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Kyoto University
American Options with Uncertainty of the Stock Prices: The Discrete-Time Model

1. Introduction

A discrete-time mathematical model for American put option with uncertainty is presented, and the randomness and fuzziness are evaluated by both probabilistic expectation and $\lambda$-weighted possibilistic mean values.

2. Fuzzy stochastic processes

First we give some mathematical notations regarding fuzzy numbers. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a $\sigma$-field and $P$ is a non-atomic probability measure. $\mathbb{R}$ denotes the set of all real numbers, and let $\mathcal{C}(\mathbb{R})$ be the set of all non-empty bounded closed intervals. A 'fuzzy number' is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [12] regarding fuzzy set theory. $\mathcal{R}$ denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with its corresponding membership functions. The $\alpha$-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0, 1])$$ and

$$\tilde{a}_0 := \text{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}^-_{\alpha}, \tilde{a}^+_{\alpha}] \ \text{for} \ \alpha \in [0, 1].$$

Hence we introduce a partial order $\succeq$, so called the 'fuzzy max order', on fuzzy numbers $\mathcal{R}$: Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers.

$$\tilde{a} \succeq \tilde{b} \ \text{means that} \ \tilde{a}^-_{\alpha} \geq \tilde{b}^-_{\alpha} \ \text{and} \ \tilde{a}^+_{\alpha} \geq \tilde{b}^+_{\alpha} \ \text{for all} \ \alpha \in [0, 1].$$

Then $(\mathcal{R}, \succeq)$ becomes a lattice. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a} \vee \tilde{b}$ with respect to the fuzzy max order $\succeq$ by the fuzzy number whose $\alpha$-cuts are

$$(\tilde{a} \vee \tilde{b})_{\alpha} = \{\max\{\tilde{a}^-_{\alpha}, \tilde{b}^-_{\alpha}\}, \max\{\tilde{a}^+_{\alpha}, \tilde{b}^+_{\alpha}\}\}, \ \alpha \in [0, 1]. \ (2.1)$$
An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: For \( \tilde{a}, \tilde{b} \in \mathcal{R} \) and \( \lambda \geq 0 \), the addition and subtraction \( \tilde{a} \pm \tilde{b} \) of \( \tilde{a} \) and \( \tilde{b} \) and the scalar multiplication \( \lambda \tilde{a} \) of \( \lambda \) and \( \tilde{a} \) are fuzzy numbers given by

\[
(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad (\tilde{a} - \tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-]
\]

and \( (\lambda \tilde{a})_\alpha := [\lambda \tilde{a}_\alpha^- , \lambda \tilde{a}_\alpha^+] \) for \( \alpha \in [0, 1] \).

A fuzzy-number-valued map \( \tilde{X} : \Omega \rightarrow \mathcal{R} \) is called a ‘fuzzy random variable’ if the maps \( \omega \mapsto \tilde{X}_\alpha^-(\omega) \) and \( \omega \mapsto \tilde{X}_\alpha^+(\omega) \) are measurable for all \( \alpha \in [0, 1] \), where \( \tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \) (see [10]). Next we need to introduce expectations of fuzzy random variables in order to describe an optimal stopping model in the next section. A fuzzy random variable \( \tilde{X} \) is called integrably bounded if both \( \omega \mapsto \tilde{X}_\alpha^-(\omega) \) and \( \omega \mapsto \tilde{X}_\alpha^+(\omega) \) are integrable for all \( \alpha \in [0, 1] \). Let \( \tilde{X} \) be an integrably bounded fuzzy random variable. The expectation \( E(\tilde{X}) \) of the fuzzy random variable \( \tilde{X} \) is defined by a fuzzy number (see [7])

\[
E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min \{\alpha, 1_{E(\tilde{X})_{\alpha}}(x)\}, \quad x \in \mathbb{R},
\]

(2.2)

where closed intervals \( E(\tilde{X})_{\alpha} := \left[ \int_{\Omega} \tilde{X}_\alpha^-(\omega) dP(\omega), \int_{\Omega} \tilde{X}_\alpha^+(\omega) dP(\omega) \right] (\alpha \in [0, 1]) \).

In the rest of this section, we introduce stopping times for fuzzy stochastic processes. Let \( T (T > 0) \) be an ‘expiration date’ and let \( \mathcal{T} := \{0, 1, 2, \cdots, T\} \) be the time space. Let a ‘fuzzy stochastic process’ \( \{\tilde{X}_t\}_{t=0}^{T} \) be a sequence of integrably bounded fuzzy random variables such that \( E(\max_{t \in \mathcal{T}} \tilde{X}_{t,0}) < \infty \), where \( \tilde{X}_{t,0}(\omega) \) is the right-end of the 0-cut of the fuzzy number \( \tilde{X}_t(\omega) \). For \( t \in \mathcal{T} \), \( \mathcal{M}_t \) denotes the smallest \( \sigma \)-field on \( \Omega \) generated by all random variables \( \tilde{X}_{s,\alpha}^-(\omega) \) and \( \tilde{X}_{s,\alpha}^+(\omega) \) \( (s = 0, 1, 2, \cdots, t; \alpha \in [0, 1]) \). We call \( (\tilde{X}_t, \mathcal{M}_t)_{t=0}^{\infty} \) a fuzzy stochastic process. A map \( \tau : \Omega \rightarrow \mathcal{T} \) is called a ‘stopping time’ if

\[
\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{M}_t \quad \text{for all } t = 0, 1, 2, \cdots, T.
\]

Then, the following lemma is trivial from the definitions ([11]).

**Lemma 2.1.** Let \( \tau \) be a stopping time. We define

\[
\tilde{X}_\tau(\omega) := \tilde{X}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \cdots, T \text{ and } \omega \in \Omega.
\]

Then, \( \tilde{X}_\tau \) is a fuzzy random variable.

**3. American put option with uncertainty of stock prices**

In this section, we formulate American put option with uncertainty of stock prices by fuzzy random variables. Let \( \mathcal{T} := \{0, 1, 2, \cdots, T\} \) be the time space with an expiration date \( T (T > 0) \) similarly to the previous section, and take a probability space \( \Omega := \mathbb{R}^{T+1} \). Let \( r (r > 0) \) be an interest rate of a bond price, which is riskless asset, and put a discount
rate $\beta = 1/(1 + r)$. Define a ‘stock price process’ $\{S_t\}_{t=0}^T$ as follows: An initial stock price $S_0$ is a positive constant and stock prices are given by

$$S_t := S_0 \prod_{s=1}^{t} (1 + Y_s) \quad \text{for } t = 1, 2, \cdots, T,$$

where $\{Y_t\}_{t=1}^T$ is a uniform integrable sequence of independent, identically distributed real random variables on $[r-1, r+1]$ such that $E(Y_t) = r$ for all $t = 1, 2, \cdots, T$. The $\sigma$-fields $\{\mathcal{M}_t\}_{t=0}^T$ are defined as follows: $\mathcal{M}_0$ is the completion of $\{\emptyset, \Omega\}$ and $\mathcal{M}_t(t = 1, 2, \cdots, T)$ denote the complete $\sigma$-fields generated by $\{Y_1, Y_2 \cdots Y_t\}$.

We consider a finance model where the stock price process $\{S_t\}_{t=0}^T$ takes fuzzy values. Now we give fuzzy values by triangular fuzzy numbers for simplicity. Let $\{a_t\}_{t=0}^T$ be an $\mathcal{M}_t$-adapted stochastic process such that $0 < a_t(\omega) \leq S_t(\omega)$ for $\omega \in \Omega$. A ‘stock price process with fuzzy values’ are represented by a sequence of fuzzy random variables $\{\tilde{S}_t\}_{t=0}^T$:

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega))$$

for $t \in T, \omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ ($x \in \mathbb{R}$) is the triangle shape function. Hence, $a_t(\omega)$ is a spread of triangular fuzzy numbers $\tilde{S}_t(\omega)$ and corresponds to the amount of fuzziness in the process. Then, $a_t(\omega)$ should be an increasing function of the stock price $S_t(\omega)$ (see Assumption S in the next section).

Let $K$ ($K > 0$) be a ‘strike price’. The ‘price process’ $\{\tilde{P}_t\}_{t=0}^T$ of American put option under uncertainty is represented by

$$\tilde{P}_t(\omega) := \beta^t(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t = 0, 1, 2, \cdots, T,$$

where $\vee$ is given by (2.1), and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp number $K$ and zero respectively. An ‘exercise time’ in American put option is given by a stopping time $\tau$ with values in $T$. For an exercise time $\tau$, we define

$$\tilde{P}_\tau(\omega) := \tilde{P}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \cdots, T, \text{ and } \omega \in \Omega.$$

Then, from Lemma 2.1, $\tilde{P}_\tau$ is a fuzzy random variable. The expectation of the fuzzy random variable $\tilde{P}_\tau$ is a fuzzy number (see (2.2))

$$E(\tilde{P}_\tau)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{P}_\tau)_\alpha}(x)\}, \quad x \in \mathbb{R},$$

where $E(\tilde{P}_\tau)_\alpha = \left[\int_{\Omega} \tilde{P}_{\tau,a}^{-}(\omega) dP(\omega), \int_{\Omega} \tilde{P}_{\tau,a}^{+}(\omega) dP(\omega)\right]$. In American put option, we must maximize the expected values (3.5) of the price process by stopping times $\tau$, and we need to evaluate the fuzzy numbers (3.5) since the fuzzy max order (2.1) on $\mathcal{R}$ is a partial order and not a linear order. In this paper, we consider the following estimation regarding the price process $\{\tilde{P}_t\}_{t=0}^T$ of American put option. Let $g : \mathcal{C}(\mathbb{R}) \mapsto \mathbb{R}$ be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda)y, \quad [x, y] \in \mathcal{C}(\mathbb{R}),$$
where $\lambda$ is a constant satisfying $0 \leq \lambda \leq 1$. This scalarization is used for the evaluation of fuzzy numbers, and $\lambda$ is called a ‘pessimistic-optimistic index’ and means the pessimistic degree in decision making. We call $g$ a ‘$\lambda$-weighting function’ and we evaluate fuzzy numbers $\tilde{a}$ by “$\lambda$-weighted possibilistic mean value”

$$\int_{0}^{1} 2\alpha g(\tilde{a}_\alpha) \ d\alpha, \quad (3.7)$$

where $\tilde{a}_\alpha$ is the $\alpha$-cut of fuzzy numbers $\tilde{a}$. (see Carlsson and Fullér [1], Goetschel and Voxman [4]) When we apply a $\lambda$-weighting function $g$ to (3.5), its evaluation follows

$$\int_{0}^{1} 2\alpha g(E(\tilde{P}_{\tau})_\alpha) \ d\alpha. \quad (3.8)$$

Now we analyze (3.8) by $\alpha$-cuts technique of fuzzy numbers. The $\alpha$-cuts of fuzzy random variables (3.2) are

$$\tilde{S}_{t,\alpha}(\omega) = [S_t(\omega) - (1 - \alpha)a_t(\omega), S_t(\omega) + (1 - \alpha)a_t(\omega)], \quad \omega \in \Omega, \quad (3.9)$$

and so

$$\tilde{S}_{t,\alpha}^\pm(\omega) = S_t(\omega) \pm (1 - \alpha)a_t(\omega), \quad \omega \in \Omega \quad (3.10)$$

for $t \in T$ and $\alpha \in [0, 1]$. Therefore, the $\alpha$-cuts of (3.3) are

$$\tilde{P}_{t,\alpha}(\omega) = [\tilde{P}_{t,\alpha}^-(\omega), \tilde{P}_{t,\alpha}^+(\omega)] := [\beta^t \max\{K - \tilde{S}_{t,\alpha}^+(\omega), 0\}, \beta^t \max\{K - \tilde{S}_{t,\alpha}^-(\omega), 0\}], \quad (3.11)$$

and we obtain $E(\max_{t \in T} \sup_{\alpha \in [0, 1]} \tilde{P}_{t,\alpha}) \leq K < \infty$ since $\tilde{S}_{t,\alpha}^-(\omega) \geq 0$, where $E(\cdot)$ is the expectation with respect to some risk-neutral equivalent martingale measure([2],[6]). For a stopping time $\tau$, the expectation of the fuzzy random variable $\tilde{P}_{\tau}$ is a fuzzy number whose $\alpha$-cut is a closed interval

$$E(\tilde{P}_{\tau})_{\alpha} = E(\tilde{P}_{\tau,\alpha}) = [E(\tilde{P}_{\tau,\alpha}^-), E(\tilde{P}_{\tau,\alpha}^+)] \quad \text{for } \alpha \in [0, 1], \quad (3.12)$$

where $\tilde{P}_{\tau,\alpha}(\omega) = [\tilde{P}_{\tau,\alpha}^-(\omega), \tilde{P}_{\tau,\alpha}^+(\omega)]$ is the $\alpha$-cut of fuzzy number $\tilde{P}_{\tau}(\omega)$. Using the $\lambda$-weighting function $g$, from (3.7) the evaluation of the fuzzy random variable $\tilde{P}_{\tau}$ is given by the integral

$$\int_{0}^{1} 2\alpha g(E(\tilde{P}_{\tau,\alpha})) \ d\alpha. \quad (3.13)$$

Put the value by $P(\tau)$. Then, from (2.2), the terms (3.8) and (3.13) coincide:

$$P(\tau) = \int_{0}^{1} 2\alpha g(E(\tilde{P}_{\tau,\alpha})) \ d\alpha = \int_{0}^{1} 2\alpha g(E(\tilde{P}_{\tau})_{\alpha}) \ d\alpha. \quad (3.14)$$

Therefore $P(\tau)$ means an evaluation of the expected price of American put option when $\tau$ is an exercise time. Further, we have the following equality.
Lemma 3.1. For a stopping time $\tau$ ($\tau \leq T$), it holds that

\[ P(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) \, d\alpha = \int_0^1 2\alpha E(g(\tilde{P}_{\tau,\alpha})) \, d\alpha = E \left( \int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\cdot)) \, d\alpha \right). \tag{3.15} \]

We put the 'optimal expected price' by

\[ V := \sup_{\tau\tau \leq T} P(\tau) = \sup_{\tau\tau \leq T} \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha})) \, d\alpha. \tag{3.16} \]

In the next section, this paper discusses the following optimal stopping problem regarding American put option with fuzziness.

Problem P. Find a stopping time $\tau^*(\tau^* \leq T)$ and the optimal expected price $V$ such that

\[ P(\tau^*) = V, \tag{3.17} \]

where $V$ is given by (3.16).

Then, $\tau^*$ is called an 'optimal exercise time'.

4. The optimal expected price and the optimal exercise time

In this section, we discuss the optimal fuzzy price $V$ and the optimal exercise time $\tau^*$, by using dynamic programming approach. Now we introduce an assumption.

Assumption S. The stochastic process $\{a_t\}_{t=0}^T$ is represented by

\[ a_t(\omega) := cS_t(\omega), \quad t = 0, 1, 2, \ldots, T, \quad \omega \in \Omega, \]

where $c$ is a constant satisfying $0 < c < 1$.

Assumption S is reasonable since $a_t(\omega)$ means a size of fuzziness and it should depend on the volatility and the stock price $S_t(\omega)$ because one of the most difficulties is estimation of the actual volatility ([8, Sect.7.5.1]). In this model, we represent by $c$ the fuzziness of the volatility, and we call $c$ a 'fuzzy factor' of the process. From now on, we suppose that Assumption S holds. For a stopping time $\tau$ ($\tau \leq T$), we define a random variable

\[ \Pi_\tau(\omega) := \int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\omega)) \, d\alpha, \quad \omega \in \Omega. \tag{4.1} \]

From Lemma 3.1, $P(\tau) = E(\Pi_\tau)$ is the evaluated price of American put option when $\tau$ is an exercise time. Then we have the following representation about (4.1).

Lemma 4.1. For a stopping time $\tau$ ($\tau \leq T$), it holds that

\[ \Pi_\tau(\omega) = \beta^{\tau(\omega)} f^{P}(S_\tau(\omega)), \quad \omega \in \Omega, \tag{4.2} \]
where $f^P$ is a function on $(0, \infty)$ such that
\[
f^P(y) := \begin{cases} 
K - y - \frac{1}{3}cy(2\lambda - 1) + \lambda\varphi^1(y) & \text{if } 0 < y < K \\
(1 - \lambda)\varphi^2(y) & \text{if } y \geq K,
\end{cases}
\] (4.3)
and
\[
\varphi^1(y) := \frac{1}{(cy)^2}((-K + y + cy)\max\{0, -K + y + cy\}^2 - \frac{2}{3}\max\{0, -K + y + cy\}^3), \quad y > 0, 
\] (4.4)
\[
\varphi^2(y) := \frac{1}{(cy)^2}((K - y + cy)\max\{0, K - y + cy\}^2 - \frac{2}{3}\max\{0, K - y + cy\}^3), \quad y > 0. 
\] (4.5)

Now we give an optimal stopping time for Problem P and we discuss an iterative method to obtain the optimal expected price $V$ in (3.16). To analyze the optimal fuzzy price $V$, we put
\[
V_t^P(y) = \sup_{t \leq \tau \leq T} E(\beta^{-t}\Pi_{\tau}|S_t = y) 
\] (4.6)
for $t = 0, 1, 2, \cdots, T$ and an initial stock price $y \ (y > 0)$. Then we note that $V = V_0^P(y)$.

**Theorem 4.1** (Optimality equation).

(i) The optimal expected price $V = V_0^P(y)$ with an initial stock price $y \ (y > 0)$ is given by the following backward recursive equations (4.7) and (4.8):
\[
V_t^P(y) = \max\{\beta E(V_{t+1}^P(y(1+Y_1))), f^P(y)\}, \quad t = 0, 1, \cdots, T-1, \ y > 0, 
\] (4.7)
\[
V_T^P(y) = f^P(y), \quad y > 0.
\] (4.8)

(ii) Define a stopping time
\[
\tau^P(\omega) := \inf\{t \in \mathbb{T} \mid V_0^P(S_t(\omega)) = f^P(S_t(\omega))\}, \quad \omega \in \Omega,
\] (4.9)
where the infimum of the empty set is understood to be $T$. Then, $\tau^P$ is an optimal exercise time for Problem P, and the optimal value of American put option is
\[
V = V_0^P(y) = P(\tau^P) 
\] (4.10)
for an initial stock price $y > 0$.

5. A numerical example

Now we give a numerical example to illustrate our idea in Sections 3 and 4.

**Example 5.1.** We consider CRR type American put option model (see Ross [8, Sect.7.4]). Put an expiration date $T = 10$, an interest rate of a bond $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 30$ and a strike price $K = 35$. Assume that
$\{Y_t\}_{t=1}^{T}$ is a uniform sequence of independent, identically distributed real random variables such that

$$Y_t := \begin{cases} e^\sigma - 1 & \text{with probability } p \\ e^{-\sigma} - 1 & \text{with probability } (1-p) \end{cases}$$

for all $t = 1, 2, \cdots, T$, where $\sigma = 0.25$ and $p = (1 + r - e^{-\sigma})/(e^\sigma - e^{-\sigma})$. Then we have $E(Y_t) = r$. The corresponding optimal exercise time is given by

$$\tau^P(\omega) = \inf\{t \in T \mid V_0^P(S_t(\omega)) = f^P(S_t(\omega))\}.$$

In the following Table, the optimal expected price $V = V_0^P(y)$ at initial stock price $y = 30$ changes with the pessimistic-optimistic index $\lambda$ of the $\lambda$-weighting function $g$.

Table. The optimal expected price $V = V_0^P(y)$ at initial stock prices $y = 30$.

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References


