, \ldots, y_n] \otimes H^{**}(pt; \mathbb{Z}/p)
\end{equation}

(2.8)

\begin{equation}
H^{**}(B\mathbb{Z}/p \times \ldots \times B\mathbb{Z}/p; \mathbb{Z}/p) \cong \mathbb{Z}[y_1, \ldots, y_n] \otimes \Lambda(x_1, \ldots, x_n) \otimes H^{**}(pt; \mathbb{Z}/p)
\end{equation}

(2.9)

(when $p = 2$, $x_i^2 = y_i + x_i$).

This fact is used to defined the reduced power operation $P^i$ in (C3). Since the Sylow $p$ subgroup of the symmetric group $S_p$ of $p$-letters is isomorphic to $\mathbb{Z}/p$, we know the isomorphism

\begin{equation}
H^*(BS_i; \mathbb{Z}/p) \cong H^*(B\mathbb{Z}/p; \mathbb{Z}/p)^{F_p} \cong \mathbb{Z}[y][\Lambda(x)]
\end{equation}

with identifying $Y = y^{p-1}$ and $X = xy^{p-2}$. If $X$ is smooth (and suppose $p$ is odd for easy of arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

\begin{equation}
H^{2*}(X; \mathbb{Z}/p) \longrightarrow H^{2*}(X \times \mathbb{S}^p ES_p) \longrightarrow H^*(X; \mathbb{Z}/p) \otimes H^{**}(BS_p; \mathbb{Z}/p)
\end{equation}

where $i$ is the Gysin map for $p$-th external power, and $\Delta$ is the diagonal map. For $deg(x) = (2i, n)$, the reduced powers are defined as

\begin{equation}
\Delta^i(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes XY^{n-i-1}.
\end{equation}

Hence note $deg(P^i) = deg(Y^i) = deg(y^{i(1-1)}) = (2i, i)$.

Voevodsky defined $i$ for non smooth $X$ and by using suspensions maps, he defined reduced poweres for all degree elements in $H^{**}(X; \mathbb{Z}/p)$ for all $X$ [Vo 3].

Moreover we can see (Ho-Kriz [H-K])

\begin{equation}
H^{**}(BGL_n; \mathbb{Z}/p) \cong \mathbb{Z}[c_1, \ldots, c_n] \otimes H^{**}(pt; \mathbb{Z}/p)
\end{equation}

where the Chern class $c_i$ with $deg(c_i) = (2i, i)$ are identified with the elementary symmetric polynomial in $H^{**}(P^\infty \times \ldots \times P^\infty; \mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*}(BG; \mathbb{Z}/p)$ for each algebraic group $G$ and for each representation $\rho : G \rightarrow GL_n$.

3. $H^{**}(X; \mathbb{Z}/p)/Ker(t_c)$ and operation $Q_i$

In this section we always assume that $X$ is smooth and $k = \mathbb{C}$. Define a bidegree algebra by

\begin{equation}
h^{**}(X; \mathbb{Z}/p) = \oplus_{m,n} H^{m,n}(X; \mathbb{Z}/p)/Ker(t_{c}^{m,n}).
\end{equation}

Suppose that $B(n, p)$ condition holds. By isomorphisms $(B, p), (L-E), (E1)$ and (E2), we have

\begin{equation}
H^{m,n}(X; \mathbb{Z}/p) \cong H^{m,n}_L(X; \mathbb{Z}/p) \cong H^{m,n}_E(X; \mathbb{Z}/p) \cong H^{m,n}_p(X; \mathbb{Z}/p) \cong H^{n}(X(\mathbb{C}); \mathbb{Z}/p).
\end{equation}

The realization map $t_{c}^{m,n}$ induces this isomorphism. Let $F_i = Im(t_{c}^{m,n})$. Then $\bigcup_i F_i = H^*(X(\mathbb{C}); \mathbb{Z}/p)$ and define the graded algebra $gr H^*(X(\mathbb{C}); \mathbb{Z}/p) = \oplus F_{i+1}/F_i$. Thus we get the additive isomorphism

\begin{equation}
h^{**}(X; \mathbb{Z}/p) \cong gr H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]
\end{equation}

of bigraded rings. However the ring structures of both rings are different, in general. The cohomology $h^{**}(X; \mathbb{Z}/p)$ is isomorphic to a $\mathbb{Z} [\tau]$-subalgebra $B$ of $H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, r^{-1}]$.
with $\deg(x) = (|x|, |x|)$ such that $B[\tau^{-1}] \cong H^*(X(C); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}]$. Namely there is a $\mathbb{Z}/p$-basis $\{a_i\}$ of $H^*(X(C); \mathbb{Z}/p)$ such that $B = \mathbb{Z}/p\{a_i\} \otimes \mathbb{Z}/p[\tau]$ for some $t_f \geq 0$.

Here we recall the Milnor primitive operation $Q_i = [Q_{i-1}, P^{p^{i-1}}]$

$$Q_i : H^{*,*}(X; \mathbb{Z}/p) \to H^{*,*+2p^{i-1}+p^{i-1}X}_p(X; \mathbb{Z}/p)$$

which is derivative, $Q_i(xy) = Q_i(x)y + xQ_i(y)$. Note also $Q_i(\tau) = 0$ by dimensional reason of $H^*(pt; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau]$.

**Lemma 3.1.** If $0 \neq Q_i_1 ... Q_i_n x \in H^2^{*,*}(X; \mathbb{Z}/p)$, then $x$ is a $\mathbb{Z}/p[\tau]$-module generator.

**Proof.** If $x = x' \tau$ then $\tau Q_i_1 ... Q_i_n (x') \neq 0$. But $Q_i_1 ... Q_i_n (x') = 0 \in H^{2*,*}(X; \mathbb{Z}/p)$ since $H^{m,n}(X; \mathbb{Z}/p) = 0$ for $m > 2n$.

Define the weight by $w(x) = 2n - m$ for an element $x \in H^{m,n}(X; \mathbb{Z}/p)$ so that $w(x') = 0$ for $x' \in CH^*(X)$. Of course we get $w(xy) = w(x) + w(y)$, $w(P^s x) = w(x)$ and $w(Q_i(x)) = w(x) - 1$.

**Corollary 3.2.** Suppose that $B(n, p)$ holds. If $x \in H^n(X(C); \mathbb{Z}/p)$ and $Q_{i_1} ... Q_{i_n}(x) \neq 0$, then there is a $\mathbb{Z}/p[\tau]$-module generator $x' \in H^{n,n}(X; \mathbb{Z}/p)$ so that $Q_i(x') = x$ and for each $0 \leq k \leq n$, $Q_{i_1} ... Q_{i_k}(x')$ is also a $\mathbb{Z}/p[\tau]$-module generator of $H^{*,*}(X; \mathbb{Z}/p)$.

**Proof.** By $B(n, p)$ condition, $H^{n,n}(X; \mathbb{Z}/p) \cong H^n(X(C); \mathbb{Z}/p)$. Hence there is an element $x' \in H^{n,n}(X; \mathbb{Z}/p)$ with $Q_i(x') = x$. This means $w(x') = n$ and $w(Q_i(x)) = w(x) - 1$.

From the above lemma, we get the corollary.

Now we consider the examples. The mod 2 cohomology of $BO(n)$ is $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_n]$ where the Stiefel-Whitney class $w_i$ restricts the elementary symmetric polynomial in $H^*(BZ/(2^n); \mathbb{Z}/2) \cong \mathbb{Z}/2[z_1, ..., z_n]$. Each element $w_t^2$ is represented by Chern class $c_t$ of the induced representation $O(n) \subset U(n)$. Hence $c_t \in CH^*(BS(n); \mathbb{Z}/2) = H^{2*,*}(BO(n); \mathbb{Z}/2)$.

**Proposition 3.3.** $h^{*,*}(BO(n); \mathbb{Z}/p) \supset \mathbb{Z}/2[c_1, ..., c_n] \otimes \Delta(w_1, ..., w_n) \otimes \mathbb{Z}/2[\tau]$ where $\deg(c_t) = (2i, i)$, $\deg(w_i) = (i, i)$ and $w_t^2 = \tau^t c_t$.

Since $Q_{i-1} ... Q_0(w_i) \neq 0$, each $w_i$ is a $\mathbb{Z}/2[\tau]$-module generator. However even $h^{*,*}(BO(n); \mathbb{Z}/2)$ seems very complicated. Consider the case $X = BO(3)$. The cohomology operations act by

$$
\begin{align*}
& w_2 \xrightarrow{S^1} w_1w_2 + w_3 \quad w_2 \xrightarrow{S^2} w_2w_3 + w_1w_2 + w_2w_3 \xrightarrow{S^1} w_1w_2 + w_3
\end{align*}
$$

**Theorem 3.4.** There is the isomorphism

$$h^{*,*}(BO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, c_3][1, w_1, w_2, Q_0w_2, Q_1w_2, w_3, Q_0w_3, Q_1w_3] \otimes \mathbb{Z}/2[\tau].$$

where $Q_0w_2 = \tau^{-1}(w_1w_2 + w_3), ...$
Computations of Chow rings and the Mod $p$ motivic cohomology of classifying SR.

W.S. Wilson ([W], [K-Y]) found a good $Q(i) = \Lambda(Q_0, \ldots, Q_i)$-module decomposition for $X = BO(n)$, namely,

$$H^*(X; \mathbb{Z}/2) = \oplus_{-1} Q^i G$$

with $Q_0 \cdots Q_i G_i \in \mathcal{C}(CH^*(X))$. Here $G_{k-1}$ is quite complicated, namely, it is generated by symmetric functions

$$\sum x^2_{k+1} \cdots x_{k+j} \cdots x_{k+q}, \quad k + q \leq n,$$

with $0 \leq i_1 \leq \ldots \leq i_k$ and $0 \leq j_1 \leq \ldots \leq j_q$; and if the number of $j$ equal to $j_u$ is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

Then $w(G_i) \geq i$ in $h^{**}(X; \mathbb{Z}/p)$, that means

Proposition 3.5. Given the weight by $w(G_i) = i+1$, we have the inclusion for $X = BO(n)$

$$h^{**}(X; \mathbb{Z}/2) \subset (\oplus_{i} Q(i)G_i) \otimes \mathbb{Z}/2[\tau].$$

One problem is that the above inclusion is really isomorphism or not. The similar decomposition holds for $X = (B\mathbb{Z}/p)^n$ and the above inclusion is an isomorphism. The case $X = BO(3)$ is also isomorphism. Since the direct decomposition of $BO(3)$ is complicated to write, we only write here that of $SO(3)$ since $O(3) \cong SO(3) \times \mathbb{Z}/2$.

$$H^*(BSO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3] \cong \mathbb{Z}/2[c_2, c_3] \{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus \mathbb{Z}/2[c_2].$$

Since there is the isomorphism $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}/2$, the cohomology of $BSO(2n+1)$ is reduced from that of $BO(2n+1)$. However note that the situation for $BO(2n)$ is different.

The extraspecial 2-group $2^{1+2n}_+ \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is the $n$-th central product of the dihedral group $D_8$ of order 8. It has a central extension

$$(3.4) \quad 0 \twoheadrightarrow \mathbb{Z}/2 \twoheadrightarrow G \twoheadrightarrow V = \mathbb{Z}/2 \twoheadrightarrow 0$$

Let $H^*(BV; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \ldots, x_{2n}]$. Then Quillen proved [Q2]

$$(3.5) \quad H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \ldots, x_{2n}]/(f, Q_0f, \ldots, Q_{n-2}f) \otimes \mathbb{Z}/2[w_{2n}].$$

Here $w_{2n}$ is the Stiefel-Whitney class of the real $2^n$ dimensional irreducible representation restricting non zero on the center and $f = \sum x_{2i-1}x_{2i} \in H^2(BV; \mathbb{Z}/2)$ represents the central extension (3.4).

Letting $y_i = x_i^2$ in $H^*(BG; \mathbb{Z}/2)$, we can write

$$f^2 = \sum y_{2i-1}y_{2i}, \quad (Q_k f)^2 = Q_0Q_k f = \sum y_{2i-1}^{Q_k}y_{2i} - y_{2i-1}y_{2i}^{Q_k}.$$
Now we consider in the motivic cohomology $H^{**}(BG; \mathbb{Z}/2)$ and change $y_i = -1$. Since $f = 0 \in H^{2,2}(BG; \mathbb{Z}/2)$, we can see that $Q_{k-1}f = 0$ and $Q_k Q_0(f) = 0$ also in $H^{**}(BG; \mathbb{Z}/2)$. However for general $n$, $\sum y_i \neq 0$ in $H^{**}(BG; \mathbb{Z}/2)$. Let

\[ (3.6) \quad A = (Z/2[y_1, \ldots, y_{2n}, c_{2n}]/(Q_0 Q_k f, \ldots, Q_0 Q_n f)) \otimes Z/2[r]. \]

**Lemma 3.6.** For $G = \mathbb{Z}/2^{a+2n}$, there is a map $A \rightarrow H^{**}(BG; \mathbb{Z}/2)$ which induces the injection $A/(f^2) \subset h^{**}(BG; \mathbb{Z}/2)$.

When $m = 0, 1, -1 \mod 8$ and $m > 0$, we say that Spin$(m)$ is real type $[Q2]$. When Spin$(m)$ is real type, from Quillen, we know that $H^*(BSpin(m); \mathbb{Z}/2) \subset H^*(BG; \mathbb{Z}/2)$ where $G = \mathbb{Z}/2^{a+1}$. Let $x$ be the Hurwitz number (for details see $[Q2]$).

**Corollary 3.7.** Let $G = Spin(m)$ be real type and the Hurwitz number $h$, and let

\[ A = (Z/2[c_2, c_3, \ldots, c_m, c_{2h}]/(Q_1 Q_0 w_2, \ldots, Q_h Q_0 w_2)) \otimes \Delta(w_2, \ldots, w_{2n})/(c_2, Q_0 c_7) \otimes Z/2[r]. \]

where $w_i, i \leq m$ (resp. $w_{2h}$) is the Stiefel-Whitney class of the usual SO$(m)$ representation (resp. of the irreducible $2^h$-dimensional spin representation). Then we have a map $A \rightarrow H^{**}(BG; \mathbb{Z}/2)$ which induces the injection $A/(c_2) \subset h^{**}(BG; \mathbb{Z}/2)$.

We study Spin$(7)$ and the exceptional Lie group $G_2$. The cohomology of $G_2$ is given by $H^*(BG_2; \mathbb{Z}/2) \cong Z/2[w_4, w_6, w_7]$, and $w_i$ is the Stiefel-Whitney class of the inclusion $G_2 \subset SO(7)$. The cohomology $H^*(BSpin(7); \mathbb{Z}/2) \cong H^*(BG_2; \mathbb{Z}/2) \otimes Z/2[w_6]$.

**Corollary 3.8.** Let $A = Z/2[c_2, c_4, c_6, c_7] \otimes \Delta(w_4, w_6, w_7) \otimes Z/2[r]$. Then there is the map $A \rightarrow H^{**}(BG_2; \mathbb{Z}/2)$ which induces the injection $A/(c_2) \subset h^{**}(BG_2; \mathbb{Z}/2)$. Similar facts hold for BSpin$(7)$ tensoring $Z/2[c_8]$.

The cohomology operations are given

\[
\begin{align*}
Q_1 Q_0 (w_4 w_6) &= w_8^2, \\
Q_2 Q_1 Q_0 (w_4 w_6 w_7) &= w_6^4.
\end{align*}
\]

**Proposition 3.9.** Let $w(w_4) = 2, w(w_{(4,6)}) = 2$ and $w(w_{(4,6,7)}) = 3$ with $tc(w_{i_1 \ldots i_n}) = w_{i_1 \ldots i_n}$. Then we have the injection

\[ h^{**}(BG_2; \mathbb{Z}/2) \subset Z/2[c_4, c_6, c_7] \]

\[ \otimes Z/2[1, w_4, S_2 w_4, Q_1 w_4, Q_2 w_4, S_2 Q_2 w_4, w_{(4,6)}, w_6, w_{(4,6,7)}] \otimes Z/p[r]. \]

Remark. If $tc^{3,4} \otimes Q$ is epic, then we can take $w_4 \in h^{4,3}(BG_2; \mathbb{Z}/2)$, i.e., $w(w_4) = 2$. The kernel $Ker(tc^{3,4})$ is not so big for $X = BG_2$. Indeed, it is known that

\[ CH^*(BG_2) \cong Z_2[c_2, c_4, c_6, c_7]/(2^*(c_2^2 - 4c_4), 2c_7, c_2 c_7), \quad \text{for some } r \geq 0. \]

The cohomology operations are given in $H^*(BSO(7); \mathbb{Z}/2)$

\[ Q_1 Q_0 w_2 = w_4^2, \quad Q_2 Q_0 w_2 = w_6^2, \quad Q_3 Q_0 w_2 = w_6^2 w_6^2 + w_6^2 w_6^2 + w_6 w_6^2. \]

Hence we have $c_3 = 0, c_5 = 0, c_2 c_7 = 0$ in $CH^*(BG_2)$ but $c_2 \neq 0$. 

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From here we consider the case $p = \text{odd}$. One of the easiest examples is the case $G = \text{PGL}_3$ and $p = 3$. The mod 3 cohomology is given by $([\text{K-Y}],[\text{Ve}])$

$$
\left( \mathbb{Z}/3[y_2]\right) \oplus \mathbb{Z}/3(1) \oplus \mathbb{Z}/3[y_8] \oplus \mathbb{Z}/3[y_{12}]
$$

It is known that $y_2, y_3, y_8$ and $y_{12}$ are represented by Chern classes. Moreover $Q_1 Q_0 (y_2) = y_8$. Hence these elements are in the Chow ring, namely,

$$
h^{2n,2n}(\text{BPGL}_3; \mathbb{Z}/3) \cong \left( \mathbb{Z}/3[y_2]\right) \oplus \mathbb{Z}/3[y_8] \oplus \mathbb{Z}/3[y_{12}].
$$

The cohomology operations are given

$$
y_2 \xrightarrow{\beta} y_3 \xrightarrow{p^1} y_7 \xrightarrow{\beta} y_8
$$

Thus we get $h^{*,*}(\text{PGL}_3; \mathbb{Z}/3)$ completely.

**Theorem 3.10.**

$h^{*,*}(\text{BPGL}_3; \mathbb{Z}/3) \cong \left( \mathbb{Z}/3[y_2]\right) \oplus \mathbb{Z}/3(1) \oplus \mathbb{Z}/3[y_8] \oplus \mathbb{Z}/3[y_{12}]$  

Next consider the extraspecial $p$-group $G = p^{1+2n}$. When $n > 2$, even the cohomology ring $H^\ast (G(C); \mathbb{Z}/p)$ are unknown, while it contains the subring

$$
B = \mathbb{Z}/p[y_1, \ldots, y_{2n}, c_p]/(Q_1 Q_0 f, \ldots Q_n Q_0 f).
$$

where $f = \sum x_{2i-1} x_{2i}$ for $\beta x_i = y_i$ and $Q_k Q_0 f = \sum y_{2i-1} y_{2i}^p - y_{2i-1} y_{2i}$. Since $f = 0 \in H^{2,2}(BG; \mathbb{Z}/p)$, we have

**Proposition 3.11.** Let $G = p^{1+2n}$ and $A = B \otimes \mathbb{Z}/p [\tau]$. Then there is an injection $A \subset H^\ast (BG; \mathbb{Z}/p)$

We consider the case $n = 1$ here. Let us write $E = p^{1+2}$. The ordinary cohomology is known by Lewis [Lj], [Te-Y3], namely,

$$
H^{\text{even}}(BE)/p \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^2 y_2 - y_1 y_2^p)) \oplus \mathbb{Z}/p\{c_2, \ldots, c_{p-1}\} \otimes \mathbb{Z}/p[\mathbb{C}]$

$$
H^{\text{odd}}(BE) \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1, a_2\}/(y_1 a_2 - y_2 a_1, y_1 a_2 - y_2 a_1) \mid |a_i| = 3.
$$

**Theorem 3.12.**

$h^{*,*}(BE; \mathbb{Z}/p) \cong (\{1, \beta^{-1}\})(H^{\text{even}}(BE)/p - \{\beta^{-1}\}) \otimes \mathbb{Z}/p[\tau]$  

where $w(H^{\text{even}}(BE)/p) = 0, w(H^{\text{odd}}(BE)) = 1$ and $\beta^{-1}$ ascends the weight one.

**Proof.** Since all elements in $H^{\text{even}}(BE)$ are generated by Chern classes, we have the isomorphism $h^{2n,2n}(BG; \mathbb{Z}/3) \cong H^{2,2}(BE)/p$. We know $H^{\text{odd}}(BE; \mathbb{Z}/p)$ is generated as a $H^{\text{even}}(BE)$-$p$-module by two elements $a_1, a_2$ such that $Q_1 a_1 = y_1 c_p$ [Te-Y3].

The mod $p$-cohomology is written additively $H^{*}(BE; \mathbb{Z}/p) \cong \{1, \beta_p\} H^*(BE)/p$. Here $\beta_p$ is the (higher) Bockstein. All elements in $H^{\text{odd}}(BE)$ are just $p$-torsion and we can take $a'_1 \in H^2(BE; \mathbb{Z}/p)$ such that $\beta(a'_1) = a_1$. Thus we take $a'_1 \in H^2(BE; \mathbb{Z}/p)$ so that $a_1 = H^2(BE; \mathbb{Z}/p)$.

Next consider elements $x = \partial_p^{-1}(y), y \in H^{\text{even}}(BE)/p$. If $y \in (\text{Ideal}(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_i b_i$ for $b_i \in H^{\text{even}}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c y_c$ with $c \in \mathbb{Z}/p[c_p]$, we can prove ([Lj]) that the elements are represented by transfer from a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Therefore we can also prove that $w(\partial_p^{-1}(y)) = 1$. Thus we complete the proof.  


References


[Sc-Y] B. Schuster and N. Yagita. Transfer of chern classes in $BP$-cohomology and Cow rings. To appear in *Trans. AMS.*


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