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AN INFILTRATION GAME WITH TWO CABLES

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1. The Game Models.

The game theoretic models we develop are based on the following scenario. One side called Infiltrator needs to get supplies to one of its installations which is accessible only via a channel under the control of another side called Defender. Defender has placed a number of static underwater devices which can be placed in the channel and which will detect any vessel going near them. Furthermore Defender will be able to destroy any vessel it detects before the vessel can deliver its supplies to the installation. From now on we will use the terms cable and agent for static underwater device and vessel respectively.

The scenario can be modelled as a two-person zero-sum game of ambush as follows. Common sense tells us that it would be natural for Defender to place the cables at the narrowest part of the channel so, by choosing units appropriately, the game can be thought of as taking place in the unit interval [0,1]. Assuming that Defender has m cables with detection radii of $a_1/2, a_2/2, \ldots, a_m/2$, they can then be represented as line segments of lengths $a_1, a_2, \ldots, a_m$ which are placed in [0,1]; we will assume that $a_1 \geq \cdots \geq a_m$. The position of the cable can be uniquely defined by giving the coordinate of the left-hand endpoint of the interval it covers. Thus we can think of a Defender strategy as a choice of m points in $[0,1]$ where the i-th point x_i represents the i-th cable being laid in the closed interval $[x_i, x_i + a_i]$. From a strictly practical point of view, it would be necessary to impose the condition $x_i + a_i \leq 1$, but, to analyse the game, it is convenient to ignore this restriction and merely stipulate that the x_i lie in $[0,1]$. In the implementation of a strategy, one would then interpret an interval $[x_i, x_i+a_i]$ with $x_i+a_i > 1$ as the interval $[1-a_i, 1]$; this new interval clearly detects all the agents detected by the one it replaces. Agents attempting to pass through the channel will be detected or not depending on the position they have at the narrowest part of the channel. Thus, if Infiltrator has r agents at his disposal, we can think of Infiltrator as choosing r points, one for each agent, in [0,1]. Choosing an appropriate payoff function is not so clearcut but, if we take the stance that the Infiltrator installation desperately needs some supplies and that a single agent reaching it would provide relief for a satisfactory period, a natural payoff for the Defender is 1 if all the agents are detected and 0 otherwise. Such a payoff would represent the situation where the installation can be relieved via other means but these means will take some time to activate and the channel is the only way to provide supplies in the meantime.

More formally we let $\Gamma(a_1, a_2 \ldots, a_m; r)$ denote the two-person zero-sum game in [0,1] in which the Defender strategy space is $[0,1]^m$, the Infiltrator strategy space is $[0,1]^r$ and the payoff to Defender when the players use $x \in [0,1]^m$ and $y \in [0,1]^r$ is 1 if $y_j \in \cup_{i=1}^m [x_i, x_i + a_i]$ for $j = 1, \ldots, r$ and 0 otherwise. The value of $\Gamma(a_1, \ldots, a_m; r)$ is denoted by $v(a_1, \ldots, a_m; r)$.

This game with $r = 1$ was introduced, and the case $m = 1$ solved, by Ruckle et al. The special case $\Gamma(a_1, a_2; 1)$ has proved more challenging and a complete solution of it has not been found. Baston and Bostock and Lee have solved the cases $a_1 \geq 1/2$ and $1/3 \leq a_1 < 1/2$ respectively while Zoroa et al have obtained a number of results when various relations hold between a_1 and a_2. Very recently Woodward, a research student of the first author, has proved that $\Gamma(a_1, \ldots, a_m; r)$ can be solved by considering an "equivalent" (finite) matrix game. This means that linear programming can be employed on individual cases and, using linear programming results, Woodward has obtained solutions of $\Gamma(a_1, a_2, a_3; 1)$ for the case when $a_1 \geq 1/3$ and $a_3 \geq 1/5$. Woodward has also shown that his matrix game is closely related to the finite game introduced by Garnaev. Garnaev's game is similar to $\Gamma(a_1, \ldots, a_m; 1)$ but play takes place on an integer interval and integer intervals replace the intervals of length $a_1, \ldots, a_m$; details of the results on this game can be found in Garnaev.

The second model we develop differs from $\Gamma(a_1, \ldots, a_m; r)$ in just one respect, namely the information that is available to Defender. In this new model, which we denote by $\Gamma_r(a_1, \ldots, a_m)$, Defender does not know how many agents Infiltrator has at his disposal. This change radically alters the situation and $\Gamma_r(a_1, \ldots, a_m)$ is not a game in the same sense as $\Gamma(a_1, \ldots, a_m; r)$ is. However there are natural definitions of optimal strategy and value for $\Gamma_r(a_1, \ldots, a_m)$.

Definition 1.1. A Defender strategy D in $\Gamma_r(a_1, \ldots, a_m)$ is *optimal* if D is an optimal strategy in $\Gamma(a_1, \ldots, a_m; r)$ for all positive integers r.

In Section 4 we prove that Defender has an optimal strategy in $\Gamma_r(a_1, \ldots, a_m)$ when the $a_1, \ldots, a_m$ satisfy certain conditions and also produce evidence that it is unlikely that he always has one. If Defender does not have an optimal strategy in $\Gamma_r(a_1, \ldots, a_m)$, the strategy he thinks "most effective" will depend on his estimate of the probabilities p_r $(r = 1, 2, \ldots)$ of Infiltrator having r agents. Suppose, for example, Defender believes that Infiltrator has only one or two agents but that each is equally likely, then Defender may choose a strategy which is neither optimal against one agent nor against two but is "reasonably effective" against both. If Infiltrator has, in fact, two agents, it is not obvious that he should play a strategy which is optimal in $\Gamma(a_1, \ldots, a_m; 2)$ because Defender is not playing optimally in that game. Clearly any solution concept that copes with this situation has to have a more complicated structure than the standard zero-sum one. In fact the appropriate context for these problems is that of Bayesian games but, in this paper, we restrict attention to a simpler solution concept which, although less widely applicable than the Bayesian one, still yields solutions in a variety of cases.

In the light of the foregoing discussion, we introduce the following definitions.

Definition 1.2. An Infiltrator strategy S in $\Gamma_r(a_1, \ldots, a_m)$ is *optimal* if Defender has an optimal strategy in $\Gamma_r(a_1, \ldots, a_m)$ and S is optimal in $\Gamma(a_1, \ldots, a_m; r)$.

Definition 1.3. If both Defender and Infiltrator have optimal strategies in $\Gamma_r(a_1, \ldots, a_m)$, we say that $\Gamma_r(a_1, \ldots, a_m)$ has a value $v_r(a_1, \ldots, a_m)$ given by $\Gamma_r(a_1, \ldots, a_m) = \Gamma(a_1, \ldots, a_m; r)$.

It should be noted that, even when $\Gamma_r(a_1, \ldots, a_m)$ has a value, it is qualitatively different from $\Gamma(a_1, \ldots, a_m; r)$ because, although Defender knows how to play optimally in $\Gamma_r(a_1, \ldots, a_m)$, he does not know its value, only that it lies in the set $\{v(a_1, \ldots, a_m; r) : r = 1, 2, \ldots\}$.

Having addressed the situation in which Defender has incomplete information, it is natural to also consider the scenario in which Infiltrator has incomplete information. The position in this case is more complex and it appears difficult to obtain meaningful results if Infiltrator has no information concerning the number of cables Defender has or their lengths. However we see in Section 2 that Infiltrator can act optimally without knowing the number of cables provided he knows their lengths lie in particular ranges. Section 4 also contains a result which involves Infiltrator having only partial information concerning the cable lengths although he does need to know the number of cables.

2. Infiltrator Does Not Know the Number of Cables.

For all our models the easiest case to consider is the one in which all the cable lengths are equal and, if there is an emphasis on standardized production, it is also the most natural. Even so, this special case illustrates some of the difficulties which can arise and which make the general case challenging. We first consider the particular example with $a_1 = a_2 = 1/4$. It is easy to see that an optimal strategy for Defender in $\Gamma(1/4, 1/4; 1)$ is to choose $(0, 1/4)$ and $(1/2, 3/4)$ at random but, in $\Gamma(1/4, 1/4; 2)$, this is a very bad Defender strategy because Infiltrator can always get an agent through undetected by using the strategy $(0, 1)$; we will show later that $v(1/4, 1/4; 2) = 1/6 = v_2(1/4, 1/4)$ so there is an optimal Defender strategy in $\Gamma(1/4, 1/4; 1)$ which is also optimal in $\Gamma(1/4, 1/4; 2)$. This illustrates that, to prove that Defender does *not* have an optimal strategy in $\Gamma_r(a_1, \ldots, a_m)$, we effectively have to check that, for *every* optimal strategy D in $\Gamma(a_1, \ldots, a_m; 1)$, there is a t such that D is non-optimal in $\Gamma(a_1, \ldots, a_m; t)$. As is often the case, proving non-existence is likely to be difficult, apart (perhaps) from some very particular examples.

Our first result does not require that all the cable lengths are equal, only that they all lie in a particular range.

Theorem 2.1. Suppose, for some positive integer n, $1/n \le a_i < 1/(n-1)$ for $i = 1, \ldots, m$, then

$$v(a_1, \ldots, a_m; r) = \binom{n-r}{m-r} / \binom{n}{m}.$$

An optimal strategy $\mathcal{D}$ chooses m out of the $I_t = [(t-1)/n, t/n]$, $(t = 1, \ldots, n$ at random and covers these m intervals with his cable lengths. An optimal Infiltrator strategy $\mathcal{I}$ chooses r of the points $P_t = t/(n-1)$, $(t = 1, \ldots, n)$ at random and sends his infiltrators down the channel at these points.

Proof. Let the Defender use the strategy $\mathcal{D}$ and $(y_1 \ldots, y_r)$ be a strategy for the Infiltrator. If the y_i $(i = 1, \ldots, r)$ are in r distinct intervals $I_1 \ldots, I_n$, then the number of Defender (pure) strategies catching all r of the infiltrators is $\binom{n-r}{m-r.}$ If y_i and y_j with $i \neq j$ are in the same I_k, it is straightforward to verify that more than $\binom{n-r}{m-r}$ Defender (pure) strategies catch all r of the infitrators. Hence $v(a_1, \ldots, a_m; r) \ge \binom{n-r}{m-r} / \binom{n}{m}$.

Let Infiltrator use the strategy $\mathcal{D}$; note that a cable length can cover at most one point of the P_i $(i = 1, \ldots, n)$. Given a pure strategy for the Defender, it therefore follows that it can detect all of the infiltrators in at most $\binom{m}{r}$ of the Infiltrator (pure) strategies. Hence $v(a_1, \ldots, a_m; r) \le \binom{m}{r} / \binom{n}{r}$. The result follows because $\binom{m}{r} / \binom{n}{r} = \binom{n-r}{m-r} / \binom{n}{m}$. ∎

Notice that the Defender strategy $\mathcal{D}$ does not involve r so that the Defender does not need to know the value of r to play optimally. We therefore have the following corollary.

Corollary 2.2. Suppose, for some integer n, $1/n \le a_i < 1/(n-1)$ for $i = 1, \ldots, m$, then

$$v_r(a_1, \ldots, a_m) = \binom{n-r}{m-r} / \binom{n}{m}.$$

Note also that the Infiltrator strategy $\mathcal{I}$ only uses n and not the particular values of the a_i. Thus Infiltrator can play optimally without knowing the precise values of the a_i provided he knows that the values all lie in an interval $[1/n, 1/(n-1)]$ and the value of n.

The theorem tells the Defender that, if all the cable lengths are within an interval of the form $[1/n, 1/(n-1)]$ for some integer n, and the Infiltrator has this information, then he gets no benefits from having different different lengths of cable nor from having any cable lengths other than the smallest one $1/n$. This would be useful knowledge if, instead of having cables to hand, the Defender was having to decide on what cable lengths should be manufactured to defend a channel.

3. Game with Two Cables.

In this section we obtain upper and lower bounds on $v(a_1, a_2; 2)$ and show that, for some pairs (a_1, a_2), the bounds are equal. For a subset of these pairs, we go on to prove that $v_r(a_1, a_2)$ exists and determine its value.

Theorem 3.1. Let $\mu_1 = \lceil (1-a_1)/a_2 \rceil$, $\mu_2 = \lceil (1-2a_1)/a_2 \rceil$ and $\rho = \lfloor a_1/a_2 \rfloor$. If $\rho \ge \mu_2$, then $v(a_1, a_2; r) \ge 1/(\mu_1 + \mu_2)$.

Theorem 3.2. Let $\mu_i = \lceil (1 - ia_1)/a_2 \rceil$ for positive integers i and $P = max\{i : \mu_i > 0\}$, then $v(q_1, a_2; 2) \ge 1/(\mu_1 + \cdots + \mu_P)$.

Proof. We introduce pure Defender strategies which will be used in a mixed strategy which will be shown to be guarantee the Defender an expectation of at least $1/(\mu_1 + \cdots + \mu_P)$. For $t = 1, \ldots, P$ and $i(t) = 0, \ldots, \mu_t - 1$, put $I^{(t)}_{i(t)} = ((t-1)a_1, ta_1 + i(t)a_2)$ and $J^{(t)}_{i(t)} = (1 - ta_1, 1 - ta_1 - (i(t)+1)a_2)$. Let $\mathcal{I}^{(t)} = \cup_{i(t)=0}^{\mu_t - 1}\{I^{(t)}_{i(t)}\}$, $\mathcal{J}^{(t)} = \cup_{i(t)=0}^{\mu_t-1}\{J^{(t)}_{i(t)}\}$ and $\mathcal{S} = \cup_{t=1}^{P}\mathcal{I}^{(t)} \cup \mathcal{J}^{(t)}$. Consider the mixed Defender strategy which chooses a member of $\mathcal{S}$ at random. Since $\mathcal{S}$ contains $2\sum_{i=1}^{P}\mu_i$ pure strategies, the result follows if, for any pure Infiltrator strategy (y_1, y_2), both infiltrators are detected by at least two members of $\mathcal{S}$. We may clearly assume that $y_1 \le y_2$.

If $y_1 \in [(\beta-1)a_1, \beta a_1]$ for some β satisfying $1 \le \beta \le P$, it is easy to check that both infiltrators are detected by at least one member of $\mathcal{I}^{(\beta)}$ and by every member of $\mathcal{I}^{(\beta)}$ if $y_2 \le \beta a_1$. Now $\mu_1 > 1$ because $a_1 + a_2 < 1$ so $|\mathcal{I}^{(1)}| \ge 2$ and we can suppose $y_2 > a_1 \ge 1 - Pa_1$ by the definition of P. Thus $y_2 \in [1 - ta_1, 1 - (t-1)a_1]$ for some t satisfying $1 \le t \le P$ and it then follows that both infiltrators are also detected by a member of $\mathcal{J}^{(t)}$.

Hence the only case remaining to consider is that when $y_1 > Pa_1 \ge 1 - a_1$. But then both infiltrators are detected by the members of $\mathcal{J}^{(1)}$ and the result follows. ∎

Theorem 3.3. Let $\lambda = \lceil \frac{a_1}{a_2} \rceil$ and $k^* = \lceil \frac{1-a_1}{\lambda a_2} \rceil$. For $k = 1, \ldots, k^*$, let $\alpha_k = \lceil \frac{1-a_1-(k-1)\lambda a_2}{a_2} \rceil$. Then $v(a_1, a_2; 2) \le 1/\sum_{k=1}^{k^*}\alpha_k$.

Proof. Let $1 - a_1 - (k-1)\lambda a_2 = (\alpha_k - 1)a_2 + \eta_k$, then $\eta_k > 0$ by the definition of α_k. Take $\eta = min\{\eta_k : k = 1, \ldots, k^*\}$ and choose $\epsilon > 0$ and $\delta > 0$ such that $\alpha_1\epsilon < \delta$ and $k^*\delta + \alpha_1\epsilon < \eta$. For

$k = 1, \ldots, k^*$ and $j = 1, \ldots, \alpha_k$, let $x_j = (j-1)a_2 + j\epsilon$, $y_{kj} = x_j + a_1 + (k-1)\lambda a_2 + k\delta_k$ and $P_{kj} = (x_j, y_{kj})$. Note that $y_{kj} = 1 - (1 - a_1 - (k-1)\lambda a_2) + (j-1)a_2 + k\delta + j\epsilon \leq 1 - (\alpha_k - j)a_2 - \eta_k + k^*\delta + \alpha_1\epsilon < 1$ and $x_j < y_{kj}$ so P_{kj} is a valid pure strategy for Infiltrator. We will show that, for any placement of the cables, at most one of the P_{kj} has both of its agents detected.

Assume that there is a pure Defender strategy which detects both agents in P_{kj} and both in $P_{k'j'}$ where $(k, j) \neq (k', j')$; without loss of generality we may assume $j \leq j'$. First note that $y_{kj} - x_j > b$ so one cable length cannot detect both agents of P_{kj} nor both agents of $P_{k'j'}$. Thus each cable length must detect an agent of P_{kj} and an agent of $P_{k'j'}$. If $j = j'$, then $k \neq k'$ and a cable length must therefore detect both y_{kj} and $y_{k'j'}$ but this is impossible because $|y_{kj} - y_{k'j'}| > \lambda a_2 \geq a_1$ by the definition of λ. Hence we may assume $j' > j$. Since $x_{j'} > x_j + a_2$ and $y_{k'j'} > x_{j'}$, the cable of length a_2 cannot detect agent x_j. Thus the cable of length a_1 detects agent x_j and, because $y_{k'j'} > a_1 + x_{j'} \geq a_1 + x_j$, it must then also detect agent $x_{j'}$. Thus, using $a_1 \leq \lambda a_2$, $(j'-1)a_2 + j'\epsilon = x_{j'} \leq x_j + \lambda a_2 = (j - 1 + \lambda)a_2 + j\epsilon$ and so $j' \leq j - 1 + \lambda$ because $\epsilon > 0$. Hence $x_{j'} - x_j \leq (\lambda - 1)a_2 + (j' - j)\epsilon$. Clearly $|y_{kj} - y_{k'j'}| > a_2$ if $k' \geq k$ so we can assume $k > k'$ but then $|y_{kj} - y_{k'j'}| > (k - k')(\lambda a_2 + \delta) - (x_{j'} - x_j) \geq \lambda a_2 + \delta - \{(\lambda - 1)a_2 + (j' - j)\epsilon\} \geq a_2 + \delta - (\alpha_1 - 1)\epsilon > a_2$.

Hence, for any pure Defender strategy, at most one of the P_{kj} has both its agents detected. The P_{kj} are clearly distinct so, if the Infiltrator chooses each point P_{kj} with equal probability, the searcher will discover both agents with probability of at most $1/\sum_{k=1}^{k^*} \alpha_k$. ∎

Theorems 3.2 and 3.3 give a lower bound and an upper bound for the value of the game and a natural question is whether they are best possible. The answer is yes in the sense that the bounds coincide in some cases as the following theorem demonstrates.

Theorem 3.4. For a positive integer m, suppose $a_1 = a_2 m$. Then

$$v(a_1, a_2; 2) = \frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_{k^*}} = \frac{1}{\mu_1 + \mu_2 + \cdots + \mu_P},$$

where μ_i, P, α_k and k^* are defined in the previous theorems.

By a usual consideration in the two-person zero-sum games, we see that $v(a, b; 2)$ is increasing in both a_1 and a_2. From this and from Theorem 3.4, we obtain the next corollary.

Corollary 3.5. For a positive integer m, suppose $a_2(m-1) < a_1 < a_2 m$. Then

$$\frac{1}{\lfloor \frac{1}{a_2(m-1)} \rfloor \{\lceil \frac{1}{a_2} \rceil - \frac{(m-1)(\lfloor \frac{1}{a_2(m-1)} \rfloor + 1)}{2}\}} \leq v(a_1, a_2; 2) \leq \frac{1}{\lfloor \frac{1}{a_2 m} \rfloor \{\lceil \frac{1}{a_2} \rceil - \frac{m(\lfloor \frac{1}{a_2 m} \rfloor + 1)}{2}\}}.$$

Suppose $a_1 = 1/n$ and $a_2 = a_1/k$ for some positive integers n and k. It follows from Theorem 3.4 that the mixed Defender strategy $\mathcal{D}^*$ in the proof of Theorem 4.1 is optimal in $\Gamma(1/n, 1/(kn); 2)$; furthermore $\mu_i = k(n-i)$ so $v(1/n, 1/(kn); 2) = 2/(kn(n-1))$. If $n \geq 3$ and Infiltrator has at least three agents, Infiltrator can always get an agent through undetected by sending agents through at the points 0, 1/2 and 1. Thus, for $n \geq 3$, $v(1/n, 1/(kn); r) = 0$ for $r \geq 3$ and any Defender strategy is optimal in $\Gamma(1/n, 1/(kn); r)$ for $r \geq 3$. Using Remark 1, we therefore have the following theorem.

Theorem 3.6. For positive integers $n \geq 3$ and k, $v_1(1/n, 1/(kn)) = (k+1)/(kn)$, $v_2(1/n, 1/(kn)) = 2/(kn(n-1))$ and $v_r(1/n, 1/(kn)) = 0$ for $r \geq 3$.

The case when $a_1 = ka_2$ for some positive integer k and $a_1 \neq 1/n$ for any positive integer n cannot be dealt with so easily. To illustrate this, $\mathcal{D}^*$ is not optimal in $\Gamma(2/7, 1/7; 1)$ because, against $\mathcal{D}^*$, Infiltrator can restrict Defender's payoff to 7/18 by sending an agent through the point 5/18 whereas it is known that $v(2/7, 1/7; 1) = 3/7 > 7/18$. We suspect that, for this case, Defender does not have an optimal strategy against an unknown number of agents and that an effective Defender strategy will need to take into account his beliefs (probabilities) concerning the number of agents available to Infiltrator.

4. Guaranteed Infiltrator Penetration.

The question of how many agents Infiltrator needs to ensure that at least one of them gets through undetected is an interesting and potentially important one for both players. To obtain an answer, it seems that the Infiltrator does have to have some knowledge of the cable lengths. Trivially, if he knows that Defender has m cable lengths all of which are less than $1/m$, he can ensure that at least one gets through undetected by sending $m+1$ infiltrators down the channel at points t/m $(t = 0, 1, \ldots, m)$. If the Infiltrator has more precise information, we can have a less trivial result.

Theorem 4.1. If the Infiltrator knows all the cable lengths (and they sum to less than one), then Infiltrator can ensure at least one agent gets through undetected if he has at least 2^m agents.

Proof. Let $Z = \{\lambda_1 a_1 + \cdots + \lambda_m a_m : \lambda_i \in \{0,1\}\ i = 1, \ldots, m\}$ and arrange the members of Z as increasing sequence $z_0 = 0, z_1, \ldots, z_{2^m-1}$. Suppose the Infiltrator uses the strategy in which infiltrators are sent down the channel at the points $w_i = z_i + i\delta$ $(i = 0, 1 \ldots, 2^m - 1)$ where $2^m\delta < 1 - \sum_{i=1}^m a_i$. Suppose a Defender strategy detects all of these infiltrators, then, in particular, one length say a_{t_1} must cover the point $w_0 = 0$. Now $a_{t_1} = z_{s_1}$ for some s_1 and so a_{t_1} cannot cover the point w_{s_1}. Hence this point must be covered by a length a_{s_2} which in turn cannot cover the point w_{s_2} where s_2 is given by $z_{s_2} = a_{t_1} + a_{t_2}$. Continuing this process we see that $w_{s_m} = w_{2^m-1}$ will not be covered by any length a_i if all the other w_i are covered. ∎

If Infiltrator knows the individual lengths of the cables, he also knows the sum of their lengths and, if this sum is significantly less than one, we shall see below that a better upper bound than that in the theorem can be obtained. However the theorem as it stands is sharp and, for any positive integer m, its upper bound of 2^m cannot be improved as the following example demonstrates.

Example. Let $a_i = 2^{i-1}/2^m$ for $i = 1, \ldots, m$, then Infiltrator cannot guarantee an agent gets through undetected if he has at most 2^{m-1} agents.

Proof. We prove the result by giving a Defender strategy which guarantees that, for any pure Infiltrator strategy involving 2^{m-1} agents, there is a probability of at least $1/2^m$ that all agents are detected. Note that, given any integer r satisfying $0 \leq r \leq 2^m - 1$, there is a unique subset of the a_i which sum to $r/2^m$. Thus, for any such r, Defender can use the pure strategy S_r which places the cables so that they cover the intervals $[0, r/2^m]$ and $[(r+1)/2^m, 1]$. There are 2^m such strategies and we let $\mathcal{S}$ denote the mixed Defender strategy which selects one of them at random. Now consider a pure Infiltrator strategy; if this strategy has no agent in the interval $(r/2^m, (r+1)/2^m)$, every agent is detected by S_r and there is a probability of at least $1/2^m$ that all the agents are detected by $\mathcal{S}$. Hence, to ensure that at least one agent gets through undetected against $\mathcal{S}$, an Infiltrator strategy must place an agent in every one of the intervals $(r/2^m, (r+1)/2^m)$ $r = 0, 1, \ldots, 2^m - 1$. Since these intervals are disjoint, this cannot be achieved with $2^m - 1$ agents.

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